ANNALES
POLONICI MATHEMATICI
LVII.1 (1992)

Asymptotic behaviour of a transport equation

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Abstract. We study the asymptotic behaviour of the semigroup of Markov operators generated by the equation $u_t + bu_x + cu = a\int_0^{ax} u(t, ax-y)\mu(dy)$. We prove that for $a > 1$ this semigroup is asymptotically stable. We show that for $a \le 1$ this semigroup, properly normalized, converges to a limit which depends only on a.

1. Introduction. In this paper we investigate the integro-differential equation

$$u_t + bu_x + cu = a\int_0^{ax} u(t, ax-y)\,\mu(dy)\,, \tag{1.1}$$

where a and b are positive constants, c is a real number, μ is a finite Borel measure on the interval $[0,\infty)$, and $u : [0,\infty)\times[0,\infty) \to \mathbb{R}$ satisfies the initial-boundary condition

$$\begin{cases} u(0,x) = v(x)\,, \\ u(t,0) = 0\,. \end{cases} \tag{1.2}$$

Equation (1.1) has a probabilistic interpretation in the case when $c = \mu([0,\infty))$. Namely, consider a particle moving with speed b in the interval $[0,\infty)$. Assume that in every time interval $[t, t+\Delta t]$ the particle has the probability $c\Delta t + o(\Delta t)$ of changing its position from x to $(x+\xi)/a$, where ξ is a random variable with distribution $c^{-1}\mu$, i.e. $\mathrm{Prob}(\xi \in A) = c^{-1}\mu(\xi \in A)$. Denote by $u(t,x)$ the probability density function of the position of the particle at time t. Then (1.1) describes the evolution of $u(t,x)$ in time. If $a = 1$ and $c = \mu([0,\infty))$ then (1.1) is known as the integro-differential Takacs equation and plays an important role in the theory of jump processes.

By means of a suitable substitution equation (1.1) may be converted into a special case with $b = c = 1$ and $\mu([0,\infty)) = 1$. In this case (1.1) generates a semigroup of Markov operators on $L^1[0,\infty)$ given by $S^t v(x) = u(t,x)$. The asymptotic behaviour of this semigroup as $t \to \infty$ strongly depends on a.

1991 *Mathematics Subject Classification*: Primary 45K05.

For $a > 1$ this semigroup was studied by Klaczak [3]. He proved that if the measure μ is absolutely continuous with respect to the Lebesgue measure and $\int x\,\mu(dx) < \infty$, then the semigroup $\{S^t\}$ is asymptotically stable. In his proof he used the method of the lower bound function introduced by Lasota and Yorke [4] and developed by Dłotko and Lasota [1].

The main aim of this paper is to give the full description of the asymptotic properties of this semigroup. This description is given in Theorem 1 of Section 2. Sections 3 and 4 contain the proof of this theorem.

2. Main result. We denote by D the set of all nonnegative elements of $L^1[0,\infty)$ with norm one. The elements of D will be called *densities*. We will assume that $v \in D$. By setting $u(t,x) = 0$ for $t \ge 0$, $x < 0$ and $\mu(A) = 0$ for $A \subset (-\infty, 0)$ equation (1.1) can be rewritten as

$$u_t + bu_x + cu = Pu(t,x)\,, \tag{2.1}$$

where $P : L^1(\mathbb{R}) \to L^1(\mathbb{R})$ is given by

$$Pf(x) = a \int f(ax-y)\,\mu(dy) = a(f*\mu)(ax)\,. \tag{2.2}$$

From the Phillips perturbation theorem [2] equation (1.1) with the initial-boundary condition (1.2) generates a semigroup $\{S^t\}$ of linear operators on $L^1(\mathbb{R})$ given by

$$S^t v(x) = u(t,x) = e^{-ct}\sum_{n=0}^{\infty} T_n(t)v(x)\,, \tag{2.3}$$

where $T_0(t)v(x) = v(x-bt)$ and

$$T_{n+1}(t)v(x) = \int_0^t T_0(t-s)PT_n(s)v(x)\,ds\,. \tag{2.4}$$

It is easy to check that if $v(x) = 0$ for $x < 0$ then $Pv(x) = 0$ and $T_0(t)v(x) = 0$ for $x < 0$ and $t \ge 0$. Consequently, $S^t v(x) = 0$ for $x < 0$ and $t \ge 0$, which implies that $\{S^t\}$ is the semigroup generated by equation (1.1).

