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Nonnegative solutions of a class of second order nonlinear differential equations

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Abstract. A differential equation of the form

$$(q(t)k(u)u')' = \lambda f(t)h(u)u'$$

depending on the positive parameter λ is considered and nonnegative solutions u such that $u(0) = 0$, $u(t) > 0$ for $t > 0$ are studied. Some theorems about the existence, uniqueness and boundedness of solutions are given.

1. Introduction. In [6] the equation

$$(k(u)u')' = f(t)u' \tag{1}$$

was considered and the author has given sufficient conditions for the existence and uniqueness of nonnegative solutions u such that $u(0) = 0$, $u(t) > 0$ for $t > 0$. This problem is connected with the description of the mathematical model of the infiltration of water. For more details see e.g. [3]–[5].

In [4] and [5] the existence and uniqueness of nonnegative solutions was proved for the differential equations

$$(uu')' = (1-t)u' \quad (t \in [0,1])$$

and

$$(uu')' = A^{-t}u' \quad (A > 1).$$

The methods are based on the special form of the equations and on the Banach fixed point theorem. In [1] and [2], the following equation was considered:

$$(k(u)u')' = (1-t)u'.$$

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In this paper we consider the equation

$$(q(t)k(u)u')' = f(t)h(u)u' \tag{2}$$

which is a generalization of (1), and give sufficient conditions for the existence and uniqueness of solutions u of (2) satisfying $u(0) = 0$, $u(t) > 0$ for $t > 0$, as well as for their boundedness and unboundedness. In the last section we discuss the dependence of solutions of the equation $(q(t)k(u)u')' = \lambda f(t)h(u)u'$ on the positive parameter λ and we consider the boundary value problem $(q(t)k(u)u')' = \lambda f(t)h(u)u'$, $\lim_{t\to\infty} u(t;\lambda) = a$ $(\in (0,\infty))$. In accordance with [6] the proof of the existence theorem is based on an iterative method and a monotone behaviour of some operator. The proof of the uniqueness is different from the one in [6]. For the special case of (2), namely (1), we obtain the same results as in [6] (where $\int_0^\infty (k(s)/s)\,ds = \infty$ should be required).

2. Notations, lemmas. We will consider the differential equation (2) in which q, k, f, h satisfy the following assumptions:

(H$_1$) $q \in C^0([0,\infty))$, $q(t) > 0$ for all $t > 0$ and $\displaystyle\int_0 \frac{dt}{q(t)} < \infty$;

(H$_2$) $k \in C^0([0,\infty))$, $k(0) = 0$, $k(u) > 0$ for all $u > 0$;

(H$_3$) $\displaystyle\int_0 \frac{k(s)}{s}\,ds < \infty$ and $\displaystyle\int^\infty \frac{k(s)}{s}\,ds = \infty$;

(H$_4$) $f \in C^1([0,\infty))$, $f(t) > 0$, $f'(t) \le 0$ for all $t \ge 0$;

(H$_5$) $h \in C^0([0,\infty))$, $h(u) \ge 0$ and the function $H(u) := \displaystyle\int_0^u h(s)\,ds$ is strictly increasing for all $u \ge 0$;

(H$_6$) $\displaystyle\int_0 \frac{k(u)}{H(u)}\,du < \infty$ and $\displaystyle\int^\infty \frac{k(u)}{H(u)}\,du = \infty$.

By a solution of (2) we mean a function $u \in C^0([0,\infty)) \cap C^1((0,\infty))$ such that $u(0) = 0$, $u(t) > 0$ for all $t > 0$, $\lim_{t\to 0^+} q(t)k(u(t))u'(t) = 0$, $q(t)k(u(t))u'(t)$ is continuously differentiable for all $t > 0$ and (2) is satisfied on $(0,\infty)$.

