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Decomposition and disintegration of positive definite kernels on convex $*$-semigroups

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Abstract. The paper deals with operator-valued positive definite kernels on a convex $*$-semigroup $\mathcal{S}$ whose Kolmogorov–Aronszajn type factorizations induce $*$-semigroups of bounded shift operators. Any such kernel Φ has a canonical decomposition into a degenerate and a nondegenerate part. In case $\mathcal{S}$ is commutative, Φ can be disintegrated with respect to some tight positive operator-valued measure defined on the characters of $\mathcal{S}$ if and only if Φ is nondegenerate. It is proved that a representing measure of a positive definite holomorphic mapping on the open unit ball $\mathcal{A}_\bullet$ of a commutative Banach $*$-algebra $\mathcal{A}$ is supported by the holomorphic characters of $\mathcal{A}_\bullet$. A relationship between positive definiteness and complete positivity is established in the case of commutative W^*-algebras.

Contents

Introduction. The general dilation theorem of Sz.-Nagy (cf. [58] and [53]) states that a Hilbert space operator-valued mapping $\Theta : \mathcal{S} \to B(\mathcal{H})$ defined on a $*$-semigroup $\mathcal{S}$ with a unit is dilatable if and only if it is positive definite and satisfies the boundedness condition. In case $\mathcal{S}$ has no unit, the positive definiteness and the boundedness condition are insufficient for Θ to be dilatable. To preserve dilatability we have to replace the positive definiteness by a stronger condition called the extension property (cf. [52] and [42]).

Nevertheless, the positive definiteness and the boundedness condition are necessary and sufficient for Θ to be predilatable. The latter means that there are a mapping $X : \mathcal{S} \to B(\mathcal{H}, \mathcal{K})$ and a $*$-representation Π of $\mathcal{S}$ in $\mathcal{K}$ such that $\Theta(s^*t) = X(s)^*X(t)$ and $\Pi(s) \circ X(t) = X(st)$ for $s, t \in \mathcal{S}$. From this point of view it is natural to go a step further, namely to consider kernels instead of mappings. We also admit some other involutory algebraic structures, like convex $*$-semigroups, $*$-multiplicative cones and $*$-algebras. A predilatable kernel whose $*$-representation Π vanishes globally (resp. has a trivial null space) is called degenerate (resp. nondegenerate). It turns out that any predilatable kernel has a canonical decomposition into a degenerate and a nondegenerate part.

Here our goal is to represent a predilatable kernel as an integral with respect to a tight positive operator-valued measure defined on $\Sigma(\mathcal{S})$, the set of all characters of the underlying algebraic structure $\mathcal{S}$ ($\mathcal{S}$ is assumed to be commutative). We show that this is possible if and only if the kernel in question is nondegenerate. In case $\mathcal{S}$ has a unit, any predilatable (or equivalently dilatable) kernel has an integral representation with respect to a regular positive operator-valued measure defined on the σ-algebra of all Borel subsets of $\Sigma(\mathcal{S})$ (see [10], [22], [17], [5], [56], [6] and [39] for the case of scalar functions and [26], [25], [56] and [40] for the case of operator mappings). Otherwise, the representing (operator-valued) measure is defined on a δ-ring which is neither a σ-algebra nor a σ-ring. The latter is a consequence of the fact that, in general, predilatable scalar kernels (or functions) can be represented via Borel measures taking extended real values (see [18], [3], [28] and [13] for the case of $*$-algebras and [31], [44] and [32] for the case of $*$-semigroups). This is why we outline in the appendix the theory of integration with respect to a tight positive operator-valued measure defined on a δ-ring of Borel subsets of a given topological Hausdorff space.

Recently Ando and Choi [1] have extended the notion of complete positivity to the context of nonlinear operator-valued mappings. Basing on the classical Schoenberg theorem (cf. [38] and [33]), they have generalized the Stinespring dilation theorem [41] to the case of completely positive nonlinear mappings defined on C^*-algebras. In general, positive definite mappings need not be completely positive. However, this is the case for holomorphic mappings defined on commutative W^*-algebras. Some particular results of that sort have been established in [46], [11] and [50].

A substantial part of the present paper, concerning the question of decomposition and disintegration, has been announced without proofs in [46].

1. Preliminaries. In the sequel $\mathbb{K}$ stands either for the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Given two complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, we denote by $B(\mathcal{H},\mathcal{K})$ the linear space of all bounded linear mappings from $\mathcal{H}$ into $\mathcal{K}$. Set $B(\mathcal{H}) := B(\mathcal{H},\mathcal{H})$ and $I_{\mathcal{H}} :=$ the identity operator on $\mathcal{H}$. The space $B(\mathbb{C},\mathcal{K})$ will be identified with $\mathcal{K}$. Given $\mathcal{A} \subset B(\mathcal{H})$, denote by $\mathcal{N}_{\mathcal{A}}$ the null space of $\mathcal{A}$ (i.e. $\mathcal{N}_{\mathcal{A}} = \{f \in \mathcal{H} : Tf = 0 \text{ for all } T \in \mathcal{A}\}$) and by $W^*(\mathcal{A})$ the smallest strongly closed complex $*$-subalgebra of $B(\mathcal{H})$ containing $\mathcal{A}$. If $\{\mathcal{A}_\omega : \omega \in \Omega\}$ is a family of subsets of $\mathcal{H}$, then $\bigvee\{\mathcal{A}_\omega : \omega \in \Omega\}$ stands for the closed linear span of $\bigcup\{\mathcal{A}_\omega : \omega \in \Omega\}$.