Now observe that substituting $\overline{u}(t,x) = e^{\lambda t}u(pt, rx)$ into (1.1), where $p = 1/d$, $r = b/d$, $\lambda = c/d - 1$, and $d = \mu([0,\infty))$ we obtain

$$\overline{u}_t + \overline{u}_x + \overline{u} = a \int \overline{u}(t, ax-y)\,\overline{\mu}(dy)\,, \tag{2.5}$$

where $\overline{\mu}$ is the probability measure on $[0,\infty)$ given by $\overline{\mu}(A) = \mu(rA)/d$. Since the properties of u can easily be deduced from the properties of $\overline{u}$, in the remainder of this paper we assume that $b = c = 1$ and $\mu([0,\infty)) = 1$. Let u be the solution of (1.1) satisfying the initial condition $u(0,x) = v(x)$ and let $U(t,x) = \int_0^x u(t,y)\,dy$. Let $\Phi(x) = \int_{-\infty}^x \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\,dy$ and $\varphi = \Phi'$.

THEOREM 1. *Assume that $v \in D$ and that v is bounded.*

(a) *If* $\int_0^\infty \ln(1+x)\,\mu(dx) < \infty$ *and* $a < 1$, *then* $U(t, a^{-\sqrt{t}\,x-t})$ *converges uniformly to* $\Phi(x)$ *on* $\mathbb{R}$ *as* $t \to \infty$.

(b) *If* $\int_0^\infty \ln(1+x)\,\mu(dx) < \infty$ *and* $a > 1$, *then there exists a continuous density* v_0 *independent of* v *such that* $u(t,x)$ *converges uniformly to* v_0 *as* $t \to \infty$. *Moreover*, $S^t v_0 = v_0$ *for* $t > 0$.

(c) *If* $a = 1$, $m = \int x\,\mu(dx) < \infty$, $m > 0$, *and* $k = \int x^2\,\mu(dx) < \infty$, *then* $U(t, \sqrt{kt}\,x + mt + t)$ *converges uniformly to* $\Phi(x)$. *Moreover*, *if* μ *has a bounded density then* $\sqrt{kt}u(t, \sqrt{kt}\,x + mt + t)$ *converges uniformly to* $\varphi(x)$ *as* $t \to \infty$.

Remark 1. In the case $b = c = 1$ and $\mu([0,\infty)) = 1$, $\{S^t\}$ is a semigroup of Markov operators, i.e. $S^t D \subset D$ for every $t > 0$. From this and from Theorem 1(b), it follows immediately that if $a > 1$ and $\int \ln(1+x)\,\mu(dx) < \infty$, then for every $v \in D$, $S^t v \to v_0$ in L^1. This generalizes the result of Klaczak [3].

We divide the proof of Theorem 1 into a sequence of lemmas. In this section we give a formula for $T_n(t)v$.

LEMMA 1. *Let*

$$\varphi_1(t,x,a) = \begin{cases} \dfrac{a}{1-a}\mathbf{1}_{[t,t/a]}(x) & \text{for } a \in (0,1)\,, \\ \dfrac{a}{a-1}\mathbf{1}_{[t/a,t]}(x) & \text{for } a > 1\,. \end{cases}$$

Define

$$\varphi_n(t,x,a) = \int\limits_0^t a\varphi_{n-1}(s, a(x-t+s), a)\,ds \tag{2.6}$$

for $t \ge 0$, $x \ge 0$, $a > 0$, $a \ne 1$, *and* $n \ge 2$. *Then for* $n \ge 1$,

$$T_n(t)v(x) = (\varphi_n * P^n v)(t,x) = \int\limits_{-\infty}^{\infty} \varphi_n(t,y,a)P^n v(x-y)\,dy\,. \tag{2.7}$$

Lemma 1 follows immediately from (2.4) and the definition of P. Using induction arguments it is easy to check the following lemma.

