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On the mean values of an analytic function

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Abstract. Let $f(z)$, $z = re^{i\theta}$, be analytic in the finite disc $|z| < R$. The growth properties of $f(z)$ are studied using the mean values $I_\delta(r)$ and the iterated mean values $N_{\delta,k}(r)$ of $f(z)$. A convexity result for the above mean values is obtained and their relative growth is studied using the order and type of $f(z)$.

1. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, $z = re^{i\theta}$, be analytic in the disc $|z| < R$, $0 < R < \infty$. For $0 \le r < R$, we set $M(r) = \max_{|z|=r} |f(z)|$. Then the *order* ϱ and *lower order* λ of $f(z)$ are defined as (see [4])

$$\lim_{r\to R} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\log^+ \log^+ M(r)}{\log x} = \begin{cases} \varrho, \\ \lambda, \end{cases} \qquad 0 \le \lambda \le \varrho \le \infty\,, \tag{1.1}$$

where $x = Rr/(R-r)$ and $\log^+ t = \max\{0, \log t\}$. When $0 < \varrho < \infty$, we define the *type* T and *lower type* τ $(0 \le \tau \le T \le \infty)$ of $f(z)$ as

$$\lim_{r\to R} \begin{Bmatrix} \sup \\ \inf \end{Bmatrix} \frac{\log^+ M(r)}{x^\varrho} = \begin{cases} T, \\ \tau. \end{cases} \tag{1.2}$$

Let $m(r) = \max_{n\ge 0}\{|a_n| r^n\}$ be the maximum term in the Taylor series expansion of $f(z)$ for $|z| = r$. If $f(z)$ is of finite order ϱ, then ([1], [3])

$$\log m(r) \simeq \log M(r) \quad \text{as } r \to R\,. \tag{1.3}$$

Hence $m(r)$ can be used in place of $M(r)$ in (1.1) and (1.2) for defining ϱ, λ etc.

The following mean value of an analytic function $f(z)$ was introduced by Hardy [2]:

$$I_\delta(r) = [J_\delta(r)]^{1/\delta} = \left[\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta\right]^{1/\delta} \tag{1.4}$$

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where $0 < \delta < \infty$. We introduce the following weighted mean of $f(z)$:

$$N_{\delta,k}(r) = x^{-k} \int_0^r I_\delta(y)\left(\frac{Ry}{R-y}\right)^{k+1}\frac{dy}{y^2}\,, \tag{1.5}$$

where $x = Rr/(R-r)$ and $0 < k < \infty$.

In this paper we have studied the growth properties of the analytic function $f(z)$ through its mean values $I_\delta(r)$ and $N_{\delta,k}(r)$. In the sequel, we also derive some convexity properties of these means and also study their relative growths. We shall assume throughout that $\varrho < \infty$.

2. We now prove

LEMMA. *For every r, $0 < r < R$, $[x^k I_\delta(r)/(R-r)]$ is an increasing convex function of $[x^k N_{\delta,k}(r)]$.*

Proof. From (1.5) we have

$$\frac{d[x^k I_\delta(r)/(R-r)]}{d[x^k N_{\delta,k}(r)]} = \frac{rI'_\delta(r)}{RI_\delta(r)} + \frac{r}{R(R-r)} + \frac{k}{R-r}\,,$$

where $I'_\delta(r)$ denotes the derivative of $I_\delta(r)$ with respect to r. Since R and k are fixed, the last two terms on the right hand side of the above equation are increasing functions of r. Further, it is well known that $\log I_\delta(r)$ is an increasing convex function of $\log r$. Hence the right hand side of the above equation is an increasing function of r and the Lemma follows.