A set $\mathcal{S}$ equipped with an associative composition $(\cdot)$ and a mapping $* : \mathcal{S} \to \mathcal{S}$ satisfying $(s^*)^* = s$ and $(st)^* = t^*s^*$ for $s, t \in \mathcal{S}$ is called a $*$-*semigroup* (if $\mathcal{S}$ has a neutral element e, then $e^* = e$). We say that a $*$-semigroup $\mathcal{S}$ is *convex* if it is a convex subset of some (real or complex) linear space such that $s(\alpha t + \beta u) = \alpha st + \beta su$, $(\alpha t + \beta u)s = \alpha ts + \beta us$ and $(\alpha s + \beta t)^* = \alpha s^* + \beta t^*$ for $s, t, u \in \mathcal{S}$, $\alpha, \beta \geq 0$, $\alpha + \beta = 1$. If, moreover, $\mathcal{S}$ is a convex cone satisfying the last-mentioned equalities for all $\alpha, \beta \geq 0$ and $s, t, u \in \mathcal{S}$, then $\mathcal{S}$ is called a $*$-*multiplicative cone*. Finally, $\mathcal{S}$ is said to be a $*$-*algebra* over $\mathbb{K}$ if $\mathcal{S}$ is a linear space over $\mathbb{K}$ such that $s(\alpha t + \beta u) = \alpha st + \beta su$, $(\alpha t + \beta u)s = \alpha ts + \beta us$ and $(\alpha s + \beta t)^* = \overline{\alpha} s^* + \overline{\beta} t^*$ for $s, t, u \in \mathcal{S}$ and $\alpha, \beta \in \mathbb{K}$.

Note. *Further on, $\mathcal{S}$ always stands for any of the algebraic structures defined in the previous paragraph.*

Denote by $\mathcal{S}^{(n)}$, $n \geq 1$, the set of all products $s_1 \ldots s_n$ with $s_1, \ldots, s_n \in \mathcal{S}$. If $\mathcal{S}$ is a convex $*$-semigroup (resp. a $*$-multiplicative cone; a $*$-algebra over $\mathbb{K}$), then $[\mathcal{S}^{(n)}]$ stands for the convex hull of $\mathcal{S}^{(n)}$ (resp. the set of all linear combinations with nonnegative coefficients of elements from $\mathcal{S}^{(n)}$; the linear span of $\mathcal{S}^{(n)}$).

We say that a mapping $X : \mathcal{S} \to B(\mathcal{H},\mathcal{K})$ defined on a convex $*$-semigroup (resp. a $*$-multiplicative cone; a $*$-algebra over $\mathbb{K}$) is *affine* if for all $s, t \in \mathcal{S}$, the equality $X(\alpha s + \beta t) = \alpha X(s) + \beta X(t)$ holds for every $\alpha, \beta \geq 0$, $\alpha + \beta = 1$ (resp. $\alpha, \beta \geq 0$; $\alpha, \beta \in \mathbb{K}$). It will be convenient to call any $B(\mathcal{H},\mathcal{K})$-valued mapping defined on a $*$-semigroup affine. We say that a mapping $\Pi : \mathcal{S} \to B(\mathcal{H})$ is *multiplicative* if $\Pi(st) = \Pi(s)\Pi(t)$ for $s, t \in \mathcal{S}$, and *symmetric* if $\Pi(s^*) = \Pi(s)^*$ for $s \in \mathcal{S}$. Π is said to be a $*$-*representation* of $\mathcal{S}$ in $\mathcal{H}$ if Π is symmetric, multiplicative and affine.

Assume $\mathcal{S}$ is commutative. A nonzero $*$-representation of $\mathcal{S}$ in $\mathbb{C}$ will be called a *character* of $\mathcal{S}$. Denote by $\Sigma(\mathcal{S})$ the collection [1] of all characters of $\mathcal{S}$. The set $\Sigma(\mathcal{S})\cup\{\mathbf{0}\}$ equipped with the topology of pointwise convergence on $\mathcal{S}$ is a completely regular Hausdorff space and the mapping $\widehat{s}:\Sigma(\mathcal{S})\cup\{\mathbf{0}\}\to\mathbb{C}$, $s\in\mathcal{S}$, defined by $\widehat{s}(x)=x(s)$ for $x\in\Sigma(\mathcal{S})\cup\{\mathbf{0}\}$, is continuous. Put $\widehat{\mathcal{S}}:=\{\widehat{s}:s\in\mathcal{S}\}$. A subset C of $\Sigma(\mathcal{S})$ is said to be $\widehat{\mathcal{S}}$-*bounded* if for every $s\in\mathcal{S}$, $\sup\{|\widehat{s}(x)|:x\in C\}<\infty$. By the Tikhonov theorem, a closed subset C of $\Sigma(\mathcal{S})$ is $\widehat{\mathcal{S}}$-bounded if and only if $C\cup\{\mathbf{0}\}$ is compact. Any closed $\widehat{\mathcal{S}}$-bounded subset of $\Sigma(\mathcal{S})$ is locally compact but not conversely. Denote by $M_+(\Sigma(\mathcal{S}))$ the convex cone of all positive Radon measures ν on $\Sigma(\mathcal{S})$ such that the closed support of ν is $\widehat{\mathcal{S}}$-bounded and $\widehat{\mathcal{S}}\subset L^2(\Sigma(\mathcal{S}),\nu)$.