LEMMA 2. *Let* $x_+ = x$ *if* $x > 0$ *and* $x_+ = 0$ *if* $x \le 0$. *Then for* $n \ge 1$ *we have*

$$\varphi_n(t,x,a) = \sum_{k=0}^n a_{n,k,a}(x - ta^{-k})_+^{n-1}\,, \tag{2.8}$$

where

$$a_{n,k,a} = \frac{(-1)^k a^{(n(n+1)+k(k-1))/2}}{(n-1)!(1-a)\dots(1-a^k)(1-a)\dots(1-a^{n-k})}\,. \tag{2.9}$$

Corollary 1. *For every $n \ge 1$ and $a > 0$, $a \ne 1$,*

$$\varphi_n(t,x,a^{-1}) = a^{-n}\varphi_n(t,xa^{-n},a)\,. \tag{2.10}$$

3. Properties of φ_n. Although the functions φ_n are given explicitly it is difficult to investigate their behaviour as $n \to \infty$ using only formula (2.8). Therefore we define, by induction, an auxiliary sequence of functions $\eta_n : [0,\infty) \to \mathbb{R}$, $n = 1,2,\ldots$ Let $a \in (0,1)$, $\eta_1(x) = (1-a)^{-1}\mathbf{1}_{[a,1]}(x)$ and

$$\eta_n(x) = \begin{cases} 0 & \text{for } x \le a^n\,, \\ n\int\limits_x^\infty \dfrac{(x-a^n)^{n-1}}{(z-a^n)^n}\eta_{n-1}(z)\,dz & \text{for } x > a^n\,. \end{cases} \tag{3.1}$$

Lemma 3. *For every $n \ge 1$,*

$$\varphi_n(t,x,a) = \frac{t^{n-1}a^n}{n!}\eta_n\left(\frac{xa^n}{t}\right). \tag{3.2}$$

Proof. Since $\varphi_1(t,x,a) = 0$ for $x \le t$, it follows from (2.6) that $\varphi_n(t,x,a) = 0$ for $x \le t$, $n > 1$. For $n = 1$ formula (3.2) is obvious. Assume that (3.2) holds for $n-1$. Then for $x > t$ we have

$$\begin{aligned}\varphi_n(t,x,a) &= \int\limits_0^t a\varphi_{n-1}(s,a(x-t+s),a)\,ds \\ &= \int\limits_0^t \frac{a^n s^{n-2}}{(n-1)!}\eta_{n-1}\left(\frac{(x-t+s)a^n}{s}\right)ds \\ &= \frac{a^n t^{n-1}}{n!}\int\limits_{a^n x/t}^\infty \frac{n(a^n x/t - a^n)^{n-1}}{(z-a^n)^n}\eta_{n-1}(z)\,dz = \frac{a^n t^{n-1}}{n!}\eta_n\left(\frac{xa^n}{t}\right). \ \blacksquare\end{aligned}$$

Now we give a probabilistic interpretation of $\{\eta_n\}$. Let $Y_1, Y_2, \ldots$ be a sequence of independent random variables such that

$$h_n(x) = (n+1)x^n\mathbf{1}_{[0,1]}(x)$$

is the density of Y_n.

Lemma 4. *Let X_1 be a random variable independent of $Y_1, Y_2, \ldots$ and with density η_1. Then the random variables X_n, $n \ge 2$, defined inductively by*

$$X_n = (X_{n-1} - a^n)Y_{n-1} + a^n\,, \tag{3.3}$$

have densities η_n.

