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Asymptotic stability of densities for piecewise convex maps

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Abstract. We study the asymptotic stability of densities for piecewise convex maps with flat bottoms or a neutral fixed point. Our main result is an improvement of Lasota and Yorke's result ([5], Theorem 4).

1. Introduction. Lasota and Yorke [5] studied the following piecewise convex maps $T : [0,1] \to [0,1]$.

(i) There exists a partition $0 = a_0 < a_1 < \ldots < a_r = 1$ such that the restriction of T to (a_{i-1}, a_i) is C^1 and convex; let T_i be a continuous extension to $[a_{i-1}, a_i)$ of this restriction for $i = 1, \ldots, r$.
(ii) $T_i(a_{i-1}) = 0$ for $i = 1, \ldots, r$.
(iii) $T_i'(a_{i-1}) > 0$ for $i = 1, \ldots, r$.
(iv) $T_1'(0) > 1$.

They showed that the Frobenius–Perron operator associated with the map above is asymptotically stable in the sense of Lasota and Mackey [4], which means that the dynamics of densities is asymptotically stable and that there exists a unique invariant exact probability measure. In this paper we improve the conditions (iii) and (iv), that is, we allow $T_i'(a_{i-1}) = 0$ and $T_1'(0) = 1$ under some extra conditions.

In §2 we give some preliminary definitions. In §3 we state our main result. In §5 we prove it, using the first return map which is studied in §4.

2. Preliminaries. In this section we first give the definition of the Frobenius–Perron operator and of its asymptotic stability. Let $(X, \mathcal{F}, m)$ be a σ-finite measure space and let $T : X \to X$ be a nonsingular transformation, that is, a measurable transformation satisfying $m(T^{-1}(A)) = 0$ for all $A \in \mathcal{F}$ with $m(A) = 0$.

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Definition 2.1. The operator $P : L^1 \to L^1$ defined by

$$\int_A Pf(x)\, m(dx) = \int_{T^{-1}(A)} f(x)\, m\,(dx) \quad \text{for } A \in \mathcal{F},\ f \in L^1(m)$$

is called the *Frobenius–Perron operator* associated with (T, m).

By $D(m) = D(X, \mathcal{F}, m)$ we shall denote the set of all densities associated with m on X, that is,

$$D(m) := \{f \in L^1(m)\,;\, f \ge 0 \text{ and } \|f\|_{L^1(m)} = 1\}\,.$$

For $f \in D(m)$ we define a probability measure m_f on $(X, \mathcal{F})$ by

$$m_f(A) = \int_A f\, dm, \quad A \in \mathcal{F}\,.$$

An $f \in D(m)$ is called a *stationary density* of P if $Pf = f$ m-a.e.

Definition 2.2. $\{P^n\}$ is called *asymptotically stable* if there exists a unique density g such that

$$\lim_{n\to\infty} \|P^n f - g\|_{L^1(m)} = 0 \quad \text{for all } f \in D(m)\,.$$

Lemma 2.1. *If there exists a density $g \in D(m)$ such that*

$$\lim_{n\to\infty} \|P^n f - g\|_{L^1(m)} = 0 \quad \textit{for } f \in D(m) \textit{ with } \operatorname{supp}(f) \subset \operatorname{supp}(g)\,, \tag{2.1}$$

and $m(X \setminus \bigcup_{n=0}^{\infty} T^{-n}\operatorname{supp}(g)) = 0$, then P is asymptotically stable.

For the proof of this, see the proof of Proposition 5.3 in [3].

Now we define exactness of a nonsingular transformation and we state a condition for exactness using Frobenius–Perron operators.

Definition 2.3. Let $(X, \mathcal{F}, \mu)$ be a probability space and $T : X \to X$ a measure preserving transformation, that is, μ is T-invariant. If $\bigcap_{n=0}^{\infty} T^{-n}\mathcal{F}$ is trivial, then (T, μ) is called *exact*.