If Π is a $*$-representation of $\mathcal{S}$ in $\mathcal{K}$, then there exists (cf. [44], Theorem 1) a unique regular spectral measure E in $\mathcal{K}$ defined on Borel subsets of $\Sigma(\mathcal{S})$ such that the closed support of E is $\widehat{\mathcal{S}}$-bounded and

$$\Pi(s)=\int\limits_{\Sigma(\mathcal{S})}\widehat{s}\,dE\,,\qquad s\in\mathcal{S}\,.$$

Call E the *spectral measure* of Π. Notice that $E(\Sigma(\mathcal{S}))\mathcal{K}=\mathcal{K}\ominus\mathcal{N}_{\Pi(\mathcal{S})}$, so $E(\Sigma(\mathcal{S}))=I_{\mathcal{K}}$ if and only if $\mathcal{N}_{\Pi(\mathcal{S})}=\{0\}$.

In case $\mathcal{S}$ is also a topological space, $\Sigma^{\mathrm{c}}(\mathcal{S})$ stands for the set of all continuous characters of $\mathcal{S}$. Notice that if $\mathcal{S}$ is a $*$-algebra which is a metrizable topological vector space then $\Sigma^{\mathrm{c}}(\mathcal{S})$ is a Borel subset of $\Sigma(\mathcal{S})$. To show this take a metric ϱ inducing the topology of $\mathcal{S}$ and set $(m,n\ge 1)$

$$C_{m,n}=\{x\in\Sigma(\mathcal{S})\cup\{\mathbf{0}\}:|x(s)|\le m^{-1}\text{ for every }s\in\mathcal{S}\text{ such that }\varrho(s,0)\le n^{-1}\}\,.$$

Then each $C_{m,n}$ is closed in $\Sigma(\mathcal{S})\cup\{\mathbf{0}\}$, and $\Sigma^{\mathrm{c}}(\mathcal{S})=\bigcap_{m=1}^{\infty}\bigcup_{n=1}^{\infty}\Sigma(\mathcal{S})\cap C_{m,n}$ is a Borel set in $\Sigma(\mathcal{S})$. We refer the reader to the appendix for further information concerning measurability and integrability.

2. Predilatable kernels. In this section we recall some basic concepts from dilation theory. Most of the facts presented below can be found either in [27] or in [24] (see also [49]).

Let Ω be a nonempty set. A kernel $\Phi:\Omega\times\Omega\to B(\mathcal{H})$ is said to be *positive definite* if

$$\sum_{k=1}^{n}\sum_{l=1}^{n}\langle\Phi(\omega_k,\omega_l)f_l,f_k\rangle\ge 0$$

for all finite sequences $\omega_1,\dots,\omega_n\in\Omega$ and $f_1,\dots,f_n\in\mathcal{H}$. It is well known

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[1] It may happen that $\Sigma(\mathcal{S})=\emptyset$ for some involutory algebraic structures (even in the case of Banach $*$-algebras).

(cf. [27], p. 18) that a positive definite kernel Φ is *hermitian symmetric*, i.e.

$$\Phi(\omega_1,\omega_2)^* = \Phi(\omega_2,\omega_1)\,, \quad \omega_1,\omega_2 \in \Omega\,, \tag{2.1}$$

and *positive*, i.e. $\Phi(\omega,\omega) \geq 0$ for every $\omega \in \Omega$. Given two positive definite kernels $\Phi, \Psi : \Omega \times \Omega \to B(\mathcal{H})$, we write $\Phi \ll \Psi$ in case $\Psi - \Phi$ is positive definite. The relation $\ll$ is a partial order in the class of $B(\mathcal{H})$-valued positive definite kernels on Ω.

