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On the spectrum of $A(\Omega)$ and $H^\infty(\Omega)$

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Abstract. We study some properties of the maximal ideal space of the bounded holomorphic functions in several variables. Two examples of bounded balanced domains are introduced, both having non-trivial maximal ideals.

1. Introduction. Let Ω be a domain (open, bounded and connected) in $\mathbb{C}^n$. Denote by $H(\Omega)$ the analytic functions on Ω, $H^\infty(\Omega) = H(\Omega)\cap L^\infty(\Omega)$ and by $A(\Omega)$ the functions in $H(\Omega)$ that are continuously extendable to $\overline{\Omega}$.

Let M be the spectrum of $H^\infty(\Omega)$ and M^A the spectrum of $A(\Omega)$. If $m \in M^A$, we have the projection $\pi m = (m(z_1), \ldots, m(z_n))$, $\widetilde{\Omega} = \{m \in M : \pi m \in \Omega\}$ and $X = \overline{\widetilde{\Omega}} \setminus \widetilde{\Omega}$ where we take the closure in the Gelfand topology. We write $\operatorname{Sh} M$ ($\operatorname{Sh} M^A$) for the Shilov boundary of M (M^A) and $\widehat{f}$ for the Gelfand transform of $f \in H^\infty(\Omega)$. Note that $\operatorname{Sh} M^A \subset X$, $\pi X \subset \partial\Omega$ and that $\pi(M) \subset \pi(M^A)$ for if $m \in M$ then m operates on A so that $z \in \pi(M) \Rightarrow z \in \pi(M^A)$.

The purpose of this paper is to study the following statements:

1. $\pi M = \overline{\Omega}$.

1′. $\pi M^A = \overline{\Omega}$.

2. If $\pi m \in \Omega$, then $mf = f(\pi m)$, $\forall f \in H^\infty(\Omega)$.

2′. If $\pi m \in \Omega$, then $mf = f(\pi m), \forall f \in A(\Omega)$.

3. If $\pi m \in \partial\Omega, f \in H^\infty(\Omega)$, then there is an $m_0 \in X$ so that $\pi(m_0) = \pi(m)$ and $\widehat{f}(m_0) = \widehat{f}(m)$.

The *Gleason problem* is to decide if the coordinate functions $z_1 - z_1^0, \ldots, z_n - z_n^0$ generate every maximal ideal $\{f \in A(\Omega) : f(z^0) = 0\}$, $z^0 \in \Omega$. An obvious obstruction to the Gleason problem is the failure of 2′, which is one of the motivations for us to study the statements above.

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2. The case $\Omega \Subset \mathbb{C}$. In the case when Ω is a domain in $\mathbb{C}$, we will prove that all statements are true.

PROPOSITION 1. *If* $\psi \in L^\infty(\Omega)$ *then*

$$V(z) = \int_\Omega \frac{\psi(\xi)\, d\xi \wedge d\bar{\xi}}{\xi - z}$$

is continuous on $\mathbb{C}$ *and* $\partial V/\partial \bar{z} = \psi$ *on* Ω.

Proof of 1 and 1′. If $m \in M^A$ then $mz \in \bar{\Omega}$ for otherwise $\frac{1}{z-mz} \in A(\Omega)$ and so

$$1 = m1 = m(z - mz)m\left(\frac{1}{z - mz}\right) = 0\,,$$

which is a contradiction.

Proof of 2. If $mz = \xi \in \Omega$, then

$$f(z) = f(\xi) + (z - \xi)\frac{f(z) - f(\xi)}{z - \xi}$$

where $\frac{f(z)-f(\xi)}{z-\xi} \in H^\infty(\Omega)$. Hence, $mf = f(\xi) = f(\pi m)$, $\forall f \in H^\infty(\Omega)$.

In the same way 2′ follows.

Proof of 3. Assume that 3 is not true. Then there exist $m \in M$ with $\pi m = \xi \in \partial\Omega$ and $g \in H^\infty(\Omega)$ with $mg = 0$ but $|g(z)| \ge d$ near ξ in Ω. Choose $\chi \in \mathcal{D}(B(\xi, r))$ so that $\chi \equiv 1$ near ξ and $|g(z)| \ge \delta > 0$ on $\Omega \cap B(\xi, r)$. Then

$$\frac{\frac{\partial \chi}{\partial \bar{z}}}{(z - \pi m)g} \in C(\bar{\Omega})$$

so by Proposition 1, there is a $\lambda \in C(\mathbb{C})$ with

$$\frac{\partial \lambda}{\partial \bar{z}} = \frac{\frac{\partial \chi}{\partial \bar{z}}}{(z - \pi m)g}\,.$$

Define

$$g_1 = \frac{\chi}{g} - (z - \pi m)\lambda\,, \qquad g_2 = \frac{1 - \chi}{z - \pi m} + \lambda g\,.$$

Then $\partial g_1/\partial \bar{z} = \partial g_2/\partial \bar{z} = 0$ so $g_1, g_2 \in H^\infty(\Omega)$ and $g_1 g + g_2(z - \pi m) = 1$, which is a contradiction since $mg = 0$.