THEOREM 1. *For $\varphi(r) = I_\delta(r)$, $J_\delta(r)$ and $N_{\delta,k}(r)$, we have*

$$\lim_{r\to R}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right. \frac{\log\log\varphi(r)}{\log x} = \left\{\begin{matrix}\varrho,\\ \lambda,\end{matrix}\right. \qquad 0 \le \lambda \le \varrho < \infty\,. \tag{2.1}$$

Proof. It is known that for $n \ge 0$,

$$a_n = \frac{1}{2\pi i}\oint_C \frac{f(z)}{z^{n+1}}\,dz\,,$$

where C is the circle $|z| = r$, $0 < r < R$. Hence

$$|a_n|r^n \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta\,.$$

Since the right hand side is independent of n, we can choose n suitably to obtain

$$m(r) \le \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|\,d\theta\,.$$

For $\delta \ge 1$, we apply Hölder's inequality to the right hand side. Then

$$\begin{aligned} m(r) &\le \frac{1}{2\pi}\Big\{\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta\Big\}^{1/\delta}\Big\{\int_0^{2\pi} d\theta\Big\}^{(\delta-1)/\delta} \\ &= \bigg[\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^\delta \, d\theta\bigg]^{1/\delta}. \end{aligned}$$

Hence $m(r) \le I_\delta(r)$. From (1.4) we obviously have $I_\delta(r) \le M(r)$. Hence for $r > 0$ and $\delta \ge 1$, we have

$$m(r) \le I_\delta(r) \le M(r)\,. \tag{2.2}$$

If $0 < \delta < 1$, then

$$\begin{aligned} 2\pi[I_{1+\delta}(r)]^{1+\delta} &= \int_0^{2\pi} |f(re^{i\theta})|^{1+\delta} \, d\theta \le M(r) \int_0^{2\pi} |f(re^{i\theta})|^{\delta} \, d\theta \\ &= 2\pi M(r)[I_\delta(r)]^\delta \le 2\pi[M(r)]^{1+\delta}\,. \end{aligned}$$

Thus

$$I_{1+\delta}(r) \le [M(r)]^{1/(1+\delta)}[I_\delta(r)]^{\delta/(1+\delta)} \le M(r)\,. \tag{2.3}$$

From (2.2) we have, in view of (1.3),

$$\log I_\delta(r) \simeq \log M(r) \quad \text{as } r \to R\,,\ \delta \ge 1\,.$$

Hence $\log I_{(1+\delta)}(r) \simeq \log M(r)$ as $r \to R$, $0 < \delta < 1$. Thus from (2.3) we have

$$\log I_\delta(r) \simeq \log M(r) \quad \text{as } r \to R\,,\ 0 < \delta < 1\,.$$

Combining these two asymptotic relations, we get

$$\log I_\delta(r) \simeq \log M(r) \quad \text{as } r \to R\,,\ \delta > 0\,. \tag{2.4}$$

From (1.4) and (2.4) we immediately have

$$\lim_{r\to R}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right. \frac{\log\log I_\delta(r)}{\log x} = \lim_{r\to R}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right. \frac{\log\log J_\delta(r)}{\log x} = \left\{\begin{matrix}\varrho,\\ \lambda.\end{matrix}\right.$$

To prove (2.1) for $\varphi(r) = N_{\delta,k}(r)$, we take

$$r' = R\bigg[1 - \frac{1}{\alpha}\Big(1 - \frac{r}{R}\Big)\bigg]$$

where $\alpha > 1$ is an arbitrary constant. Then from (1.5) we have

$$\begin{aligned} N_{\delta,k}(r') &= (x')^{-k} \int_0^{r'} I_\delta(y)\Big(\frac{Ry}{R-y}\Big)^{k+1}\frac{dy}{y^2} \\ &> (x')^{-k} \int_r^{r'} I_\delta(y)\Big(\frac{Ry}{R-y}\Big)^{k+1}\frac{dy}{y^2}\,, \end{aligned}$$

where $x' = Rr'/(R-r')$. Since $I_\delta(r)$ is an increasing function of r, we have

$$N_{\delta,k}(r') > \frac{I_\delta(r)}{k}\frac{(x')^k - x^k}{(x')^k} = O(1)I_\delta(r)\,. \tag{2.5}$$