Proof. Since $\operatorname{supp}\eta_1 = [a,1]$ and $\operatorname{supp} h_n = [0,1]$, we may assume that $a \le X_1 \le 1$ and $0 \le Y_n \le 1$. This implies that $a^n \le X_n \le 1$. Let

$x \in (a^n, 1)$. Then

$$\begin{aligned}\operatorname{Prob}(X_n < x) &= \operatorname{Prob}((X_{n-1} - a^n)Y_{n-1} + a^n < x) \\ &= \iint_A \eta_{n-1}(z) h_{n-1}(y)\, dy\, dz\,,\end{aligned}$$

where

$$A = \{(y,z) : a^{n-1} \le z \le 1\,,\ 0 \le y \le 1\,,\ (z - a^n)y + a^n < x\}\,.$$

Hence

$$\operatorname{Prob}(X_n < x) = \int_0^x \int_0^1 \eta_{n-1}(z) h_{n-1}(y)\, dy\, dz + \int_x^\infty \eta_{n-1}(z) \left(\frac{x - a^n}{z - a^n} \right)^n dz.$$

This implies that the density of X_n is given by (3.1). ∎

Lemma 5. *Let*

$$g_n(x) = \frac{1}{n+1} \eta_n \left(\frac{x}{n+1} \right).$$

Then there exists a continuous density g vanishing at ∞ such that g_n converges uniformly to g on $[0, \infty)$.

Proof. First we check that the sequence $\{g_n\}$ is relatively compact in the topology of uniform convergence on $[0, \infty)$. Indeed, from (3.1) it follows that

$$\eta_n(x) \le \int_x^\infty \frac{n(x - a^n)^{n-1}}{(z - a^n)^n}\, dz \sup \eta_{n-1} \le \frac{n}{n-1} \sup \eta_{n-1}\,.$$

This implies that

$$\sup \eta_n \le n \sup \eta_1\,. \tag{3.4}$$

Integrating (3.1) by parts we obtain

$$\eta_n(x) = \frac{n}{n-1} \eta_{n-1}(x) + \frac{n}{n-1} (x - a^n)^{n-1} \int_x^\infty \frac{\eta'_{n-1}(z)}{(z - a^n)^{n-1}}\, dz\,.$$

Consequently,

$$\eta'_n(x) = n(x - a^n)^{n-2} \int_x^\infty \frac{\eta'_{n-1}(z)}{(z - a^n)^{n-1}}\, dz\,,$$

and

$$\sup |\eta'_n| \le \frac{n}{n-2} \sup |\eta'_{n-1}|\,.$$

This implies that

$$\sup |\eta'_n| \le Cn^2 \tag{3.5}$$

for $n = 3, 4, \ldots$, and some constant C. From the definition of g_n, (3.4) and (3.5) it follows that the sequences $\{g_n\}$ and $\{g'_n\}$ are bounded. Let $X'_n = (n+1)X_n$ and $Y'_n = \frac{n+2}{n+1} Y_n$. Then g_n is the density of X'_n,

$$EY'_n = 1, \quad EX'^{\,2}_n = 1 + \frac{1}{(n+1)(n+3)}, \quad Y'_n \le \frac{n+2}{n+1}$$

and

(3.6) $$X'_{n+1} = X'_n Y'_n - (n+1)a^{n+1} Y'_n + (n+2)a^{n+1} .$$

Since X'_n and Y'_n are independent, we have $EX'_{n+1} = EX'_n + a^{n+1}$ and, consequently,