COROLLARY. *Suppose* $f_1, \ldots, f_{m-1} \in A(\Omega)$ *and* $f_m \in H^\infty(\Omega)$ *such that* $\sum_{j=1}^m |f_j|^2 \ge \delta > 0$ *on* Ω. *Then there exist* $g_1, \ldots, g_m \in H^\infty(\Omega)$ *such that* $\sum_{j=1}^m f_j g_j = 1$.

3. The case $\Omega \Subset \mathbb{C}^n$

THEOREM 1. *Suppose Ω is a domain in $\mathbb{C}^n$ such that*

(i) $\pi M^A = \overline{\Omega}$,

(ii) *for every $z_0 \in \partial\Omega$, there is a ball $B(z^0, r)$ such that the analytic polynomials are dense in $A(\Omega \cap B(z^0, r))$.*

Then $M^A \cong \overline{\Omega}$.

Proof. Given $f \in A(\Omega)$, consider the uniform algebra B on M^A generated by $f \circ \pi$ and $A(\Omega)$. It then follows from [3, Lemma 9.1, p. 93] that B and $A(\Omega)$ have the same Shilov boundary. But $f = f \circ \pi$ on $\mathrm{Sh}\, M^A$, which proves the theorem.

Remark 1. If Ω is pseudoconvex with C^1-boundary, then by [5, Lemma 3], (ii) holds true everywhere.

Remark 2. If Ω is pseudoconvex with smooth boundary, then using the estimates for $\overline{\partial}$ from [6] and the Koszul complex one can prove that (i) holds true (cf. [5]). Therefore, in this case $M^A \cong \overline{\Omega}$.

Remark 3. Let Ω be strictly pseudoconvex with C^3-boundary. Then 2 and 2′ hold (cf. [8, pp. 292 and 350]). Note that, via Remark 2, 1′ holds for all pseudoconvex sets of type S_δ.

Also, every boundary point is a peak point ([8, 1.14]) so Theorem 2 of [1] applies, which means that 3 is true in this case.

4. The case $\Omega \Subset \mathbb{C}^2$.

In this section, we consider domains in $\mathbb{C}^2$.

PROPOSITION 2. *Suppose that Ω has the property that for every $\overline{\partial}$-closed $(0,1)$-form t with coefficients in $L^\infty(\Omega)$ ($C(\overline{\Omega})$), there is a function $T \in L^\infty(\Omega)$ ($C(\overline{\Omega})$) with $\overline{\partial}T = t$. Then 1 and 2 (1′ and 2′) hold true. Furthermore, 3 holds at all peak points for $A(\Omega)$.*

Proof. Let $(z_1^0, z_2^0) \notin \overline{\Omega}$ be given. Consider

$$g_1 = \frac{\overline{z_1 - z_1^0}}{|z_1 - z_1^0|^2 + |z_2 - z_2^0|^2} - (z_2 - z_2^0)\lambda\,,$$

$$g_2 = \frac{\overline{z_2 - z_2^0}}{|z_1 - z_1^0|^2 + |z_2 - z_2^0|^2} + (z_1 - z_1^0)\lambda$$

where $\lambda \in L^\infty(\Omega)$ and solves

$$\overline{\partial}\lambda = t = \frac{\overline{z_2 - z_2^0}}{(|z_1 - z_1^0|^2 + |z_2 - z_2^0|^2)^2}\, d\bar z_1 - \frac{\overline{z_1 - z_1^0}}{(|z_1 - z_1^0|^2 + |z_2 - z_2^0|^2)^2}\, d\bar z_2\,.$$

Since $\overline{\partial}t = 0$ and since the coefficients of t are uniformly bounded on Ω, λ exists.

This means that g_1 and g_2 are analytic and bounded on Ω. Furthermore, $1 = g_1(z_1 - z_1^0) + g_2(z_2 - z_2^0)$ so $(z_1^0, z_2^0) \notin \pi M$.