Proposition 2.2 ([3], Proposition 2.3). *If there exists $g \in D(m)$ such that* (2.1) *holds*, *then T preserves the measure m_g and (T, m_g) is exact. Conversely*, *if there exists a stationary density g such that (T, m_g) is exact*, *then* (2.1) *holds.*

3. Main result.

Now we state the main result in this paper.

Theorem 3.1. *Assume that a map $T : [0, 1] \to [0, 1]$ satisfies the following conditions*:

(1) *There exists a partition $0 = a_0 < a_1 < \ldots < a_r = 1$ such that the restriction of T to (a_{i-1}, a_i) is a C^1 function*; *let T_i be a continuous extension to $[a_{i-1}, a_i)$ of this restriction for $i = 1, \ldots, r$.*

(2) $T_i(a_{i-1}) = 0$ *for* $i = 1, \ldots, r$.

(3) $T'(x) > 1$ *for* $x \in (0, a_1)$ *and* $T'(x) > 0$ *for* $x \in (a_{i-1}, a_i)$, $i = 2, \ldots, r$.

(4) $T_i'(x)$ *is an increasing function for* $i = 1, \ldots, r$.

(5) *There exists* n_0 *such that*

$$\sum_{n=n_0}^{\infty} (T_j^{-1} T_1^{-n}(a_1) - a_{j-1}) < \infty$$

for all j *satisfying* $a_{j-1} \in \bigcup_{n=1}^{\infty} T^n(0, a_1)$.

Then the Frobenius–Perron operator associated with T *is asymptotically stable.*

Remark 3.1. Suppose that the condition (5) of the above theorem is invalid. Then there exist no m-absolutely continuous T-invariant ergodic probability measures, but there does exist an m-absolutely continuous T-invariant ergodic σ-finite measure ([2], Theorem 1.1).

Theorem 3.1 and Remark 3.1 imply that asymptotic stability of the Frobenius–Perron operator associated with a map T satisfying the conditions (1)–(4) of Theorem 3.1 is characterized by the finiteness of an m-absolutely continuous T-invariant σ-finite measure.

In the case of $T_1'(0) = 1$, the following corollary gives a useful criterion for the asymptotic stability of the Frobenius–Perron operator associated with T.

Corollary 3.2. *Assume that a map* $T : [0, 1] \to [0, 1]$ *satisfies the following conditions*:

(1)–(4): *same as in Theorem* 3.1.

(5) $T_1(x) \ge x + Mx^p$ *for some* $M > 0$ *and* $1 < p < 2$ *and* $T_j(x) \ge L(x - a_{j-1})^q$ *for* $q < (p-1)^{-1}$ *and for all* j *with* $a_{j-1} \in \bigcup_{n=1}^{\infty} T^n[0, a_1)$ *and* $L > 0$.

Then the Frobenius–Perron operator associated with T *is asymptotically stable.*

Remark 3.2. Under the conditions (1)–(4) of Theorem 3.1, there are no m-absolutely continuous T-invariant probability measures if $T_1(x) \le x + Mx^2$ for some $M > 0$ ([2], Corollary 1.1.2).

In the case of $T_1'(0) > 1$, the following corollary is useful.

Corollary 3.3. *Assume that a map* $T : [0, 1] \to [0, 1]$ *satisfies the following conditions*:

(1)–(4): *same as in Theorem* 3.1.

(5) $T_1'(0) > 1$ *and* $\int_{a_{j-1}}^{a_j} \log T(x)\, dm > -\infty$ *for all* j *with* $a_{j-1} \in \bigcup_{n=1}^{\infty} T^n[0, a_1)$.

Then the Frobenius–Perron operator associated with T is asymptotically stable.

Remark 3.3. Suppose that the integral condition of the above corollary is invalid. Then there exist no m-absolutely continuous T-invariant ergodic probability measures ([2], Corollary 1.1.3).

This integral condition corresponds to the condition (A) for S-unimodal maps studied by Benedicks and Misiurewicz [1].