For $u \in [0,\infty)$ we define the strictly increasing functions K and V by

$$K(u) = \int_0^u k(s)\,ds\,, \qquad V(u) = \int_0^u \frac{k(s)}{H(s)}\,ds\,.$$

Clearly $K \in C^1([0,\infty))$, $V \in C^0([0,\infty)) \cap C^1((0,\infty))$, $\lim_{u\to\infty} K(u) = \infty = \lim_{u\to\infty} V(u)$.

Set $M = \{u\,;u \in C^0([0,\infty)),\ u(0)=0,\ u(t)>0 \text{ for } t>0\}$.

LEMMA 1. *If u is a solution of* (2), *then u is a solution of the integral equation*

$$K(u(t)) = \int\limits_0^t \left(\frac{f(s)}{q(s)} - f'(s)\int\limits_s^t \frac{dz}{q(z)}\right) H(u(s))\,ds \tag{3}$$

and conversely, if $u \in M$ is a solution of (3), *then u is a solution of* (2).

Proof. Let u be a solution of (2). Integrating (2) from a (>0) to t, we obtain

$$\begin{aligned} q(t)k(u(t))u'(t) - q(a)k(u(a))u'(a) &= \int\limits_a^t f(s)h(u(s))u'(s)\,ds \\ &= f(t)H(u(t)) - f(a)H(u(a)) - \int\limits_a^t f'(s)H(u(s))\,ds\,. \end{aligned}$$

Let $a \to 0^+$. We get

$$(K(u(t)))' = \frac{1}{q(t)}\Big[f(t)H(u(t)) - \int\limits_0^t f'(s)H(u(s))\,ds\Big] \tag{4}$$

for $t>0$, and integrating (4) from 0 to t, we have

$$\begin{aligned} K(u(t)) &= \int\limits_0^t \frac{1}{q(s)}\Big[f(s)H(u(s)) - \int\limits_0^s f'(z)H(u(z))\,dz\Big]\,ds \\ &= \int\limits_0^t \left(\frac{f(s)}{q(s)} - f'(s)\int\limits_s^t \frac{dz}{q(z)}\right) H(u(s))\,ds\,, \end{aligned}$$

and consequently, u is a solution of (3).

Now, let $u \in M$ be a solution of (3). Then

$$u(t) = K^{-1}\Big[\int\limits_0^t \left(\frac{f(s)}{q(s)} - f'(s)\int\limits_s^t \frac{dz}{q(z)}\right) H(u(s))\,ds\Big] \tag{5}$$

for $t \ge 0$, where K^{-1} denotes the inverse function to K on $[0,\infty)$. From (4) it follows that $u' \in C^0((0,\infty))$ and

$$u'(t) = \frac{1}{q(t)k(u(t))}\Big[f(t)H(u(t)) - \int\limits_0^t f'(s)H(u(s))\,ds\Big]\,,$$

therefore

$$q(t)k(u(t))u'(t) = f(t)H(u(t)) - \int_0^t f'(s)H(u(s))\,ds\,. \tag{6}$$

Hence

$$\lim_{t\to 0^+} q(t)k(u(t))u'(t) = 0\,, \qquad q(t)k(u(t))u'(t) \in C^1((0,\infty))\,,$$
$$(q(t)k(u(t))u'(t))' = f(t)h(u(t))u'(t) \qquad \text{for } t > 0\,,$$

consequently, u is a solution of (2).

Remark 1. It follows from Lemma 1 that solving (2) is equivalent to solving the integral equation (3) in the set M.