To prove 2, we prove that Gleason's problem can be solved. Assume that $\pi m = (0,0) \in \Omega$, $f \in H^\infty(\Omega)$ and $f(0) = 0$. We wish to prove that $mf = 0$. If p is the Taylor expansion of f around zero of degree three, then $f - p$ vanishes to order four near zero. Therefore, the $\overline{\partial}$-closed $(0,1)$-form

$$t = \frac{(f-p)\overline{z}_2}{(|z_1|^2 + |z_2|^2)^2}\, d\overline{z}_1 - \frac{(f-p)\overline{z}_1}{(|z_1|^2 + |z_2|^2)^2}\, d\overline{z}_2$$

has uniformly bounded coefficients on Ω.

By assumption, there is a function $\lambda \in L^\infty(\Omega)$ so that $\overline{\partial}\lambda = t$. Therefore, the functions

$$g_1 = \frac{(f-p)\overline{z}_1}{|z_1|^2 + |z_2|^2} - z_2\lambda\,, \quad g_2 = \frac{(f-p)\overline{z}_2}{|z_1|^2 + |z_2|^2} + z_1\lambda$$

are in $H^\infty(\Omega)$. Furthermore, $f - p = z_1 g_1 + z_2 g_2$ so $f = p + z_1 g_1 + z_2 g_2$, which solves the Gleason problem, gives $mf = 0$ and completes the proof of 2.

$1'$ and $2'$ are proved analogously. The proof of 3 is in [1].

Remark 4. Fornæss and Øvrelid [2] proved the Gleason property for $A(\Omega)$ when Ω is real-analytic, and Noell [7] when Ω is of finite type. Thus $2'$ holds true in those cases.

5. Balanced H^∞-domains. A domain in $\mathbb{C}^n$ is called an *H^∞-domain* if it is its own H^∞-envelope of holomorphy. A subset Ω in $\mathbb{C}^n$ is said to be *balanced* if $\lambda\Omega \subset \Omega$, $\forall \lambda \in \mathbb{C}$, $|\lambda| \le 1$.

Let H_m denote the homogeneous polynomials of degree m and

$$\mathrm{Ho}(\mathbb{C}^n) = \{\varphi \in \mathrm{PSH}(\mathbb{C}^n) : \varphi \not\equiv 0,\ \varphi(\lambda x) = |\lambda|\varphi(x),\ \forall \lambda \in \mathbb{C},\ \forall x \in \mathbb{C}^n\}$$

where $\mathrm{PSH}(\mathbb{C}^n)$ denote the plurisubharmonic functions on $\mathbb{C}^n$. For a domain in $\mathbb{C}^n$ containing 0 we define the homogeneous extremal function

$$\psi_\Omega = \sup\{|Q|^{1/\deg Q} : Q \in H_{\deg Q},\ |Q| \le 1 \text{ on } \overline{\Omega}\}\,.$$

Then $\psi_\Omega^* \in \mathrm{Ho}$ where $*$ denotes the usual regularization.

PROPOSITION 3. *Suppose Ω is a balanced domain. Then every $f \in H^\infty(\Omega)$ extends to $\Omega^* = \{z \in \mathbb{C}^n : \psi_\Omega^* < 1\}$ and Ω^* is an H^∞-domain. Every $f \in A(\Omega^*)$ can be uniformly approximated on $\overline{\Omega}$ by polynomials.*

Proof. Let $f \in H^\infty(\Omega)$. We can assume that f is non-constant and $|f| \le 1$ on Ω. Near zero, we have

$$f(z) = \sum_{j=0}^{\infty} \Big(\sum_{|\alpha|=j} a_\alpha z^\alpha \Big)\,.$$

Let $z \in \Omega$; then

$$f(\lambda z) = \sum_{j=0}^{\infty} \sum_{|\alpha|=j} (a_\alpha z^\alpha)\lambda^j, \qquad |\lambda| \le 1,\ \lambda \in \mathbb{C}.$$

By Cauchy's inequality,

$$\Big| \sum_{|\alpha|=j} a_\alpha z^\alpha \Big| \le \sup_{|\lambda|=1} |f(\lambda z)| < 1$$

so $|\sum_{|\alpha|=j} a_\alpha z^\alpha|^{1/j} < 1$ on Ω and therefore on Ω^* by the definition of ψ^*_Ω. Hence, the series $\sum_{j=0}^\infty \sum_{|\alpha|=j} a_\alpha z^\alpha$ converges normally in Ω^* and extends f to $H^\infty(\Omega^*)$, which proves the first part of the proposition.

