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Even coefficient estimates for bounded univalent functions

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Abstract. Extremal coefficient properties of Pick functions are proved. Even coefficients of analytic univalent functions f with $|f(z)| < M$, $|z| < 1$, are bounded by the corresponding coefficients of the Pick functions for large M. This proves a conjecture of Jakubowski. Moreover, it is shown that the Pick functions are not extremal for a similar problem for odd coefficients.

Let S denote the class of functions f,

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \,, \tag{1}$$

analytic and univalent in the unit disk $E = \{z : |z| < 1\}$. Let S^M, $M > 1$, denote the family of functions $f \in S$ bounded by M: $|f(z)| < M$ for $|z| < 1$. Moreover, set $S^\infty = S$.

L. de Branges [1] proved the Bieberbach conjecture: $|a_n| \le n$, $n \ge 2$, in the class S, with equalities only for the Koebe functions K_α,

$$K_\alpha(z) = \frac{z}{(1 - e^{i\alpha} z)^2} \,, \qquad \alpha \in \mathbb{R} \,.$$

The functions $P_\alpha^M \in S^M$ which satisfy the equation

$$\frac{M^2 P_\alpha^M(z)}{(M - P_\alpha^M(z))^2} = K_\alpha(z) \,, \qquad |z| < 1 \,,\ M > 1 \,, \qquad P_\alpha^\infty = K_\alpha \,,$$

are called *Pick functions*. Let

$$P_0^M(z) = z + \sum_{n=2}^{\infty} p_{n,M} z^n \,, \qquad 1 < M \le \infty \,, \qquad p_{n,\infty} = n \,.$$

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Z. Jakubowski [4] conjectured that even coefficients of functions $f \in S^M$ are bounded by $p_{n,M}$ for large M. Namely, for every even $n \geq 2$ there exists $M_n^+ > 1$ such that for all $M \geq M_n^+$ and all $f \in S^M$,

$$|a_n| \leq p_{n,M} . \tag{2}$$

For references to earlier results due to Z. Jakubowski, A. Zielińska, K. Zyskowska, L. Pietrasik, M. Schiffer, O. Tammi, O. Jokinen, see [4]. Recently the author's student V. G. Gordenko [3] proved the Jakubowski conjecture for $n = 6$. Moreover, he showed that Pick functions do not maximize $|a_5|$ in S^M with finite M.

In this article we prove the Jakubowski conjecture for all even $n \geq 2$. Moreover, we show that odd coefficients of functions $f \in S^M$ do not necessarily satisfy (2) for sufficiently large M.

1. According to [1] only Koebe functions are extremal for the estimate of $|a_n|$ in S. Since the classes S^M are rotation invariant, it is sufficient to find an upper estimate for $\operatorname{Re} a_n$ instead of one for $|a_n|$. Thus the Jakubowski conjecture reduces to the fact that only Pick functions P_0^M and their rotations give a local maximum of $\operatorname{Re} a_n$ in the class S^M for large M.

The author [6], [7] described a constructive algorithm determining the value set V_n^M of the coefficient system $\{a_2, \ldots, a_n\}$ in the class S^M, $1 < M \leq \infty$. The set V_n^M is the set reachable at time $t = \log M$ for the dynamical control system

$$\frac{da}{dt} = -2 \sum_{s=1}^{n-1} e^{-s(t+iu)} A(t)^s a(t) , \qquad a(0) = a^0 , \tag{3}$$

where $a = a(t) \in \mathbb{C}^n$,

$$a(t) = \begin{pmatrix} a_1(t) \\ \vdots \\ a_n(t) \end{pmatrix} , \qquad A(t) = \begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ a_1(t) & 0 & \ldots & 0 & 0 \\ a_2(t) & a_1(t) & \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{n-1}(t) & a_{n-2}(t) & \ldots & a_1(t) & 0 \end{pmatrix} ,$$

$a^0 = (1, 0, \ldots, 0)^T$, $a_1(t) \equiv 1$, and $u = u(t)$ is a real control. Optimal controls satisfy the Pontryagin maximum principle. They maximize the Hamilton function

$$H(t, a, \overline{\psi}, u) = -2 \sum_{s=1}^{n-1} \operatorname{Re}[e^{-s(t+iu)} (A^s a)^T \overline{\psi}] ,$$

while the conjugate vector $\overline{\psi} = (\overline{\psi}_1, \ldots, \overline{\psi}_n)^T$ of complex-valued Lagrange multipliers satisfies the conjugate Hamilton system

$$\frac{d\overline{\psi}}{dt} = 2\sum_{s=1}^{n-1} e^{-s(t+iu)}(s+1)(A^T)^s\overline{\psi}\,, \qquad \psi(0)=\xi\,. \tag{4}$$