Lemma 2. *If* $u \in M$ *is a solution of* (3), *then*

$$V^{-1}\Bigl(\int_0^t \frac{f(s)}{q(s)}\,ds\Bigr) \le u(t) \le V^{-1}\Bigl(f(0)\int_0^t \frac{ds}{q(s)}\Bigr) \qquad \textit{for } t \ge 0\,. \tag{7}$$

Proof. Let $u \in M$ be a solution of (3). Then $u'(t) > 0$ for $t > 0$ and (cf. (6))

$$f(t)H(u(t)) \le q(t)k(u(t))u'(t) \le \Bigl[f(t) - \int_0^t f'(s)\,ds\Bigr]H(u(t))$$
$$= f(0)H(u(t))\,,$$

hence

$$\frac{f(t)}{q(t)} \le \frac{k(u(t))u'(t)}{H(u(t))} = (V(u(t)))' \le \frac{f(0)}{q(t)} \qquad \text{for } t > 0\,. \tag{8}$$

Integrating (8) from 0 to t, we obtain

$$\int_0^t \frac{f(s)}{q(s)}\,ds \le V(u(t)) \le f(0)\int_0^t \frac{ds}{q(s)} \qquad \text{for } t \ge 0 \tag{9}$$

and (7) follows.

Define the operator $T : M \to M$ by

$$(Tu)(t) = K^{-1}\Bigl[\int_0^t \Bigl(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\Bigr)H(u(s))\,ds\Bigr] \qquad \text{for } t \ge 0$$

and set

$$\underline{\varphi}(t) = V^{-1}\Bigl(\int_0^t \frac{f(s)}{q(s)}\,ds\Bigr)\,, \qquad \overline{\varphi}(t) = V^{-1}\Bigl(f(0)\int_0^t \frac{ds}{q(s)}\Bigr) \qquad \text{for } t \ge 0\,.$$

LEMMA 3. *For* $t \in [0, \infty)$,

$$(T\underline{\varphi})(t) \ge \underline{\varphi}(t)\,, \qquad (T\,\overline{\varphi})(t) \le \overline{\varphi}(t)\,. \tag{10}$$

P r o o f. Setting

$$\alpha(t) = \int\limits_0^t \left(\frac{f(s)}{q(s)} - f'(s)\int\limits_s^t \frac{dz}{q(z)}\right) H(\underline{\varphi}(s))\,ds - K(\underline{\varphi}(t))\,,$$

$$\beta(t) = \int\limits_0^t \left(\frac{f(s)}{q(s)} - f'(s)\int\limits_s^t \frac{dz}{q(z)}\right) H(\overline{\varphi}(s))\,ds - K(\overline{\varphi}(t))$$

for $t \ge 0$ we see that to prove Lemma 3 it is enough to show $\alpha(t) \ge 0$ and $\beta(t) \le 0$ on $[0, \infty)$. Since

$$\begin{aligned}\alpha'(t) &= \frac{f(t)}{q(t)} H(\underline{\varphi}(t)) - \frac{1}{q(t)}\int\limits_0^t f'(s) H(\underline{\varphi}(s))\,ds - K'(\underline{\varphi}(t))\underline{\varphi}'(t)\\ &= -\frac{1}{q(t)}\int\limits_0^t f'(s) H(\underline{\varphi}(s))\,ds \ge 0\,,\\ \beta'(t) &= \frac{f(t)}{q(t)} H(\overline{\varphi}(t)) - \frac{1}{q(t)}\int\limits_0^t f'(s) H(\overline{\varphi}(s))\,ds - K'(\overline{\varphi}(t))\overline{\varphi}'(t)\\ &\le \frac{f(t)-f(0)}{q(t)} H(\overline{\varphi}(t)) - \frac{H(\overline{\varphi}(t))}{q(t)}\int\limits_0^t f'(s)\,ds = 0\end{aligned}$$

for $t > 0$ and $\alpha(0) = 0 = \beta(0)$, we see $\alpha(t) \ge 0$, $\beta(t) \le 0$ on $[0, \infty)$ and inequalities (10) are true.