Assume now that $f \in A(\Omega^*)$. Let $\varepsilon > 0$ be given and choose $0 < r < 1$ such that $\sup_{z\in\bar\Omega} |f(z) - f(rz)| < \varepsilon$. By the above, $\sup_{z\in\Omega} |\sum a_\alpha z^\alpha| \le 1$, therefore, $\sum_{j=0}^p \sum_{|\alpha|=j} (a_\alpha z^\alpha) r^j$ uniformly convergs on $\overline{\Omega}$ to $f(rz)$. The proposition is proved.

Theorem 2 (Siciak). *Let $h \in \mathrm{Ho}(\mathbb{C}^n)$ and consider $D = \{z \in \mathbb{C}^n : h < 1\}$ where we assume D to be bounded. Then the following conditions are equivalent.*

(i) *D is an H^∞-domain*,
(ii) $h = \psi^*_D$,
(iii) $D = (\widehat{\overline{D}})^\circ$,
(iv) *the set $N(h) = \{a \in \mathbb{C}^n : h$ is discontinuous at $a\}$ is pluripolar.*

Lemma 1. *Assume U is a subharmonic function and that the Lebesgue measure of $N(U) = \{a \in \mathbb{C}^n : U$ discontinuous at $a\}$ is zero. Then $\overline{\{U < 1\}}^\circ = \{U < 1\}$.*

The proof is left to the reader. It is based on the fact that if two subharmonic functions are equal almost everywhere, they are equal.

We now construct two examples.

Example 1. Let (a_j) be a sequence of complex numbers contained and dense in $\{x \in \mathbb{C} : |x| = 1\}$. Define

$$h(x,y) = e^{\Sigma_{j=1}^\infty \alpha_j \log|x - a_j y|}, \qquad (x,y) \in \mathbb{C}^2,$$

where $(\alpha_j)_{j=1}^\infty$ is a sequence of positive numbers with $\sum_{j=1}^\infty \alpha_j = 1$. Then $h \in \mathrm{Ho}(\mathbb{C}^2)$ and

$$N(h) = \Big\{(x,y) \in \mathbb{C}^2 : |x| = |y| \text{ and } \sum_{j=1}^\infty \alpha_j \log|x - a_j y| > -\infty\Big\}$$

so $N(h)$ is a non-pluripolar set of vanishing Lebesgue measure. If we define $D = \{(x,y) \in \mathbb{C}^2 : h(x,y) + \max(|x|,|y|) < 1\}$ then

(i) D is pseudoconvex and balanced,

(ii) $\overline{D}^{\circ} = D$ by Lemma 1,

(iii) D is not an H^∞-domain, by Theorem 2. The H^∞-envelope is the bidisc by Proposition 3.

An example of this nature was given in Siciak [10, Ex. 5.3] (see also Sibony [9, Prop. 1] where a Hartogs domain Ω is given with properties (i) and (ii)) but our construction is much more elementary.

EXAMPLE 2. Let $(a_j)_{j=1}^\infty$ and $(\alpha_j)_{j=1}^\infty$ be as in Example 1. Define

$$W(x,y,z) = e^{\Sigma_{j=1}^\infty \alpha_j \max(\log|x-a_jy|, \log|z|)} + \max(|x|,|y|,|z|)\,,$$

$(x,y,z) \in \mathbb{C}^3$. Then $W \in \mathrm{Ho}(\mathbb{C}^3)$ so $D = \{W < 1\}$ is a balanced, pseudoconvex set and $N(W) \subset \{z = 0\}$, a pluripolar set. Thus

(i) $\overline{D}^{\circ} = D$ by Lemma 1,

(ii) D is an H^∞-domain by Theorem 2,

(iii) $\widehat{\overline{D}} \neq \overline{D}$ since $\{(x,y,0) : |x| \le 1,\ |y| \le 1\} \subset \widehat{\overline{D}}$ by Example 1.

This gives a counterexample to a problem of Siciak [10, Problem 4.2], who proved [Th. 4.1] that such an example cannot be found in $\mathbb{C}^2$.

Remark 5. Let Ω be a balanced domain in $\mathbb{C}^2$. Then 2′ holds true and $M^A \cong \overline{\{z \in \mathbb{C}^n : \psi_\Omega^* < 1\}}$.

Remark 6. Let Ω be a balanced domain in $\mathbb{C}^n$. Then 2′ holds true. By Example 2, πM^A may be strictly larger than $\overline{\{z \in \mathbb{C}^n : \psi_\Omega^* < 1\}}$.

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