4. The first return map. In this section we first show how to construct an invariant measure of a given transformation from an invariant measure of the first return map and next study finiteness of the invariant measure constructed. The first return map of T on A is defined as $T^{n(x)}(x)$, where $n(x)$ is $\inf\{n \ge 1; T^{n(x)} \in A\}$. In the following lemma, let T be a transformation on a measure space $(X, \mathcal{F}, m)$ and let $A \subset X$ be a measurable set with $A \subset \bigcup_{n=1}^{\infty} T^{-n}(A)$. Then the first return map is well defined.

LEMMA 4.1 ([6], Lemma 2 and [2], Lemma 3.2). *Let R_A be the first return map of T on A and let μ_A be an R_A-invariant ergodic probability measure. Then the measure μ defined by*

$$\mu(D) = \sum_{n=1}^{\infty} \mu_A(A_n \cap T^{-n} D) \quad \textit{for } D \in \mathcal{F} \tag{4.1}$$

is T-invariant ergodic, where $A_1 = A$ and $A_{n+1} = A_n \cap T^{-n}(A^C)$ for $n \ge 1$.

In the rest of this section we assume that a map $T : [0,1] \to [0,1]$ satisfies the assumptions of Theorem 3.1. Put

$$\alpha_n = T_1^{-n}(a_1) \quad \text{for } n \ge 0 \text{ and}$$
$$\beta_{in} = T_i^{-1}(\alpha_n) \quad \text{if it exists, for } i = 2, \ldots, r \ \text{ and } n \ge 0\,.$$

Consider the first return map R of T on $[a_1, 1]$. Then R can be represented in the following form. For $i = 2, \ldots, r$,

$$R(x) = \begin{cases} T(x) & \text{if } T_i(x) > a_1, \\ T^{n+1}(x) & \text{if } T_i(x) \in (\alpha_n, \alpha_{n-1}), \text{ for } n \ge 1. \end{cases}$$

It is clear that $R(x)$ is defined except on the set of the endpoints of a countable partition of $[a_1, 1]$.

LEMMA 4.2. *Assume that there exists an m-absolutely continuous R-invariant probability measure whose density g is bounded in the right neighborhood of a_{j-1} and that there exists an integer n_0 such that*

$$\sum_{n=n_0}^{\infty} |\beta_{j,n} - a_{j-1}| < \infty \tag{4.2}$$

for all j satisfying $a_{j-1} \in T^n(0,a_1)$ for some n. Let μ be the T-invariant measure defined in Lemma 4.1. *Then $\mu([0,1]) < \infty$.*

Proof. It is easy to see that there exists an integer n_1 such that

$$A_n = \bigcup_{i=2}^{r} [a_{i-1}, \beta_{i,n-2}] \quad \text{for } n \geq n_1 .$$

There exist $n_2 \geq n_1$ and $\gamma < \infty$ such that $g \leq \gamma 1_{(a_{j-1}, \beta_{j,n_2})}$ for all j satisfying $a_{j-1} \in T^n(0,a_1)$ for some n. Thus

$$\begin{aligned} \mu([0,1]) = \sum_{n=1}^{\infty} \mu_A(A_n) &\leq \sum_{n=1}^{n_2-1} \mu_A(A_n) + \sum_{i=2}^{r} \sum_{n=n_2}^{\infty} \mu_A([a_{i-1}, \beta_{i,n-2}]) \\ &\leq \sum_{n=1}^{n_2-1} \mu_A(A_n) + \gamma \sum_{j} \sum_{n=n_2}^{\infty} |a_{j-1} - \beta_{j,n}| , \end{aligned}$$

where j satisfies $a_{j-1} \in T^n(0,a_1)$ for some n. Therefore $\mu([0,1]) < \infty$.

Now we state upper estimates for a stationary density for piecewise monotonic maps with countable partitions which naturally arise from first return maps. Let X be a union of disjoint intervals with $m(X) < \infty$, S a map from X into itself and $\{I_k\}$ a countable partition of X.

Definition 4.1. $(S, X, \{I_k\})$ is called *countable piecewise C^1 with finite images* if S satisfies the following three conditions:

(a) S restricted to the interior of each I_k is a C^1 function.

(b) $1/S'$ is of bounded variation (wherever S' is not defined we define it as the right derivative).