3. Existence theorem. We define sequences $\{u_n\} \subset M$, $\{v_n\} \subset M$ by the recurrence formulas

$$\begin{aligned} u_0 &= \underline{\varphi}\,, \qquad u_{n+1} = T(u_n)\,,\\ v_0 &= \overline{\varphi}\,, \qquad v_{n+1} = T(v_n)\end{aligned}$$

for $n = 0, 1, 2, \ldots$

THEOREM 1. *Let assumptions* (H_1)–(H_6) *be fulfilled. Then the limits*

$$\lim_{n\to\infty} u_n(t) =: \underline{u}(t)\,, \qquad \lim_{n\to\infty} v_n(t) =: \overline{u}(t)$$

exist for all $t \ge 0$. *The functions* $\underline{u}$, $\overline{u}$ *are solutions of* (2), *and if* u *is any solution of* (2) *then*

$$\underline{u}(t) \le u(t) \le \overline{u}(t) \quad \textit{for } t \ge 0\,. \tag{11}$$

P r o o f. By Lemma 3 we have

$$u_0(t) \le u_1(t)\,, \qquad v_1(t) \le v_0(t) \qquad \text{for } t \ge 0\,.$$

Since $\alpha, \beta \in M$ and $\alpha(t) \le \beta(t)$ for $t \ge 0$ implies $(T\alpha)(t) \le (T\beta)(t)$ for $t \ge 0$, we deduce

$$\underline{\varphi}(t) = u_0(t) \le u_1(t) \le \ldots \le u_n(t) \le \ldots \le v_n(t) \le \ldots \le v_1(t) \le v_0(t) = \overline{\varphi}(t)$$

for $t \ge 0$ and $n \in \mathbb{N}$. Therefore the limits $\lim_{n\to\infty} u_n(t) =: \underline{u}(t)$, $\lim_{n\to\infty} v_n(t) =: \overline{u}(t)$ exist for all $t \ge 0$, $\underline{\varphi}(t) \le \underline{u}(t) \le \overline{u}(t) \le \overline{\varphi}(t)$ on $[0,\infty)$ and using the Lebesgue theorem we see that $\underline{u}$, $\overline{u}$ are solutions of (3) and $\underline{u}, \overline{u} \in M$.

If $u \in M$ is a solution of (3), by Lemma 2 we have

$$\underline{\varphi}(t) \le u(t) \le \overline{\varphi}(t) \qquad \text{for } t \ge 0$$

and (11) follows by the monotonicity of T.

Lemma 3. *If* (2) *admits two different solutions u and v, then $u(t) \ne v(t)$ for all $t > 0$.*

P r o o f. Let u, v be two different solutions of (2). First, suppose there exists a $t_1 > 0$ such that $u(t) < v(t)$ for $t \in (0, t_1)$ and $u(t_1) = v(t_1)$. Since $H(u(t)) - H(v(t)) < 0$ on $(0, t_1)$, we have

$$K(u(t_1)) - K(v(t_1)) = \int_0^{t_1} \left(\frac{f(s)}{q(s)} - f'(s) \int_s^{t_1} \frac{dz}{q(z)} \right) (H(u(s)) - H(v(s)))\, ds < 0,$$

contradicting $K(u(t_1)) = K(v(t_1))$.

Secondly, suppose there exist $0 < t_1 < t_2$ such that $u(t_n) = v(t_n)$ ($n = 1, 2$) and $u(t) \ne v(t)$ on (t_1, t_2). Suppose

$$u(t) < v(t) \qquad \text{for } t \in (t_1, t_2)\,.$$

Then $u'(t_1) - v'(t_1) \le 0$, $u'(t_2) - v'(t_2) \ge 0$, $H(u(t)) - H(v(t)) < 0$ on (t_1, t_2), therefore

$$\begin{aligned} 0 &\le q(t_2) k(u(t_2))(u'(t_2) - v'(t_2)) - q(t_1) k(u(t_1))(u'(t_1) - v'(t_1)) \\ &= - \int_{t_1}^{t_2} f'(s)(H(u(s)) - H(v(s)))\, ds \le 0 \end{aligned}$$

and consequently, $f'(t) = 0$ on $[t_1, t_2]$. Hence $u'(t_1) = v'(t_1)$, $f(t) =$const $(=: k)$ for $t \in [t_1, t_2]$ and

$$K(u(t)) - K(v(t)) = \int_{t_1}^{t} \frac{k}{q(s)} (H(u(s)) - H(v(s)))\, ds \qquad \text{for } t \in [t_1, t_2]\,.$$