The vector $(\psi_2(\log M),\ldots,\psi_n(\log M))$ is orthogonal to the boundary hypersurface ∂V_n^M of V_n^M. More precisely, it is orthogonal to a tangent plane or to a certain support plane if they exist. If $\operatorname{Re} a_n$ attains its maximum at any point $x\in\partial V_n^M$, then there exists ψ such that $(\psi_2(\log M),\ldots,\psi_n(\log M)) = (0,\ldots,0,1)$ at this point.

Points of ∂V_n^M are obtained from boundary extremal functions f, $f(z) = Mw(z,\log M)$, where $w(z,t)$ are solutions of the Cauchy problem for Loewner's differential equation

$$\frac{dw}{dt} = -w\frac{e^{iu}+w}{e^{iu}-w}\,, \qquad w|_{t=0}=z\,, \tag{5}$$

with optimal controls $u=u(t)$. Differentiating (5) with respect to z, we obtain a differential equation for $w'(z,t)$, from which we deduce differential equations for the coefficient system $b(t)=\{b_0(t),\ldots,b_{n-1}(t)\}$ of the function $f'(z)/(e^tw'(z,t))$. The system for $b(t)$ coincides with (4) with A^T replaced by A. Hence if $(\psi_2(\log M),\ldots,\psi_n(\log M)) = (0,\ldots,0,1)$, then

$$(\psi_2(t),\ldots,\psi_n(t)) = (b_{n-2}(t),\ldots,b_0(t))\,. \tag{6}$$

The initial value at $t=0$ yields that $\xi=(\xi_1,(n-1)a_{n-1},\ldots,2a_2,1)^T$.

2. Now we are able to prove the theorem for odd coefficients of $f\in S^M$.

THEOREM 1. *The Pick functions P_0^M are not extremal for the problem of estimating* $\operatorname{Re} a_{2m+1}$ *in the class S^M, for all sufficiently large finite M and natural m.*

Proof. P_0^M and K_0 correspond to the control $u(t)\equiv\pi$ in (3)–(4). In this case the condition $(\psi_2(\log M),\ldots,\psi_n(\log M)) = (0,\ldots,0,1)$ requires the initial value $(\xi_2,\ldots,\xi_n) = ((n-1)p_{n-1,M},\ldots,2p_{2,M},1)$, $1<M\le\infty$, in (4).

Put $n=2m+1$ and write the Hamilton function at $t=0$,

$$H(0,a^0,\overline{\xi},u) = -2\sum_{s=1}^{2m}\xi_{s+1}\cos(su)\,.$$

Hence

$$\frac{\partial H(0,a^0,\overline{\xi},u)}{\partial u} = 2\sum_{s=1}^{2m} s\xi_{s+1}\sin(su)\,,$$

and this derivative vanishes at $u = \pi$. Moreover,

$$\left.\frac{\partial^2 H(0, a^0, \overline{\xi}, u)}{\partial u^2}\right|_{u=\pi} = 2\sum_{s=1}^{2m} (-1)^s s^2 \xi_{s+1}\,.$$

Evidently this derivative vanishes if $M = \infty$. It must be non-positive for finite M if $u \equiv \pi$ satisfies Pontryagin's maximum principle.

Let us examine how this derivative depends on M. Write

$$h(M) = \sum_{s=1}^{2m} (-1)^s s^2 \xi_{s+1} = \sum_{s=1}^{2m} (-1)^s s^2 (2m+1-s) p_{2m+1-s,M}\,, \qquad p_{1,M} = 1\,.$$

Every coefficient $p_{j,M}$ can be found from (3). It is the jth coordinate of the vector $a(\log M)$ if $u(t) \equiv \pi$. Put $T = 1-1/M$, $h(M) = h(1/(1-T)) = g(T)$. Then by elementary calculations we find from (3) that

$$\left.\frac{dg}{dT}\right|_{T=1} = \frac{1}{3}\sum_{s=1}^{2m-1} (-1)^s s^2 (2m-s)(2m+1-s)^2(2m+2-s)\,.$$