(3.7) $$EX'_n = 1 + a + \ldots + a^n \le \frac{1}{1-a} .$$

This and the Chebyshev inequality imply

(3.8) $$\operatorname{Prob}(X'_n \ge M) \le \frac{EX'_n}{M} \le \frac{1}{M(1-a)} ,$$

which yields

(3.9) $$\int_M^\infty g_n(x)\, dx \le \frac{1}{M(1-a)} .$$

Since $\{g'_n\}$ is bounded, there exists a constant K such that $g'_n(x) \ge -K$ for $x \ge 0$, $n \ge 3$. Let $x_0 \ge M$; then

$$g_n(x) \ge g_n(x_0) - K(x - x_0) \quad \text{for } x \in [x_0, x_0 + g_n(x_0)/K] .$$

From this it follows that

$$\int_M^\infty g_n\, dx \ge g_n^2(x_0)/(2K) .$$

Using (3.9) we obtain $g_n(x_0) \le (2K/((1-a)M))^{1/2}$ and, consequently,

(3.10) $$\lim_{M\to\infty} \sup_{n\ge 1} \sup_{x \ge M} g_n(x) = 0 .$$

Condition (3.10) and boundedness of $\{g_n\}$ and $\{g'_n\}$ imply that $\{g_n\}$ is relatively compact. Moreover, from (3.9) it follows that all accumulation points of $\{g_n\}$ are densities. Now, we show that $\{g_n\}$ has only one accumulation point. Applying the inequality $Y'_n \le (n+2)/(n+1)$ to (3.6) we obtain $X'_{n+1} \ge X'_n Y'_n$. Let

$$Z_{n,k} = Y'_n Y'_{n+1} \ldots Y'_{n+k-1} .$$

Then $X'_{n+k} \ge X'_n Z_{n,k}$. Since $X'_n, Y'_n, \ldots, Y'_{n+k-1}$ are independent and $EZ_{n,k} = 1$, we have

(3.11) $$E(X'_{n+k} - X'_n Z_{n,k}) = EX'_{n+k} - EX'_n \le \frac{a^{n+1}}{1-a} .$$

Furthermore,

$$EZ_{n,k}^2 \le \prod_{j=n}^{\infty}\left(1+\frac{1}{j^2}\right).$$

Thus $D^2 Z_{n,k} \le \beta(n)$, where $\lim_{n\to\infty}\beta(n) = 0$. Let δ and ε be fixed positive numbers and choose $n_0(\delta,\varepsilon)$ such that for $n \ge n_0(\delta,\varepsilon)$

$$\beta(n) \le \varepsilon^2\delta^3(1-a)^2/108 \quad \text{and} \quad a^{n+1} < \varepsilon\delta(1-a)/6\,.$$

Then from (3.8), (3.11) and the Chebyshev inequality it follows that

$$\mathrm{Prob}(|X'_{n+k} - X'_n Z_{n,k}| > \varepsilon/2) \le 2a^{n+1}/((1-a)\varepsilon) < \delta/3\,,$$

$$\mathrm{Prob}(|Z_{n,k}-1| > \delta\varepsilon(1-a)/6) \le 36 D^2 Z_{n,k}/(\delta^2\varepsilon^2(1-a)^2) \le \delta/3\,,$$

and

$$\mathrm{Prob}(X'_n > 3/(\delta(1-a))) \le \delta/3\,.$$

The last three inequalities imply

$$\mathrm{Prob}(|X'_{n+k} - X'_n| \le \varepsilon) > 1-\delta \tag{3.12}$$

and, consequently, for every $\varepsilon > 0$ we have

$$\lim_{n\to\infty,\, m\to\infty} \mathrm{Prob}(|X'_m - X'_n| > \varepsilon) = 0\,. \tag{3.13}$$

Hence X'_n converges in probability. It follows that $\{g_n\}$ has only one accumulation point g. Since $\{g_n\}$ is relatively compact, g_n converges uniformly to g. ∎

R e m a r k 2. Since $g, g_1, g_2, \ldots$ are densities and $g_n \to g$ uniformly, g_n converges to g in L^1.