Then we have

$$0 = K(u(t_2)) - K(v(t_2)) = \int_{t_1}^{t_2} \frac{k}{q(s)}(H(u(s)) - H(v(s)))\,ds\,,$$

which contradicts $H(u(t)) - H(v(t)) \neq 0$ for $t \in (t_1, t_2)$.

4. Bounded and unbounded solutions

THEOREM 2. *Let assumptions* (H_1)–(H_6) *be fulfilled. Then*

(i) *some* (*and then any*) *solution of* (2) *is bounded if and only if*

$$\int_0^\infty \frac{ds}{q(s)} < \infty\,,$$

(ii) *some* (*and then any*) *solution of* (2) *is unbounded if and only if*

$$\int_0^\infty \frac{ds}{q(s)} = \infty\,.$$

Proof. First observe that either $\int_0^\infty ds/q(s) < \infty$ or $\int_0^\infty ds/q(s) = \infty$.

Suppose $\int_0^\infty ds/q(s) < \infty$. Then according to Lemma 2 any solution of (3) (and by Lemma 1 also any solution of (2)) is bounded.

Suppose $\int_0^\infty ds/q(s) = \infty$ and let u be a solution of (2). Then

$$K(u(t)) = \int_0^t \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right) H(u(s))\,ds \qquad \text{for } t \geq 0\,,$$

and for $t \geq t_1$, where t_1 is a positive number, we have

$$\begin{aligned}
K(u(t)) = & \int_0^{t_1} \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right) H(u(s))\,ds \\
& + \int_{t_1}^{t} \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right) H(u(s))\,ds \\
& \geq H(u(t_1)) \int_{t_1}^{t} \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right) ds \\
& = H(u(t_1)) f(t_1) \int_{t_1}^{t} \frac{dz}{q(z)}\,.
\end{aligned}$$

Therefore $\lim_{t\to\infty} K(u(t)) = \infty$ and u is necessarily unbounded.

5. Uniqueness theorem

THEOREM 3. *Let assumptions* (H_1)–(H_6) *be fulfilled. Assume that there exists* $\varepsilon > 0$ *such that the modulus of continuity* $\gamma(t)$ $(:= \sup\{|q(t_1) - q(t_2)|;\ t_1, t_2 \in [0,\varepsilon],\ |t_1 - t_2| \le t\})$ *of* q *on* $[0,\varepsilon]$ *satisfies*

$$\limsup_{t\to 0^+} \gamma(t)/t < \infty\,.$$

Then (2) *admits a unique solution.*

Proof. According to Lemma 1 and Theorem 1, it is sufficient to show that (3) admits a unique solution, that is, $\underline{u} = \overline{u}$, where $\underline{u}$, $\overline{u}$ are defined in Theorem 1. Since $0 < \underline{u}(t) \le \overline{u}(t)$ on $(0,\infty)$, we see that $\underline{u}'(t) > 0$, $\overline{u}'(t) > 0$ for $t > 0$. Set $u_1 = \underline{u}$, $u_2 = \overline{u}$, $A_i = \lim_{t\to\infty} u_i(t)$ and $w_i = u_i^{-1}$, where u_i^{-1} denotes the inverse function to u_i $(i = 1,2)$. Then

$$w_i'(x) = q(w_i(x))k(x)\Big[\int_0^x f(w_i(s))h(s)\,ds\Big]^{-1} \quad \text{for } x \in (0, A_i),\ i = 1,2$$

and

$$w_i(x) = \int_0^x q(w_i(s))k(s)\Big[\int_0^s f(w_i(z))h(z)\,dz\Big]^{-1} ds \quad \text{for } x \in [0, A_i),\ i = 1,2\,.$$