One can verify that $(1/12)(j+1)(j+2)^2(j+3)$ is the jth coefficient of the function $(1-z)^{-4} + 2z(1-z)^{-5}$ while $(-1)^s s^2$ is the $(s-1)$th coefficient of $(z-1)(z+1)^{-3}$. Thus $(-\frac{1}{4})\frac{dg}{dT}|_{T=1}$ is the $(2m-2)$th coefficient of $(1-z^2)^{-2}(1-z)^{-2}$, and it is positive. Hence $h(M)$ is decreasing for sufficiently large M. Since $h(\infty) = 0$, we conclude that $h(M) > 0$ for large M.

The last result contradicts the maximizing property of the control $u = \pi$. This proves Theorem 1.

3. Now we are going to investigate the extremal properties of even coefficients of Pick functions.

THEOREM 2. *For every natural m there exists $M^+_{2m} > 1$ such that each function $f \in S^M$ satisfies the inequalities* (2) *for $n = 2m$ and all $M \geq M^+_{2m}$.*

P r o o f. Let X denote an arbitrary neighbourhood of the function K_0 in the class S, endowed with the topology of uniform convergence on compact subsets of the unit disk. Set $X^M = X \cap S^M$. The Pick function P_0^M belongs to X^M for sufficiently large M. By Section 1, it is sufficient to show that only P_0^M gives a local maximum for $\operatorname{Re} a_n$ in X^M.

Again we have $(\psi_2(\log M), \ldots, \psi_n(\log M)) = (0, \ldots, 0, 1)$ at a point $x \in \partial V_n^M$ where $\operatorname{Re} a_n$ attains its maximum. If x comes from a function $f \in S^M$ with expansion (1), then we need the initial value $(\xi_2, \ldots, \xi_n) = ((n-1)a_{n-1}, \ldots, 2a_2, 1)$ in (4).

Put $n = 2m$, $\xi^0 = (\xi_1, (2m-1)^2, \ldots, 1)^T$. Then

$$H(0, a^0, \overline{\xi}^0, u) = -2\sum_{s=1}^{2m-1} (2m-s)^2 \cos(su)\,.$$

By elementary calculations we find that

$$H(0,a^0,\overline{\xi}^0,u)-H(0,a^0,\overline{\xi}^0,\pi)=\frac{(-\sin u)[2m\sin u-\sin(2mu)]}{(1-\cos u)^2}\,.$$

It is easy to verify that the right-hand side of this equality is negative on $[0,2\pi]$, except for $u=\pi$, where it vanishes. Thus

$$H(0,a^0,\overline{\xi}^0,u)\le H(0,a^0,\overline{\xi}^0,\pi)\,, \tag{7}$$

with equality only for $u=\pi$. Moreover,

$$\frac{\partial H(0,a^0,\overline{\xi}^0,u)}{\partial(\cos u)}=2\sum_{s=1}^{2m-1}(-1)^s s^2(2m-s)^2\,.$$

This is the $(2m-2)$th coefficient of $-2(1-z^2)^{-2}$, and it is negative.

The sign of this derivative and the inequality (7) are preserved for close points ξ. Let $\xi=(\xi_1,\dots,\xi_n)^T$ be an arbitrary point in a neighbourhood of ξ^0, with $\xi_2,\dots,\xi_n$ real. Then according to the continuity principle $H(0,a^0,\overline{\xi},u)$ attains its maximum on $[0,2\pi]$ at the single point $u=\pi$. We can choose $(\xi_2,\dots,\xi_n)=((n-1)p_{n-1,M},\dots,2p_{2,M},1)$ for sufficiently large M. The control $u=\pi$ satisfies Pontryagin's maximum principle for $t>0$ in a certain neighbourhood of the initial value $t=0$, and the corresponding solution $w(z,t)$ of Loewner's differential equation (5) has real coefficients. Hence $u=\pi$ is optimal on the whole half-axis $[0,\infty)$ (see e.g. [6], [7]). This gives the Pick function P_0^M. So P_0^M satisfies the necessary conditions for maximum of $\operatorname{Re} a_n$.

It remains to show that the necessary conditions for an extremum hold at a unique point in X^M.