It follows from an operator version of the Kolmogorov–Aronszajn factorization theorem (cf. [27], Proposition 5.1) that a kernel $\Phi : \Omega \times \Omega \to B(\mathcal{H})$ is positive definite if and only if there exists a complex Hilbert space $\mathcal{K}$ and a mapping $X : \Omega \to B(\mathcal{H},\mathcal{K})$ which factorizes Φ, i.e.

$$\Phi(\omega_1,\omega_2) = X(\omega_1)^* X(\omega_2)\,, \quad \omega_1,\omega_2 \in \Omega\,. \tag{2.2}$$

It is always possible to choose X in such a way that

$$\mathcal{K} = \overline{\mathcal{E}}_X \tag{2.3}$$

where $\mathcal{E}_X$ stands for the linear span of the set $\bigcup\{X(\omega)\mathcal{H} : \omega \in \Omega\}$. Call a pair $(\mathcal{K}, X)$ satisfying (2.2) and (2.3) a *minimal factorization* of Φ. It is well known (cf. [49], Theorem 1.1) that any two minimal factorizations $(\mathcal{K}, X)$ and $(\mathcal{L}, Y)$ of Φ are *unitarily equivalent*, i.e. there exists a (unique) unitary operator $U \in B(\mathcal{K},\mathcal{L})$ such that

$$UX(\omega) = Y(\omega)\,, \quad \omega \in \Omega\,. \tag{2.4}$$

We say that a kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is *bi-affine* if Φ has the *transfer property*, i.e.

$$\Phi(us,t) = \Phi(s,u^*t)\,, \quad u,s,t \in \mathcal{S}\,, \tag{2.5}$$

and each mapping $\Phi(s,\cdot)$, $s \in \mathcal{S}$, is affine. It turns out that minimal factorizations share some algebraic properties with positive definite bi-affine kernels. Namely, if $(\mathcal{K}, X)$ is a minimal factorization of a positive definite bi-affine kernel Φ on $\mathcal{S}$, then X is affine (use Proposition 6.2 of [27]).

If $(\mathcal{K}, X)$ is a minimal factorization of a positive definite kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ and $\Pi : \mathcal{S} \to B(\mathcal{K})$ is such that

$$\Pi(s)X(t)f = X(st)f\,, \quad s,t \in \mathcal{S}\,,\ f \in \mathcal{H}\,, \tag{2.6}$$

then the triplet $(\mathcal{K}, X, \Pi)$ is called a *minimal propagator* of Φ. Let $R \in B(\mathcal{H},\mathcal{K})$ and $\Pi : \mathcal{S} \to B(\mathcal{K})$ be given. We say that the triplet $(\mathcal{K}, R, \Pi)$ is a *minimal dilation* of Φ if $(\mathcal{K}, X, \Pi)$ is a minimal propagator of Φ with $X(s) = \Pi(s)R$, $s \in \mathcal{S}$. A positive definite bi-affine kernel $\Phi : \mathcal{S}\times\mathcal{S} \to B(\mathcal{H})$ is said to be *predilatable* (resp. *dilatable*) if it has a minimal propagator (resp. a minimal dilation). Notice that if $(\mathcal{K}, X, \Pi)$ is a minimal propagator of a predilatable kernel Φ, then Π has to be a ∗-representation of $\mathcal{S}$. Indeed, it follows from (2.6) and (2.3) that Π is multiplicative. Since Φ is a positive definite bi-affine kernel, X is affine. This in turn implies that so is Π. The symmetry of Π follows from the transfer property (2.5) (cf. [49], Lemma 3.1).

The following lemma describes the null spaces of minimal propagators.

LEMMA 2.1. *Let $(\mathcal{K}, X, \Pi)$ be a minimal propagator of a predilatable kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ and let $m \geq 1$. Then*

$$\mathcal{N}_{\Pi(\mathcal{S}^{(m)})} = \mathcal{K} \ominus Q_{m+1}\mathcal{K} = \mathcal{N}_{Q_m \Pi(\mathcal{S})}\,,$$

where Q_j is the orthogonal projection of $\mathcal{K}$ onto $\bigvee\{X(s)\mathcal{H} : s \in [\mathcal{S}^{(j)}]\}$.

Proof. Since X is affine, $g \in \mathcal{K} \ominus Q_{m+1}\mathcal{K}$ if and only if

$$\langle \Pi(s_m^* \ldots s_1^*)g, X(s_{m+1})f\rangle = \langle g, X(s_1 \ldots s_m s_{m+1})f\rangle = 0, \quad s_j \in \mathcal{S}\,,\ f \in \mathcal{H},$$

or, equivalently, if and only if

$$\begin{aligned}\langle Q_m \Pi(s_1^*)g, X(s_2 \ldots s_{m+1})f\rangle &= \langle \Pi(s_1^*)g, X(s_2 \ldots s_{m+1})f\rangle \\ &= \langle g, X(s_1 \ldots s_m s_{m+1})f\rangle = 0\,, \quad s_j \in \mathcal{S}\,,\ f \in \mathcal{H}\,.\end{aligned}$$

Thus $g \in \mathcal{K} \ominus Q_{m+1}\mathcal{K}$ if and only if $\Pi(s)g = 0$ for every $s \in \mathcal{S}^{(m)}$ or, equivalently, if and only if $Q_m\Pi(s)g = 0$ for every $s \in \mathcal{S}$. ∎