Therefore, for $x \in [0, A_1)$ we have

$$\begin{aligned}
(12)\quad (0 \le)\ & w_1(x) - w_2(x) \\
&= \int_0^x (q(w_1(s)) - q(w_2(s)))k(s)\Big[\int_0^s f(w_2(z))h(z)\,dz\Big]^{-1} ds \\
&\quad + \int_0^x \Big\{q(w_1(s))k(s)\Big[\int_0^s f(w_1(z))h(z)\,dz \int_0^s f(w_2(z))h(z)\,dz\Big]^{-1} \\
&\quad \times \int_0^s (f(w_2(z)) - f(w_1(z)))h(z)\,dz\Big\}\,ds\,.
\end{aligned}$$

Define $a = u_1(\varepsilon)$, $X(x) = \max\{w_1(t) - w_2(t);\, 0 \le t \le x\}$ for $x \in [0,a]$. Suppose $X(x) > 0$ on $(0,a]$. Then

$$|q(w_1(x)) - q(w_2(x))| \le \gamma(X(x)) \quad \text{for } x \in [0,a]$$

and using (12) we have

$$w_1(x) - w_2(x) \le (LX(x) + T\gamma(X(x)))V(x) \quad \text{for } 0 \le x \le a\,,$$

where

$$T = \frac{1}{f(\varepsilon)}\,, \qquad L = T^2 \max_{t\in[0,\varepsilon]} f'(t) \max_{t\in[0,\varepsilon]} q(t)\,.$$

Hence

$$X(x) \le (LX(x) + T\gamma(X(x)))V(x)$$

and

$$\frac{\gamma(X(x))}{X(x)}V(x) \ge (1 - LV(x))T^{-1} \qquad \text{for } x \in (0,a]\,.$$

By the assumption of Theorem 2, $\limsup_{x\to 0^+} \gamma(X(x))/X(x) < \infty$, therefore $\lim_{x\to 0^+}(\gamma(X(x))/X(x))V(x) = 0$, which contradicts the fact that $\lim_{x\to 0^+}(1 - LV(x))T^{-1} = T^{-1}$. This proves that there exists an interval $[0,b]$ $(0 < b \le \infty)$ such that $u_1 = u_2$ on $[0,b]$.

Assume $u_1 \not\equiv u_2$ on $[0,\infty)$ and let $[0,c]$ be the maximal interval where $u_1(t) = u_2(t)$. Define

$$Y(t) = \max\{u_2(s) - u_1(s)\,;\ c \le s \le t\} \qquad \text{for } t \ge c\,.$$

Then $Y(c) = 0$ and $Y(t) > 0$ for all $t > c$. Since

$$K(u_2(t)) - K(u_1(t)) = \int_c^t \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right)(H(u_2(s)) - H(u_1(s)))\,ds$$

for $t \ge c$, we have

$$u_2(t) - u_1(t) \le L_1 Y(t)\int_c^t \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right)ds \qquad \text{for } t \in [c, c+1]\,,$$

where

$$L_1 = \max\{h(u)\,; u \in [u_1(c), u_2(c+1)]\}[\min\{k(u)\,; u \in [u_1(c), u_2(c+1)]\}]^{-1}\,.$$

Hence

$$Y(t) = L_1 Y(t)\int_c^t \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right)ds$$

and

$$1 \le L_1 \int_c^t \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right)ds \qquad \text{for } t \in (c, c+1]\,,$$

which is a contradiction. This completes the proof.