LEMMA 6. *Let g be the function from Lemma 5 corresponding to $a < 1$. Then*

$$\int \left|\frac{n!}{t^n}\varphi_n(t,x,a) - a^n g(a^n x)\right| dx \to 0 \tag{3.14}$$

and

$$\frac{n!}{t^n}\varphi_n(t,x,a^{-1}) \rightrightarrows g(x) \qquad \text{on } [0,\infty) \tag{3.15}$$

as $t\to\infty$ and $n/t \to 1$.

This follows immediately from Lemmas 3 and 5, Corollary 1 and Remark 2.

4. Convergence of solutions. We first examine the operator P. In this section we assume that μ satisfies

$$\int_0^\infty \ln(1+x)\,\mu(dx) < \infty\,. \tag{4.1}$$

Let $v \in D$ and let $\mu_{n,v}$ denote the Borel measure on $[0,\infty)$ defined by

$$\mu_{n,v}(A) = \int_A P^n v\,dx\,. \tag{4.2}$$

Lemma 7. (i) *If $a > 1$, then there exists a Borel probability measure μ_0 such that for every $v \in D$ the measures $\mu_{n,v}$ converge weakly to μ_0 as $n \to \infty$.*

(ii) *If $a < 1$, then for every $v \in D$ there exists $v_0 \in D$ such that the functions $v_n(x) = a^{-n}P^n v(a^{-n}x)$ converge in L^1 to v_0 as $n \to \infty$.*

(iii) *If $a = 1$, $m = \int x\,\mu(dx) < \infty$, $k = \int x^2\,\mu(dx) < \infty$, $\sigma^2 = k - m^2 > 0$, $w_{n,v}(x) = \sqrt{n}\sigma P^n v(nm + x\sigma\sqrt{n})$ and $W_{n,v}(x) = \int_{-\infty}^{x} w_{n,v}(y)\,dy$ then $W_{n,v} \rightrightarrows \Phi$ on $\mathbb{R}$. Moreover, if μ has a bounded density then $w_{n,v} \rightrightarrows \varphi$ on $\mathbb{R}$.*

The proof is partly based on the technique developed by Łoskot [5] who investigated iterates of random variables.

Proof. Let $\xi_1, \xi_2, \ldots$ be a sequence of independent random variables with distribution μ, i.e. $\mathrm{Prob}(\xi_i \in A) = \mu(A)$, and let X be a random variable independent of $\xi_1, \xi_2, \ldots$ with density v. Then $P^n v$ is the density of

$$\zeta_n = a^{-n}X + a^{-1}\xi_1 + \ldots + a^{-n}\xi_n\,. \tag{4.3}$$

Let $a > 1$. From the Kolmogorov three series theorem (see e.g. [7]) it follows that $\sum a^{-n}\xi_n$ converges a.e. if

$$\sum_{n=1}^{\infty} E\biggl(\frac{a^{-n}\xi_n}{1 + a^{-n}\xi_n}\biggr) < \infty\,.$$

Since each ξ_n has distribution μ,

$$\begin{aligned}\sum_{n=1}^{\infty} E\biggl(\frac{a^{-n}\xi_n}{1 + a^{-n}\xi_n}\biggr) &= \sum_{n=1}^{\infty} \int_0^{\infty} \frac{a^{-n}x}{1 + a^{-n}x}\,\mu(dx) \\ &< \int_0^{\infty}\int_0^{\infty} \frac{a^{-t}x}{1 + a^{-t}x}\,\mu(dx)\,dt = \ln^{-1} a \int_0^{\infty} \ln(1+x)\,\mu(dx) < \infty\,.\end{aligned}$$

This implies that ζ_n converges a.e. to some random variable ζ and, consequently, $\mu_{n,v}$ converges weakly to μ_0, where $\mu_0(A) = \mathrm{Prob}(\zeta \in A)$.