Given $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ and $f \in \mathcal{H}$, define $\Phi_f : \mathcal{S} \times \mathcal{S} \to \mathbb{C}$ by

$$\Phi_f(s,t) = \langle \Phi(s,t)f, f\rangle\,, \qquad s,t \in \mathcal{S}\,.$$

Assume that Φ is predilatable and $(\mathcal{K}, X, \Pi)$ is a minimal propagator of Φ. Then for every $f \in \mathcal{H}$, the space $\mathcal{K}_f := \bigvee\{X(s)f : s \in \mathcal{S}\}$ reduces Π to a $*$-representation Π_f of $\mathcal{S}$ in $\mathcal{K}_f$. Define a mapping $X_f : \mathcal{S} \to \mathcal{K}_f$ by $X_f(s) = X(s)f$, $s \in \mathcal{S}$, $f \in \mathcal{H}$. It is easy to see that for every $f \in \mathcal{H}$, $(\mathcal{K}_f, X_f, \Pi_f)$ is a minimal propagator of Φ_f. Call it the restriction of $(\mathcal{K}, X, \Pi)$ to $\mathcal{K}_f$.

Assume now that a kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is dilatable and $(\mathcal{K}, R, \Pi)$ is a minimal dilation of Φ. Define $X : \mathcal{S} \to B(\mathcal{H}, \mathcal{K})$ by $X(s) = \Pi(s)R$, $s \in \mathcal{S}$. We show that $(\mathcal{K}_f, Rf, \Pi_f)$ is a minimal dilation of Φ_f for every $f \in \mathcal{H}$. Indeed, since the orthogonal projection P_f of $\mathcal{K}$ onto $\mathcal{K}_f$ commutes with Π, we have $\Pi(s)(I_{\mathcal{K}} - P_f)Rf = 0$ for every $s \in \mathcal{S}$. This implies that $(I_{\mathcal{K}} - P_f)Rf \in \mathcal{N}_{\Pi(\mathcal{S})}$. Since $Q_2\mathcal{K} = \mathcal{K}$ (cf. [49], Theorem 3.5(iii)), Lemma 2.1 leads to $\mathcal{N}_{\Pi(\mathcal{S})} = \{0\}$. Thus $(I_{\mathcal{K}} - P_f)Rf = 0$ and consequently $Rf = P_fRf$. Now it is easy to check that $(\mathcal{K}_f, Rf, \Pi_f)$ is a minimal dilation of Φ_f. Call it the restriction of $(\mathcal{K}, R, \Pi)$ to $\mathcal{K}_f$.

We say that a $B(\mathcal{H})$-valued mapping Θ defined either on $[\mathcal{S}^{(2)}]$ or on $\mathcal{S}$ is *positive* (resp. *positive definite*, *predilatable*) if so is the kernel $\Phi^\Theta : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ given by

$$\Phi^\Theta(s,t) := \Theta(s^*t)\,, \qquad s,t \in \mathcal{S}\,.$$

A mapping $\Theta : \mathcal{S} \to B(\mathcal{H})$ is said to be *dilatable* if there are a complex Hilbert space $\mathcal{K}$, an operator $R \in B(\mathcal{H}, \mathcal{K})$ and a $*$-representation Π of $\mathcal{S}$ in $\mathcal{K}$ such that $\Theta(s) = R^*\Pi(s)R$ for $s \in \mathcal{S}$.

Given $t \in \mathcal{S}$ and $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ (resp. $\Theta : \mathcal{S} \to B(\mathcal{H})$), we define a kernel ${}_t\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ (resp. a mapping ${}_t\Theta : \mathcal{S} \to B(\mathcal{H})$) by

$${}_t\Phi(u,v) := \Phi(tu,tv)\,, \quad u,v \in \mathcal{S} \quad (\text{resp. } {}_t\Theta(s) := \Theta(t^*st),\ s \in \mathcal{S})\,.$$

3. Criteria of predilatability. We begin with a result which reformulates and improves some criteria of predilatability for positive definite bi-affine kernels (see [52]–[55] and [24]). All of them can be regarded as equivalent forms of the boundedness condition introduced by Sz.-Nagy in [58].

Given a positive definite bi-affine kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$, we define functions $\varkappa^\Phi$, $\varkappa_\Phi : \mathcal{S} \to \overline{\mathbb{R}}_+$ by (2)

$$\varkappa^\Phi(s) := \sup\{\lim_{n\to\infty} \langle \Phi((s^*s)^n t, (s^*s)^n t) f, f\rangle^{1/4n} \ :\ t \in \mathcal{S}, f \in \mathcal{H}\}\,, \quad s \in \mathcal{S}\,,$$

$$\varkappa_\Phi(s) := \sup\{\lim_{n\to\infty} \langle \Phi((s^*s)^n, (s^*s)^n) f, f\rangle^{1/4n} \ :\ f \in \mathcal{H}\}\,, \quad s \in \mathcal{S}\,.$$

THEOREM 3.1. *Let $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ be a positive definite bi-affine kernel. Then the following conditions are equivalent*:

(i) *Φ is predilatable*,

(ii) *$\varkappa^\Phi(s) < \infty$, $s \in \mathcal{S}$*,

(iii) *there exist $\varrho : \mathcal{S} \to \mathbb{R}_+$ and $\gamma : \mathcal{S} \times \mathcal{H} \to \mathbb{R}_+$ such that*

$$\varrho(s^2) \le \varrho(s)^2\,, \quad s \in \mathcal{S}\,,$$
$$\langle \Phi(st,st)f,f\rangle \le \varrho(s)\gamma(t,f)\,, \quad s,t \in \mathcal{S}\,,\ f \in \mathcal{H}\,,$$