(c) There are only a finite number of different intervals in the collection $\{S(I_k)\}$.

Lemma 4.3. *Assume that $S : [v,w] \to [v,w]$ is countable piecewise C^1 with finite images and $S'(x) \geq \lambda > 1$ whenever $S'(x)$ is defined. Let P be the Frobenius–Perron operator associated with (S,m). Then there exists a bounded stationary density of P.*

This lemma is an easy consequence of the proof of Theorem 1 in [6].

Lemma 4.4. R (*the first return map of T on* $[a_1,1]$) *has an m-absolutely continuous invariant ergodic probability measure μ whose density is bounded in the right neighborhood of a_i for $i \geq 1$ and which satisfies*

$$m\Big([a_1,1] \setminus \bigcup_{n=0}^{\infty} R^{-n} \mathrm{supp}(\mu_A)\Big) = 0 . \tag{4.3}$$

Proof. Let $\{I_k\}$ be the partition of $[a_1, 1]$ with respect to the first return map R. Put $\xi = \inf\{R'(x)\,;R'(x)$ is defined$\}$. Then it is easy to see that $\xi > 0$. Put

$$\xi_n = \inf\{(R^n)'(x)\,;(R^n)'(x) \text{ is defined}\} \quad \text{and} \quad \xi_* = \inf\{\xi_n\,; n \geq 1\}\,.$$

Clearly $\xi_* > 0$ and we can consider the first return map R^* of R on A^*, where A^* is the union of I_k with $\inf_{x \in I_k} R'(x) > \xi_*^{-1}$. It is easy to check that R^* satisfies the assumption of Lemma 4.3. Thus R^* has an m-absolutely continuous invariant probability measure μ_{A^*} whose density is bounded. As a consequence, R has an m-absolutely continuous invariant probability measure μ_A whose density is bounded in the right neighborhood of a_i for $i \geq 1$. (4.3) and ergodicity follow from Proposition 5.1 in [2].

5. Proof of Theorem. Let T be a map satisfying the assumptions of Theorem 3.1 and let R be the first return map of T on $[a_1, 1]$. We begin the proof of Theorem 3.1 with the following lemmas.

LEMMA 5.1. *There exists an m-absolutely continuous T-invariant ergodic probability measure μ such that*

$$m\Big([0,1] \setminus \bigcup_{n=0}^{\infty} T^{-n}\mathrm{supp}(\mu)\Big) = 0\,.$$

This follows from Lemmas 4.1, 4.2 and 4.4.

LEMMA 5.2. *Let P be the Frobenius–Perron operator associated with T. Then $P^n f$ is a decreasing function for f in D_0 which is a dense subset of $D(m)$ and for sufficiently large n.*

This is shown in the proof of Theorem 4 in [5], where the assumptions (iii) and (iv) of the introduction are not used.

LEMMA 5.3. *Let $f \in D(m)$. Assume that there exists a nonnegative function h such that $\|h\|_{L^1(m)} > 0$ and that*

$$\lim_{n\to\infty} \|(P^n f - h)^-\|_{L^1(m)} = 0\,.$$

Then there exists a stationary density h^* such that

$$\lim_{n\to\infty} \|P^n f - h^*\|_{L^1(m)} = 0\,.$$

For the proof of this, see the proof of Theorem 2 in [5].

Proof of Theorem 3.1. Throughout the proof $\|\cdot\|$ stands for $\|\cdot\|_{L^1(m)}$. Let g be the density corresponding to the m-absolutely continuous T-invariant ergodic probability measure of Lemma 5.1 and let c be an arbitrary positive constant. First, we prove that

$$\lim_{n\to\infty} \|P^n f - g\| = 0 \quad \text{for } f \in D(m) \text{ with } f \leq cg\,. \tag{5.1}$$

Let z be a positive constant satisfying

$$\int_0^z cg\,dm < 1/2\,.$$

Then

$$P^n f \geq \tfrac{1}{2}\cdot 1_{(0,z)} \quad \text{for } f \in D_0 \text{ with } f \leq cg\,. \tag{5.2}$$