Let us consider the point $a=(1,2,\dots,n)^T$ in $\partial V_n=\partial V_n^\infty$ and its neighbourhood Q_a, $Q_a\subset\partial V_n$. Points of Q_a appear as the phase space projections of solutions of the Cauchy problem for the Hamilton system (3), (4). The neighbourhood Q_a corresponds to a neighbourhood Q_ξ of the initial value $\Lambda=(\xi_2,\dots,\xi_n)=((n-1)^2,\dots,1)$ in (4). This correspondence is not one-to-one. All points $\xi^*\in Q_\xi$ with real coordinates $\xi_2^*,\dots,\xi_n^*$ are mapped to the point a. The correspondence between the conjugate vector and the initial value is one-to-one in Q_ξ. This means that the hypersurface ∂V_n does not have any tangent hyperplane at a. It has support hyperplanes there. The initial value Λ selects the support hyperplane Π with normal vector $(0,\dots,0,1)$. But Π and ∂V_n may be tangent along some directions in the imaginary parts of coordinates of the phase vector, i.e. along the directions of the imaginary parts of $\xi_2,\dots,\xi_n$. We will show that this is at most first order tangency.

Let $(a(t), \psi(t))$ solve the Cauchy problem (3)–(4) with $u = \pi$ and with initial value Λ, and let $\Lambda^* = (\xi_2^*, \ldots, \xi_n^*) = \Lambda + \varepsilon(\delta_2, \ldots, \delta_n)$, where $\varepsilon > 0$, and $\delta_2, \ldots, \delta_n$ are constant complex numbers. Suppose that Π and ∂V_n have second order tangency along the direction determined by $(\delta_2, \ldots, \delta_n)$. The phase vector $a^*(t)$ and the conjugate vector $\psi^*(t)$ solve the Cauchy problem (3)–(4) with $\psi^*(0) = (\xi_1, \xi_2^*, \ldots, \xi_n^*)^T$ and with optimal control $u^* = u^*(t, a^*, \overline{\psi}^*)$.

Second order tangency implies that $\operatorname{Re} a_n^*(\infty) = n + O(\varepsilon^3)$. Since $|a_n^*(\infty)| \leq n$, we have $\operatorname{Im} a_n^*(\infty) = O(\varepsilon^2)$, and so $a_n^*(\infty) = n + O(\varepsilon^2)$. By E. Bombieri's result stated in [5], there are constants α_n and β_n such that $\operatorname{Re}(2 - a_2) < \alpha_n \operatorname{Re}(n - a_n)$ for n even, and $|2 - a_2| \leq \beta_n$. It follows that $\operatorname{Re} a_2^*(\infty) = 2 + O(\varepsilon^3)$, $\operatorname{Im} a_2^*(\infty) = O(\varepsilon^2)$, and so $a_2^*(\infty) = 2 + O(\varepsilon^2)$. By D. Bshouty's result [2], there exist constants c_k and d_k such that for $k \geq 2$, $\operatorname{Re}(k - a_k) \leq c_k \operatorname{Re}(2 - a_2)$ and $k - |a_k| \leq d_k \operatorname{Re}(2 - a_2)$. It follows that for $2 \leq k \leq n$, $\operatorname{Re} a_k^*(\infty) = k + O(\varepsilon^3)$, $\operatorname{Im} a_k^*(\infty) = O(\varepsilon^2)$, and so $a_k^*(\infty) = k + O(\varepsilon^2)$. Hence $(\psi_2^*(\infty), \ldots, \psi_n^*(\infty)) = (0, \ldots, 0, 1) + O(\varepsilon)$. The relation (6) at $t = 0$ implies that $\Lambda^* = \Lambda + O(\varepsilon^2)$. This contradicts our assumptions.

Thus the hyperplane Π may have at most first order tangency to ∂V_n along some directions. Π is the unique support hyperplane with normal vector $(0, \ldots, 0, 1)$ in the neighbourhood Q_a. The hypersurfaces ∂V_n^M depend analytically on M, except for manifolds of smaller dimension. Hence, passing from ∂V_n to ∂V_n^M, we have the unique support hyperplane with normal vector $(0, \ldots, 0, 1)$ in a neighbourhood $Q_a^M \subset \partial V_n^M$ of the point $a^M = (1, p_{2,M}, \ldots, p_{n,M})^T$, for M sufficiently large. This ends the proof.

Theorem 2 answers affirmatively the Jakubowski conjecture.

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