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Injective hyperbolicity of domains

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Abstract. The pseudometric of Hahn is identical to the Kobayashi–Royden pseudometric on domains of dimension greater than two. Thus injective hyperbolicity coincides with ordinary hyperbolicity in this case.

1. Introduction. The Kobayashi pseudodistance d_M and Kobayashi–Royden pseudodifferential metric K_M of a complex manifold M are defined by means of extremal problems for holomorphic mappings of the unit disk $\mathbb{D}$ into M. By restricting to *injective* holomorphic mappings in these extremal problems, one arrives at a pseudodistance τ_M and a pseudodifferential metric S_M respectively. These were considered first on plane domains by Siu [4], and in general by Hahn [1]. In the literature, they go under the names of S-metric or Hahn metric. If the pseudodifferential metric S_M satisfies an inequality

$$S_M(z,\xi) \geq c\|\xi\|, \qquad c > 0,$$

at each point of M, then M is said to be *S-hyperbolic* (alternatively *Hahn hyperbolic* or *injective hyperbolic*). In this note we consider S_M and its relationship to K_M.

From the work of Siu [4] and Minda [3] it is known that if M is a Riemann surface, then it is S-hyperbolic unless it is the plane or the extended plane, and Minda also proved that S_M and K_M are distinct unless M is simply connected. For domains of higher dimension there are results on S-hyperbolicity due to Zhang [7], Vesentini [5] and Vigué [6]. Zhang proved that if S_M is a complete metric, then M is a domain of holomorphy, and observed that the converse does not hold. Vesentini showed that a domain of the form $\mathbb{C}^* \times \Omega$ is not S-hyperbolic if Ω is a domain of dimension two or larger, thus disproving the claim by Hahn that $(\mathbb{C}^*)^n$ is S-hyperbolic for

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any positive integer n. Vigué generalized the result of Vesentini by showing that a product of two domains is S-hyperbolic only if it is hyperbolic.

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2. Domains in high dimensions

THEOREM 1. *If* $\Omega \subseteq \mathbb{C}^n$, $n \geq 3$, *is a domain*, *then* $S_\Omega \equiv K_\Omega$.

Proof. Let $a \in \Omega$, $\eta \in \mathbb{C}^n$ with $\eta \neq 0$ be given. It is enough to show that $S_\Omega(a,\eta) \leq K_\Omega(a,\eta)$. By a translation of Ω we may assume that $a = 0$, and by a rotation, we may assume that $\eta_1 \ldots \eta_n \neq 0$. Let $\varepsilon > 0$ be arbitrary.

Choose a holomorphic mapping $f : \mathbb{D} \to \Omega$ with $f(0) = 0$ and

$$f_*(0)\nu = \eta, \qquad |\nu| \leq K_\Omega(0,\eta) + \varepsilon/2$$

for some $\nu \in \mathbb{C}$. Define $f_1 : \mathbb{D} \to \Omega$ by $f_1(z) = f((1-\delta)z)$ for a suitably small $\delta > 0$; then $f_1(0) = 0$ and

$$(f_1)_*(0)\frac{\nu}{1-\delta} = \eta, \qquad \left|\frac{\nu}{1-\delta}\right| \leq K_\Omega(0,\eta) + \varepsilon,$$

say. Since f_1 is holomorphic on $\overline{\mathbb{D}}$ and $\operatorname{dist}(f_1(\overline{\mathbb{D}}), \partial\Omega) > 0$, there exists a polynomial mapping $f_2 : \overline{\mathbb{D}} \to \Omega$ with $f_2(0) = 0$ and $(f_2)_*(0) = (f_1)_*(0)$. We write out f_2 explicitly:

$$f_2(z) = \Big(\ldots, \sum_{k=1}^{m} A_{jk} z^k, \ldots\Big), \qquad 1 \leq j \leq n.$$