(iv) *for every $f \in \mathcal{H}$, the scalar kernel Φ_f is predilatable.*

If $\mathcal{S}$ is commutative, then (i) *is equivalent to either of the following two conditions*:

(v) *$\varkappa_\Phi(s) < \infty$, $s \in \mathcal{S}$*,

(vi) *for every $u \in \mathcal{S}$, the kernel ${}_u\Phi$ is predilatable.*

If $(\mathcal{K}, X, \Pi)$ is a minimal propagator of Φ, then $\|\Pi(\cdot)\| = \varkappa^\Phi(\cdot)$, and $\|\Pi(\cdot)\| = \varkappa_\Phi(\cdot)$ in case $\mathcal{S}$ is commutative.

Proof. It follows from Theorem 1 of [43] that (i) and (ii) are equivalent and $\|\Pi(\cdot)\| = \varkappa^\Phi(\cdot)$. If Φ is predilatable, then (iii) holds with $\varrho(s) = \|\Pi(s)\|^2$ and $\gamma(t,f) = \langle \Phi(t,t)f,f\rangle$. Conversely, if Φ satisfies (iii), then using the identity

$$\lim_{n\to\infty} \langle \Phi((s^*s)^n t, (s^*s)^n t) f, f\rangle^{1/4n} = \lim_{n\to\infty} \langle \Phi((s^*s)^{2^n} t, (s^*s)^{2^n} t) f, f\rangle^{2^{-(n+2)}}$$

(2) Notice that the limits appearing in the definitions of $\varkappa^\Phi$ and $\varkappa_\Phi$ always exist in $\overline{\mathbb{R}}_+$ (cf. [51]).

one can show that Φ satisfies (ii). The equivalence (i)$\Leftrightarrow$(iv) can be proved essentially in the same way as Theorem 1 of [45] (the sequence $a_n = f(t^*(s^*s)^n t; x, x)$ from [45], p. 252, has to be replaced by $a_n = \langle \Phi((s^*s)^n t, (s^*s)^n t) f, f\rangle$).

Assume now that $\mathcal{S}$ is commutative. Then, repeating the arguments used in the proofs of Remark 2 of [55] and Lemma 1 of [47], one can show that (i) and (v) are equivalent and $\|\Pi(\cdot)\| = \varkappa_\Phi(\cdot)$. Suppose that Φ is predilatable. We show that ${}_u\Phi$ is dilatable for all $u \in \mathcal{S}$. Take a minimal propagator $(\mathcal{K}, X, \Pi)$ of Φ. Then for all $g, f_1, \ldots, f_n \in \mathcal{H}$ and $s_1, \ldots, s_n, t_1, \ldots, t_n \in \mathcal{S}$ we have

$$\Big|\sum_{k=1}^n \langle {}_u\Phi(s_k, t_k) f_k, g\rangle\Big|^2 = \Big|\sum_{k=1}^n \langle \Phi(u, us_k^* t_k) f_k, g\rangle\Big|^2$$
$$= \Big|\Big\langle \sum_{k=1}^n X(us_k^*t_k) f_k, X(u)g\Big\rangle\Big|^2 \le \|X(u)g\|^2 \Big\|\sum_{k=1}^n X(us_k^*t_k) f_k\Big\|^2$$
$$= \langle \Phi(u,u)g, g\rangle \sum_{k=1}^n \sum_{j=1}^n \langle {}_u\Phi(s_k^* t_k, s_j^* t_j) f_j, f_k\rangle\,.$$

Since ${}_u\Phi$ is a positive definite bi-affine kernel which satisfies (iii) with $\varrho(s) = \|\Pi(s)\|^2$ and $\gamma(t, f) = \langle {}_u\Phi(t,t) f, f\rangle$, we deduce from Theorem 3.5 of [49] that ${}_u\Phi$ is dilatable. Suppose now that ${}_u\Phi$ is predilatable for every $u \in \mathcal{S}$. Then

$$\lim_{n\to\infty} \langle \Phi((s^*s)^n, (s^*s)^n) f, f\rangle^{1/4n}$$
$$= \lim_{n\to\infty} (\langle {}_u\Phi((s^*s)^{n-1}, (s^*s)^{n-1}) f, f\rangle^{1/4(n-1)})^{(n-1)/n}$$
$$\le \varkappa_{{}_u\Phi}(s) < \infty\,, \qquad s \in \mathcal{S}\,,\ f \in \mathcal{H}\,,$$

with $u = s^*s$. Applying (v) we conclude that Φ is predilatable. ■

Notice that in some particular cases of $*$-semigroups the condition (ii) of Theorem 3.1 is either needless or follows from the positive definiteness of the bi-affine kernel in question. This occurs when $\mathcal{S}$ is an inverse semigroup with involution determined by the equality $ss^*s = s$, a group with involution $s^* = s^{-1}$ or a complex Banach $*$-algebra (with involution which is not assumed to be continuous).