In fact, if not, it follows from Lemma 5.2 that there exists $y \in [0, z)$ such that

$$1 = \int_0^y P^n f\,dm + \int_y^1 P^n f\,dm \leq \int_0^z cg\,dm + \frac{1}{2}(1-y) < 1\,,$$

which is impossible and (5.2) is proved. (5.2) and Lemma 5.3 imply that

$$\lim_{n\to\infty} \|P^n f - h^*\| = 0 \quad \text{for } f \in D_0 \text{ with } f \leq cg\,.$$

Since (T, m_g) is ergodic, we have $h^* = g$. Thus we obtain (5.1).

Next, we prove that

$$\lim_{n\to\infty} \|P^n f - g\| = 0 \quad \text{for } f \in D(m) \text{ with } \mathrm{supp}(f) \subset \mathrm{supp}(g)\,. \tag{5.3}$$

Put $f_c = \min(f, cg)$. Then $f = \|f_c\|^{-1} f_c + r_c$, where $r_c = (1 - \|f_c\|^{-1}) f_c + f - f_c$. Since $\mathrm{supp}(f) \subset \mathrm{supp}(g)$, we have $\lim_{c\to\infty} f_c(x) = f(x)$ for each x. Hence $\|f_c - f\| \to 0$ and $\|f_c\| \to \|f\| = 1$ $(c \to \infty)$. Thus, for any $\varepsilon > 0$ we can find a constant c such that $\|r_c\| < 2^{-1}\varepsilon$. Since $\|f_c\|^{-1} f_c$ is a density bounded by $c\|f_c\|^{-1} g$, it follows from the first part of the proof that $\|P^n(\|f_c\|^{-1} f_c) - g\| \leq 2^{-1}\varepsilon$ for sufficiently large n. Therefore, $\|P^n f - g\| < \varepsilon$ for sufficiently large n.

By (5.3) and Lemma 5.1, Lemma 2.1 finishes the proof.

The following lemma is used to prove Corollary 3.2 and is easily verified.

LEMMA 5.4. *Let $T : [0, a] \to [0, 1]$ be a continuous strictly increasing function with $T(0) = 0$, where a is a positive constant. If $T(x) \geq x + Mx^p$ for some $p > 1$ and $M > 0$ on $(0, a]$, then there exists a k such that*

$$T^{-n}(a) \leq (k + 2^{-1}(p-1)Mn)^{1/(p-1)} \quad \textit{for all } n\,.$$

Proof. Put $\tau(x) = x + Mx^p$. Since $x < \tau(x) \leq T(x)$, we have $T^{-n}(a) \leq \tau^{-n}(a)$. By an elementary calculation, we have

$$\tau\left(\frac{1}{n^{1/(p-1)}}\right) \geq \frac{1}{(n - 2^{-1}(p-1)M)^{1/(p-1)}} \quad \text{for large } n\,.$$

Therefore, for a k with $k^{1/(1-p)} \geq a$, we get

$$T^{-n}(a) \leq \tau^{-n}(a) \leq \tau^{-n}\left(\frac{1}{k^{1/(p-1)}}\right) \leq \frac{1}{(k + 2^{-1}(p-1)nM)^{1/(p-1)}}\,.$$

Proof of Corollary 3.2. By Lemma 5.4 the conditions (1)–(3) and (5) of Corollary 3.2 imply the condition (5) of Theorem 3.1.

The following lemma is used to prove Corollary 3.3.

LEMMA 5.5. *Let* $T : [a, b] \to [0, T(b)]$ *be a strictly increasing* C^1 *function with* $T(a) = 0$. *If*

$$\int_a^b \log T(x)\, dm > -\infty\,,$$

then for $0 < \alpha < 1$

$$\sum_{n=1}^{\infty} T^{-1}(\alpha^n) < \infty\,.$$

For the proof of this lemma, see Lemmas 1 and 2 in [1].

Proof of Corollary 3.3. By Lemma 5.5 the conditions (1)–(3) and (5) of Corollary 3.3 imply the condition (5) of Theorem 3.1.

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