Let $a < 1$. Then v_n is the density of $a^n\zeta_n$. Since the ξ_i have the same distribution, from (4.3) it follows that v_n is the density of $X + \xi_1 + \ldots + a^{n-1}\xi_n$. The series $\sum a^{n-1}\xi_n$ is a.e. convergent. This implies that v_n converges in L^1 to some density.

Let $a = 1$. That $W_{n,v} \rightrightarrows \Phi$ follows immediately from the central limit theorem. If μ has a bounded density, then we apply the local form of the central limit theorem (see e.g. [6]).

Proof of Theorem 1. Let $N(t)$ be the Poisson process, i.e.

$$\mathrm{Prob}(N(t)=n)=p_{n,t}=t^n e^{-n}/n!\,.$$

Let

$$A_{t,\delta}=\{\,n\ge 0:|n-t|<\delta t\},\qquad \delta>0\,.$$

Since $EN(t)=t$ and $D^2N(t)=t$, the Chebyshev inequality implies

$$\sum_{n\notin A_{t,\delta}} p_{n,t}\le D^2N(t)/(\delta t)^2=1/(\delta^2 t)\,. \tag{4.4}$$

Case 1: $a<1$. Let $\varepsilon>0$. Since $T_n(t)=\varphi_n * P^n$, Lemma 7(ii) and (3.14) imply that for every $\varepsilon>0$ there exist $t_0>0$ and $\delta>0$ such that

$$\int |t^{-n}n!T_n(t)v(x)-a^n(v_0*g)(a^n x)|\,dx<\varepsilon \tag{4.5}$$

for $t>t_0$ and $n\in A_{t,\delta}$. By the definition of $T_n(t)v$, $\int T_n(t)v=t^n/n!$. Let

$$w(t,x)=\sum_{n=0}^{\infty} p_{n,t}a^n(v_0*g)(a^n x)\,. \tag{4.6}$$

From (4.4) and (4.5) it follows that

$$\begin{aligned}\int |u(t,x)-w(t,x)|\,dx &\le \sum_{n=0}^{\infty}\int |e^{-t}T_n(t)v(x)-p_{n,t}a^n(v_0*g)(a^n x)|\,dx\\ &<\sum_{n\in A_{t,\delta}}\varepsilon p_{n,t}+\sum_{n\notin A_{t,\delta}}2p_{n,t}<\varepsilon+2/(\delta^2 t)\,.\end{aligned}$$

This implies that $u(t,\cdot)-w(t,\cdot)$ converges to 0 in L^1 as $t\to\infty$. Let

$$W(t,x)=\int_{-\infty}^{x} w(t,y)\,dx,\qquad F(x)=\int_0^{a^{-x}}(v_0*g)(y)\,dy$$

and $H(t,x)=W(t,a^{-x})$. Then F is a distribution function and

$$H(t,x)=\sum_{n=0}^{\infty}p_{n,t}F(x-n)\,.$$

Let X be a random variable independent of $N(t)$ with distribution function F. Then $H(t,x)$ are the distribution functions of the process $N(t)+X$. Since $(N(t)-t)/\sqrt{t}$ converges weakly to the normal distribution,

$$H(t,t+x\sqrt{t})\rightrightarrows \Phi(x)\qquad\text{on }\mathbb{R}\text{ as }t\to\infty\,,$$

which gives

$$U(t,a^{-\sqrt{t}x-t})\rightrightarrows \Phi(x)\qquad\text{on }\mathbb{R}\,.$$

Case 2: $a > 1$. From Lemma 7(i) and (3.15) it follows that $t^{-n} n! T_n(t)v$ converges uniformly to $g * \mu_0$ as $n \to \infty$ and $t \to \infty$ in such a way that $n/t \to 1$. Since $\{g_n\}$ is bounded, there exists $C > 0$ such that

$$\sup_x T_n(t)v \le \sup_x |\varphi_n(t,x,a)| \le t^{n-1} C/(n-1)! \,. \tag{4.7}$$

Now, using a similar argument to that in Case 1 we obtain $u(t,\cdot) \rightrightarrows g * \mu_0$ on $[0,\infty)$ as $t \to \infty$.