PROPOSITION 3.2. *Let $\mathcal{S}$ be a complex Banach $*$-algebra. Then any positive definite bi-affine kernel on $\mathcal{S}$ is predilatable.*

Proof. Take a minimal factorization $(\mathcal{K}, X)$ of Φ. Denote by $\mathcal{O}^\#(\mathcal{E}_X)$ the $*$-algebra of all linear operators $L : \mathcal{E}_X \to \mathcal{E}_X$ such that $L^*(\mathcal{E}_X) \subset \mathcal{E}_X$ with involution $L^\# := L^*|_{\mathcal{E}_X}$. It follows from Theorem 3.11 of [49] that there exists a $*$-algebra-homomorphism $\Pi : \mathcal{S} \to \mathcal{O}^\#(\mathcal{E}_X)$ which satisfies (2.6).

Let $\mathcal{S}_1 = \mathcal{S} \times \mathbb{C}$ be the unitization of $\mathcal{S}$ and let $\Pi_1(s,\alpha) := \Pi(s) + \alpha I_{\mathcal{K}}$ for $s \in \mathcal{S}$ and $\alpha \in \mathbb{C}$. Then for any $g \in \mathcal{E}_X$, $\langle \Pi_1(\cdot)g, g\rangle$ is a positive linear functional on $\mathcal{S}_1$. Thus, by Lemma 37.6(iii) of [9],

$$|\langle \Pi_1(s)g, g\rangle| \le \langle \Pi_1(0,1)g, g\rangle \|s\| = \|g\|^2\|s\|, \quad g \in \mathcal{E}_X, \ s \in \mathcal{S}_1, \ s = s^*,$$

which implies boundedness of any $\Pi(s)$, $s \in \mathcal{S}$. Therefore $(\mathcal{K}, X, \overline{\Pi})$ is a minimal propagator of $(\mathcal{K}, X)$. ∎

The next result shows that continuous operator-valued positive definite linear mappings on some topological $*$-algebras always have continuous propagators (see [20] for all definitions concerning topological algebras we need in this paper). If it is not specified otherwise, the continuity of operator-valued mappings is understood with respect to the uniform operator topology.

PROPOSITION 3.3. *Let $\mathcal{S}$ be a locally multiplicatively-convex $*$-algebra with continuous involution. If $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is a jointly continuous positive definite bi-affine kernel, then Φ is predilatable and for any minimal propagator $(\mathcal{K}, X, \Pi)$ of Φ, Π is continuous. This is the case when $\Phi = \Phi^{\Theta}$ with some continuous positive definite linear mapping $\Theta : \mathcal{S} \to B(\mathcal{H})$.*

Proof. Notice first that

$$\|\Phi(s,s)\|^{1/2} = \sup\{|\langle \Phi(s,s)f, f\rangle|^{1/2} : \|f\| = 1\}, \quad s \in \mathcal{S},$$

so the function $\mathcal{S} \ni s \to \|\Phi(s,s)\|^{1/2} \in \mathbb{R}_+$ is a seminorm on $\mathcal{S}$ which, by the assumptions, is continuous. Thus there exists a continuous submultiplicative seminorm ϱ on $\mathcal{S}$ such that $\|\Phi(s,s)\| \le \varrho(s)^2$, $s \in \mathcal{S}$. Applying Theorem 3.1(iii) we see that Φ is predilatable. Take a minimal propagator $(\mathcal{K}, X, \Pi)$ of Φ. It follows from Theorem 3.1 that

$$\begin{aligned}\|\Pi(s)\| &= \sup\{\lim_{n\to\infty} \langle \Phi((s^*s)^n t, (s^*s)^n t)f, f\rangle^{1/4n} \ : \ f \in \mathcal{H}, t \in \mathcal{S}\} \\ &\le \sup\{\lim_{n\to\infty} (\varrho(t)\|f\|)^{1/2n} \varrho(s^*s)^{1/2} \ : \ f \in \mathcal{H}, t \in \mathcal{S}\} \le \varrho(s^*s)^{1/2}, \quad s \in \mathcal{S}.\end{aligned}$$

Since the involution "$*$" and the seminorm ϱ are continuous on $\mathcal{S}$, Π is also continuous. ∎

4. Degenerate and nondegenerate predilatable kernels. In general, nonzero predilatable kernels on $\mathcal{S}$ without neutral element may have zero minimal propagators (this is not the case for $\mathcal{S}$ having a neutral element). From this point of view it is natural to distinguish the class of predilatable kernels having this pathological property.

Let $(\mathcal{K}, X, \Pi)$ be a minimal propagator of a predilatable kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$. We say that Φ is *degenerate* (resp. *nondegenerate*) if $\mathcal{N}_{\Pi(\mathcal{S})} = \mathcal{K}$ (resp. $\mathcal{N}_{\Pi(\mathcal{S})} = \{0\}$). The definition does not depend on the choice of $(\mathcal{K}, X, \Pi)$. It is an easy observation that each dilatable kernel is nondegenerate (cf. [49], Theorem 3.5(iii)) and that the only predilatable kernel which is both degenerate and nondegenerate is zero. The class of all degenerate (resp. nondegenerate) kernels on $\mathcal{S}$ forms a convex cone.