6. Dependence of solutions on the parameter. Consider the differential equation

$$(q(t)k(u)u')' = \lambda f(t)h(u)u'\,, \qquad \lambda > 0\,, \tag{13}$$

depending on the positive parameter λ. Assume that assumptions (H_1)–(H_6) are satisfied. Set

$$\underline{\varphi}(t;\lambda) = V^{-1}\left(\lambda\int_0^t \frac{f(s)}{q(s)}\,ds\right), \qquad \overline{\varphi}(t;\lambda) = V^{-1}\left(\lambda f(0)\int_0^t \frac{dz}{q(z)}\right)$$

and define

$$(T_\lambda u)(t) = K^{-1}\left(\lambda \int\limits_0^t \left(\frac{f(s)}{q(s)} - f'(s) \int\limits_s^t \frac{dz}{q(z)}\right) H(u(s))\, ds\right),$$

$$u_0(t;\lambda) = \underline{\varphi}(t;\lambda)\,, \qquad u_{n+1}(t;\lambda) = (T_\lambda u_n)(t)\,,$$

$$v_0(t;\lambda) = \overline{\varphi}(t;\lambda)\,, \qquad v_{n+1}(t;\lambda) = (T_\lambda v_n)(t)$$

for $t \in [0,\infty)$, $\lambda \in (0,\infty)$ and $n \in \mathbb{N}$.

THEOREM 4. *Let assumptions* (H_1)–(H_6) *be fulfilled. Then the limits*

$$\lim_{n\to\infty} u_n(t;\lambda) =: \underline{u}(t;\lambda)\,, \qquad \lim_{n\to\infty} v_n(t;\lambda) =: \overline{u}(t;\lambda) \tag{14}$$

exist for $t \in [0,\infty)$ *and* $\lambda > 0$. *The functions* $\underline{u}(t;\lambda)$ *and* $\overline{u}(t;\lambda)$ *are solutions of* (13), *and if* $u(t;\lambda)$ *is any solution of* (13) *then*

$$\underline{u}(t;\lambda) \le u(t;\lambda) \le \overline{u}(t;\lambda) \quad \text{for } t \ge 0\,. \tag{15}$$

Moreover, for all $0 < \lambda_1 < \lambda_2$ *we have*

$$\underline{u}(t;\lambda_1) < \underline{u}(t;\lambda_2)\,, \quad \overline{u}(t;\lambda_1) < \overline{u}(t;\lambda_2) \quad \text{for } t > 0\,. \tag{16}$$

P r o o f. The proof of the existence of the limits $\lim_{n\to\infty} u_n(t;\lambda)$ and $\lim_{n\to\infty} v_n(t;\lambda)$ and of (15) is similar to the proof of Theorem 1 and therefore it is omitted here.

Let $0 < \lambda_1 < \lambda_2$. Then $\underline{\varphi}(t;\lambda_1) < \underline{\varphi}(t;\lambda_2)$, $\overline{\varphi}(t;\lambda_1) < \overline{\varphi}(t;\lambda_2)$ and $(T_{\lambda_1}u)(t) < (T_{\lambda_2}u)(t)$ for each $u \in M$ and $t > 0$. Since H is strictly increasing on $[0,\infty)$, we have

$$u_n(t;\lambda_1) < u_n(t;\lambda_2)\,, \quad v_n(t;\lambda_1) < v_n(t;\lambda_2) \quad \text{for } t > 0 \text{ and } n \in \mathbb{N}\,,$$

and consequently,

$$\underline{u}(t;\lambda_1) \le \underline{u}(t;\lambda_2)\,, \quad \overline{u}(t;\lambda_1) \le \overline{u}(t;\lambda_2) \quad \text{for } t \ge 0\,.$$