It can be easily verified that $x'/x \to \alpha$ and $(\log x')/\log x \to 1$ as $r \to R$. Hence we have

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log\log N_{\delta,k}(r)}{\log x} \ge \lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log\log I_\delta(r)}{\log x}\,. \tag{2.6}$$

For the reverse inequality we have from (1.5),

$$N_{\delta,k}(r) \le I_\delta(r)/k\,. \tag{2.7}$$

Hence

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log\log N_{\delta,k}(r)}{\log x} \le \lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log\log I_\delta(r)}{\log x}\,. \tag{2.8}$$

Combining (2.6) and (2.8) we get the relation (2.1) for $\varphi(r) = N_{\delta,k}(r)$. This proves (2.1) completely.

Theorem 2. *For* $0 < \varrho < \infty$, *we have*

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log I_\delta(r)}{x^\varrho} = \begin{cases}T\,,\\ \tau\,,\end{cases} \tag{2.9}$$

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log N_{\delta,k}(r)}{x^\varrho} = \begin{cases}T\,,\\ \tau\,.\end{cases} \tag{2.10}$$

Proof. The relation (2.9) follows easily from (2.4) and the definitions of T and τ. To prove (2.10) we have from (2.7),

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log N_{\delta,k}(r)}{x^\varrho} \le \lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log I_\delta(r)}{x^\varrho}\,. \tag{2.11}$$

Also, from (2.5) we have

$$\log N_{\delta,k}(r') > O(1) + \log I_\delta(r)\,.$$

Since $x'/x \to \alpha$ as $r \to R$, we have

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log N_{\delta,k}(r')}{(x')^\varrho} \ge \alpha^{-\varrho}\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log I_\delta(r)}{x^\varrho}\,.$$

Since $\alpha > 1$ was arbitrary, we thus have

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log N_{\delta,k}(r)}{x^\varrho} \ge \lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix}\frac{\log I_\delta(r)}{x^\varrho}\,. \tag{2.12}$$

Now combining (2.11) and (2.12), we get (2.10) in view of (2.9). This proves Theorem 2.

In the next two theorems, we obtain the relative growth of $I_\delta(r)$ and $N_{\delta,k}(r)$. We prove

THEOREM 3. *For the mean values $I_\delta(r)$ and $N_{\delta,k}(r)$ as defined before, we have*

$$\left.\begin{matrix}\varrho\\ \lambda\end{matrix}\right\} \le \lim_{r\to R}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right. \frac{\log[I_\delta(r)/(R-r)N_{\delta,k}(r)]}{\log x} \le \left\{\begin{matrix}\varrho+1\,,\\ \lambda+1\,.\end{matrix}\right. \tag{2.13}$$

P r o o f. From (1.5) we have

$$\frac{d}{dr}[x^r N_{\delta,k}(r)] = x^{k+1} I_\delta(r)/r^2$$

where $x = Rr/(R-r)$. Expanding and rearranging the terms on the left hand side, we get

$$\frac{N'_{\delta,k}(r)}{N_{\delta,k}(r)} = \frac{RI_\delta(r)}{r(R-r)N_{\delta,k}(r)} - \frac{kR}{r(R-r)}\,.$$

Integrating on both sides of this equation with respect to r, we get

$$\log N_{\delta,k}(r) = O(1) + R\int_{r_0}^{r} \frac{I_\delta(y)\,dy}{y(R-y)N_{\delta,k}(y)} - k\log[r/(R-r)] \tag{2.14}$$

where $0 < r_0 \le r < R$. Since $\varrho < \infty$, we have from Theorem 1,

$$\lim_{r\to R} \frac{\log(R-r)}{\log N_{\delta,k}(r)} = 0\,. \tag{2.15}$$

Now from the Lemma, $[I_\delta(y)/(R-y)N_{\delta,k}(y)]$ is an increasing function of y. Hence from (2.14) we have

$$\log N_{\delta,k}(r) < O(1) + \frac{RI_\delta(r)\log(r/r_0)}{(R-r)N_{\delta,k}(r)} - k\log[r/(R-r)]\,,$$

or, in view of (2.15),

$$\log N_{\delta,k}(r)\{1+o(1)\} < \frac{RI_\delta(r)\log(r/r_0)}{(R-r)N_{\delta,k}(r)}\,.$$