We shall show that there exist slight perturbations $\widetilde{A}_{jk}$ of the coefficients A_{jk}, $1 \leq j \leq n$, $2 \leq k \leq n$, such that

$$f_3(z) = \Big(\ldots, \sum_{k=1}^{m} \tilde{A}_{jk} z^k, \ldots\Big), \qquad 1 \leq j \leq n,$$

with $\widetilde{A}_{j1} = A_{j1}$, is an injective mapping $f_3 : \mathbb{D} \to \Omega$. Since $f_3(0) = 0$ and

$$(f_3)_*(0) = (\ldots, \widetilde{A}_{j1}, \ldots) = (\ldots, A_{j1}, \ldots) = (f_2)_*(0) = (f_1)_*(0),$$

the mapping f_3 is a competitor in the extremal problem that defines $S_\Omega(0,\eta)$, so

$$S_\Omega(0,\eta) \leq \left|\frac{\nu}{1-\delta}\right| \leq K_\Omega(0,\eta) + \varepsilon.$$

Letting $\varepsilon \to 0$, $S_\Omega \leq K_\Omega$ follows.

It remains to establish that it is possible to choose f_3 as required. Assume $f_3(z) = f_3(w)$ for some $z, w \in \mathbb{C}$ with $z \neq w$, thus

$$\begin{gathered}\widetilde{A}_{11}z + \ldots + \widetilde{A}_{1m}z^m = \widetilde{A}_{11}w + \ldots + \widetilde{A}_{1m}w^m,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \widetilde{A}_{n1}z + \ldots + \widetilde{A}_{nm}z^m = \widetilde{A}_{n1}w + \ldots + \widetilde{A}_{nm}w^m.\end{gathered}$$

Rearranging and dividing by $z - w$, we obtain

$$\begin{gathered}\widetilde{A}_{12}(z+w) + \widetilde{A}_{13}(z^2 + zw + w^2) + \ldots = -\widetilde{A}_{11},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ \widetilde{A}_{n2}(z+w) + \widetilde{A}_{n3}(z^2 + zw + w^2) + \ldots = -\widetilde{A}_{n1}.\end{gathered}$$

The image of $\mathbb{C}^2$ under the mapping given by

$$\begin{aligned} X_1 &= z + w,\\ X_2 &= z^2 + zw + w^2,\\ &\ldots\ldots\ldots\ldots\ldots\ldots\\ X_{m-1} &= z^{m-1} + \ldots + w^{m-1}\end{aligned}$$

lies on a projective surface V, while the equations

$$\begin{gathered}B_{12}X_1 + \ldots + B_{1m}X_{m-1} = 1,\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ B_{n2}X_1 + \ldots + B_{nm}X_{m-1} = 1,\end{gathered}$$

where $B_{jk} = \widetilde{A}_{jk}/(-\widetilde{A}_{j1})$, define a linear subspace L of the projective space $P_{m-1}(\mathbb{C})$ which is generically of dimension $m-1-n$. Thus $V \cap L = \emptyset$ generically since $\dim(V) + \dim(L) = 2 + m - 1 - n = (m-1) - (n-2) < m-1$ when $n \geq 3$. In particular, the set of B_{jk} for which $V \cap L = \emptyset$ is dense in $\mathbb{C}^{n(m-1)}$, and so the set of $\widetilde{A}_{jk}$ for which f_3 is injective on $\mathbb{C}$ is dense in $\mathbb{C}^{n(m-1)}$. Since $\mathrm{dist}(f_2(\overline{\mathbb{D}}), \partial\Omega) > 0$, we can choose the $\widetilde{A}_{jk}$ close enough to the A_{jk} so that $\mathrm{dist}(f_3(\overline{\mathbb{D}}), \partial\Omega) > 0$ while keeping f_3 injective. ∎

This theorem has some of the results of Zhang, Vesentini and Vigué as corollaries in dimension greater than two. From [2] it is known that a domain which is complete hyperbolic is a domain of holomorphy, thus the theorem of Zhang follows for domains of dimension greater than two. Theorem III of [5] follows directly, as does Corollaire 3.2 of [6] in dimension greater than two.

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