If $v(t_0;\lambda_1) = v(t_0;\lambda_2)$ for a $t_0 > 0$, where v is either $\underline{u}$ or $\overline{u}$, then in view of Lemma 1 we get

$$\lambda_1 \int\limits_0^{t_0} \left(\frac{f(s)}{q(s)} - f'(s) \int\limits_s^{t_0} \frac{dz}{q(z)}\right) H(v(s;\lambda_1))\, ds$$

$$= \lambda_2 \int\limits_0^{t_0} \left(\frac{f(s)}{q(s)} - f'(s) \int\limits_s^{t_0} \frac{dz}{q(z)}\right) H(v(s;\lambda_2))\, ds\,,$$

contradicting $\lambda_1 < \lambda_2$ and

$$\left(\frac{f(t)}{q(t)} - f'(t) \int\limits_t^{t_0} \frac{ds}{q(s)}\right) (H(v(t;\lambda_1)) - H(v(t;\lambda_2))) \le 0 \quad \text{for } t \in (0,t_0]\,.$$

Hence (16) is proved.

THEOREM 5. *Let the assumptions of Theorem* 3 *be fulfilled and* $\int_0^\infty ds/q(s) < \infty$. *Then for each* $a \in (0,\infty)$ *there exists a unique* $\lambda_0 > 0$ *such that equation* (13) *with* $\lambda = \lambda_0$ *has a* (*necessarily unique*) *solution* $u(t;\lambda_0)$ *with*

$$\lim_{t\to\infty} u(t;\lambda_0) = a\,.$$

Proof. According to Theorem 3 equation (13) has for each $\lambda > 0$ a unique solution $u(t;\lambda)$, and by Theorem 1 this solution is bounded. Since $u(t;\lambda)$ is strictly increasing in t on $[0,\infty)$, we can define $g : (0,\infty) \to (0,\infty)$ by

$$g(\lambda) = \lim_{t\to\infty} u(t;\lambda)\,.$$

According to Theorem 4, g is nondecreasing on $(0,\infty)$. If $g(\lambda_1) = g(\lambda_2)$ for some $0 < \lambda_1 < \lambda_2$, then

$$\int_0^\infty \left(\frac{f(s)}{q(s)} - f'(s)\int_s^\infty \frac{dz}{q(z)}\right)(H(u(s;\lambda_2)) - H(u(s;\lambda_1)))\,ds = 0\,,$$

contradicting $H(u(t;\lambda_1)) - H(u(t;\lambda_2)) < 0$ for $t \in (0,\infty)$. Hence g is strictly increasing on $(0,\infty)$. To prove Theorem 5 it is enough to show that g maps $(0,\infty)$ onto itself. First, we see from $\underline{\varphi}(t;\lambda) \le u(t;\lambda) \le \overline{\varphi}(t;\lambda)$ that $\lim_{\lambda\to 0^+} g(\lambda) = 0$ and $\lim_{\lambda\to\infty} g(\lambda) = \infty$. Secondly, assume to the contrary

$$\lim_{\lambda\to\lambda_0^-} g(\lambda) < \lim_{\lambda\to\lambda_0^+} g(\lambda)$$

for a $\lambda_0 > 0$. Setting

$$v_1(t) = \lim_{\lambda\to\lambda_0^-} u(t;\lambda)\,, \qquad v_2(t) = \lim_{\lambda\to\lambda_0^+} u(t;\lambda) \qquad \text{for } t \ge 0\,,$$

we get $v_1 \ne v_2$. On the other hand, using the Lebesgue dominated convergence theorem as $\lambda \to \lambda_0^-$ and $\lambda \to \lambda_0^+$ in the equality

$$u(t;\lambda) = K^{-1}\left[\lambda \int_0^t \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right)H(u(s;\lambda))\,ds\right]$$

we see that

$$v_i(t) = K^{-1}\left[\lambda_0 \int_0^t \left(\frac{f(s)}{q(s)} - f'(s)\int_s^t \frac{dz}{q(z)}\right)H(v_i(s))\,ds\right]$$

for $t \ge 0$ and $i = 1,2$.

Therefore v_1, v_2 are solutions of (13) with $\lambda = \lambda_0$, contradicting the fact that equation (13) with $\lambda = \lambda_0$ has a unique solution.

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