Hence

$$\lim_{r\to R}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right. \frac{\log\log N_{\delta,k}(r)}{\log x} \le \lim_{r\to R}\left\{\begin{matrix}\sup\\ \inf\end{matrix}\right. \frac{\log[I_\delta(r)/(R-r)N_{\delta,k}(r)]}{\log x}\,.$$

In view of (2.1), we get the left hand inequalities of (2.13). To obtain the right hand inequalities of (2.13), we again take arbitrary $\alpha > 1$ and

$r' = R[1-(1/\alpha)(1-r/R)]$. Then from (2.14), since $r' > r$,

$$\log N_{\delta,k}(r') \ge O(1) + R\int_r^{r'} \frac{I_\delta(y)\,dy}{y(R-y)N_{\delta,k}(y)} - k\log[r'/(R-r')]$$
$$\ge O(1) + \frac{RI_\delta(r)\log(r'/r)}{(R-r)N_{\delta,k}(r)} - k\log[r'/(R-r')]\,.$$

Using (2.15) we have

$$[1+o(1)]\log N_{\delta,k}(r') \ge \frac{RI_\delta(r)\log(r'/r)}{(R-r)N_{\delta,k}(r)} + O(1)\,, \tag{2.16}$$

or

$$\frac{\log\log N_{\delta,k}(r')}{\log x} \ge \frac{\log[I_\delta(r)/(R-r)N_{\delta,k}(r)]}{\log x} + \frac{\log\log(r'/r)}{\log x} + o(1)\,.$$

As before, $(\log x)/\log x' \to 1$ and $[\log\log(r'/r)]/\log x \to -1$ as $r \to R$. Hence we obtain, on proceeding to limits,

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{\log[I_\delta(r)/(R-r)N_{\delta,k}(r)]}{\log x} \le \begin{Bmatrix}\varrho+1\,,\\ \lambda+1\,.\end{Bmatrix}$$

This proves Theorem 3.

Theorem 4. *For* $0<\varrho<\infty$, *we have*

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{I_\delta(r)/N_{\delta,k}(r)}{x^\varrho} \le \begin{Bmatrix}AT\,,\\ A\tau\,,\end{Bmatrix} \tag{2.17}$$

where $A = (\varrho+1)^{\varrho+1}/\varrho^\varrho$.

Proof. From (2.16) we have

$$[1+o(1)]\frac{\log N_{\delta,k}(r')}{(x')^\varrho} \ge \frac{R\log(r'/r)I_\delta(r)}{(R-r)N_{\delta,k}(r)(x')^\varrho} + o(1)\,.$$

Since

$$\lim_{r\to R}\frac{\log(r'/r)}{R-r} = \frac{\alpha-1}{\alpha R} \quad\text{and}\quad \lim_{r\to R}\frac{x'}{x} = \alpha\,,$$

where as before $x' = Rr'/(R-r')$, we get on proceeding to limits

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{\log N_{\delta,k}(r')}{(x')^\varrho} \ge \left(\frac{\alpha-1}{\alpha}\right)\alpha^{-\varrho}\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{I_\delta(r)/N_{\delta,k}(r)}{x^\varrho}\,.$$

Since $\alpha>1$ was arbitrary, we can take $\alpha = (\varrho+1)/\varrho$. Hence, using (2.10) we obtain

$$\lim_{r\to R}\begin{Bmatrix}\sup\\ \inf\end{Bmatrix} \frac{I_\delta(r)/N_{\delta,k}(r)}{x^\varrho} \le \begin{Bmatrix}AT\,,\\ A\tau\,,\end{Bmatrix}$$

where $A = (\varrho+1)^{\varrho+1}/\varrho^\varrho$. Thus Theorem 4 follows.

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