Case 3: $a = 1$. It is easy to observe that the solution u of (1.1) and (1.2) is given by

$$u(t,x) = \sum_{n=0}^{\infty} p_{n,t} P^n v(x-t) \,. \tag{4.8}$$

Let $k > m^2$, $G_n(x) = \int_{-\infty}^{x} P^n v(y)\, dy$ and

$$Z(t,x) = U(t, \sqrt{kt}x + mt + t) = \sum_{n=0}^{\infty} p_{n,t} G_n(\sqrt{kt}x + mt) \,.$$

Let $\varepsilon > 0$. From Lemma 7(iii) it follows that there exists $n_0 > 0$ such that

$$|G_n(nm + y\sigma\sqrt{n}) - \Phi(y)| < \varepsilon \tag{4.9}$$

for $n \ge n_0$ and $y \in \mathbb{R}$. This implies that

$$\left| G_n(\sqrt{kt}x + mt) - \Phi\left(\frac{\sqrt{kt}x + m(t-n)}{\sigma\sqrt{n}} \right) \right| < \varepsilon$$

for $n \ge n_0$. Let $\delta > 0$ be such that $|\Phi(x/\sqrt{n}) - \Phi(x/\sqrt{t})| < \varepsilon$ for $n \in A_{t,\delta}$ and $x \in \mathbb{R}$. Then

$$\left| G_n(\sqrt{kt}x + mt) - \Phi\left(\frac{\sqrt{kt}x + m(t-n)}{\sigma\sqrt{t}} \right) \right| < 2\varepsilon \tag{4.10}$$

for $n \in A_{t,\delta}$, $x \in \mathbb{R}$ and sufficiently large t. Let

$$W(t,x) = \sum_{n=0}^{\infty} p_{n,t} \Phi\left(\frac{\sqrt{kt}x + m(t-n)}{\sigma\sqrt{t}} \right) . \tag{4.11}$$

Then

$$|Z(t,x) - W(t,x)| \le 2\varepsilon + 2 \sum_{n \notin A_{t,\delta}} p_{n,t} \le 2\varepsilon + 2/(\delta^2 t)$$

for sufficiently large t. This implies that

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |Z(t,x) - W(t,x)| = 0 \,. \tag{4.12}$$

Similarly, if v is a bounded function and μ has a bounded density then

$$\lim_{t\to\infty} \sup_{x\in\mathbb{R}} |z(t,x) - w(t,x)| = 0 \,, \tag{4.13}$$

where $z(t,x) = \frac{\partial Z}{\partial x}(t,x)$ and $w(t,x) = \frac{\partial W}{\partial x}(t,x)$. Now let X be a random variable independent of the process $N(t)$ and with density φ. Then for every $t > 0$ the function $w(t,x)$ is the density of the random variable

$$Y(t) = \frac{\sigma X}{\sqrt{k}} + \frac{m(N(t)-t)}{\sqrt{kt}}\,.$$

It is easy to check that the density of $Y(t)$ converges uniformly to φ as $t \to \infty$, which completes the proof in the case $k > m^2$. If $k = m^2$ then μ is concentrated at $x = m$. This implies that

$$u(t,x) = \sum_{n=0}^{\infty} p_{n,t} v(x - nm - t)\,.$$

Now suppose that ξ is a random variable independent of the process $N(t)$ and with density function v. Then $u(t,x)$ is the density of $X(t) = \xi + t + mN(t)$. Since the distribution function of $(X(t) - t - mt)/(m\sqrt{t})$ converges uniformly to Φ as $t \to \infty$, we obtain

$$U(t, m\sqrt{t}x + mt + t) \rightrightarrows \Phi(x)\,. \ \blacksquare$$

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Reçu par la Rédaction le 2.1.1991