PROPOSITION 4.1. *If* $\Phi_1, \Phi_2 : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ *are degenerate* (*resp. nondegenerate*) *predilatable kernels and* $\alpha, \beta \geq 0$, *then* $\Phi := \alpha\Phi_1 + \beta\Phi_2$ *is a degenerate* (*resp. nondegenerate*) *predilatable kernel.*

Proof. Let $(\mathcal{K}_j, X_j, \Pi_j)$ be a minimal propagator of Φ_j. Set $\mathcal{K} := \bigvee\{(\sqrt{\alpha}X_1(s)f) \oplus (\sqrt{\beta}X_2(s)f) : s \in \mathcal{S},\ f \in \mathcal{H}\}$ and define $X : \mathcal{S} \to B(\mathcal{H}, \mathcal{K})$ by

$$X(s)f := (\sqrt{\alpha}X_1(s)f) \oplus (\sqrt{\beta}X_2(s)f)\,, \qquad s \in \mathcal{S}\,,\ f \in \mathcal{H}\,.$$

Then $(\mathcal{K}, X)$ is a minimal factorization of Φ such that

$$\begin{aligned}&\Pi_1 \oplus \Pi_2(s)X(t)f \\ &\quad = (\sqrt{\alpha}X_1(st)f) \oplus (\sqrt{\beta}X_2(st)f) = X(st)f\,, \qquad s,t \in \mathcal{S}\,,\ f \in \mathcal{H}\,.\end{aligned}$$

Thus the space $\mathcal{K}$ reduces $\Pi_1 \oplus \Pi_2$ to a $*$-representation Π and $(\mathcal{K}, X, \Pi)$ is a minimal propagator of Φ. If Φ_1 and Φ_2 are degenerate, then $\Pi_1 \oplus \Pi_2 = 0$, which implies that Φ is degenerate. If Φ_1 and Φ_2 are nondegenerate, then $\mathcal{N}_{\Pi(\mathcal{S})} \subset \mathcal{N}_{\Pi_1 \oplus \Pi_2(\mathcal{S})} = \mathcal{N}_{\Pi_1(\mathcal{S})} \oplus \mathcal{N}_{\Pi_2(\mathcal{S})} = \{0\}$, so Φ is nondegenerate. ∎

Our goal here is to find characterizations of degenerate and nondegenerate predilatable kernels which are not formulated in terms of minimal propagator. Consider first the case of degenerate kernels.

THEOREM 4.2. *Let* $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ *be a positive definite kernel such that* $\Phi(s, \cdot)$ *is affine for every* $s \in \mathcal{S}$. *Then* Φ *is predilatable and degenerate if and only if* $\Phi(s, tu) = 0$ *for all* $s, t, u \in \mathcal{S}$.

Proof. Assume that $\Phi(s, tu) = 0$ for all $s, t, u \in \mathcal{S}$. Since Φ is positive definite, it is hermitian symmetric. This implies that $\Phi(tu, s) = 0$ for all $s, t, u \in \mathcal{S}$. Thus Φ has the transfer property (2.5) and consequently Φ is a bi-affine kernel. It follows from Theorem 3.1 that Φ is predilatable. Take a minimal factorization $(\mathcal{K}, X)$ of Φ. Then, by Lemma 2.1 with $m = 1$, Φ is degenerate if and only if $X(u) = 0$ for $u \in \mathcal{S}^{(2)}$. This in turn is equivalent to $\Phi(s, tu) = 0$ for all $s, t, u \in \mathcal{S}$ (use (2.2) and (2.3)). ∎

For nondegenerate kernels, the following lemma turns out to be very useful.

LEMMA 4.3. *Let* $(\mathcal{K}, X, \Pi)$ *be a minimal propagator of a predilatable kernel* $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$. *Then* Φ *is nondegenerate if and only if one of the following two conditions holds*:

(i) $\mathcal{K} = \bigvee\{X(s)\mathcal{H} : s \in [\mathcal{S}^{(2)}]\}$,
(ii) $I_{\mathcal{K}} \in W^*(\Pi(\mathcal{S}))$.

If Φ is nondegenerate, then for any $n \geq 2$, we have

(iii) $\mathcal{K} = \bigvee\{X(s)\mathcal{H} : s \in [\mathcal{S}^{(n)}]\}$,

(iv) $I_{\mathcal{K}} \in W^*(\Pi(\mathcal{S}^{(n)}))$.

Proof. Applying Lemma 2.1 with $m = 1$ we see that Φ is nondegenerate if and only if (i) holds.

Denote by J_m the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K} \ominus \mathcal{N}_{\Pi(\mathcal{S}^{(m)})}$. Then, by Lemma 2.1, we have $J_m = Q_{m+1}$. It follows from the von Neumann double commutant theorem (cf. [59], Proposition 1) that

$$J_m \in W^*(\Pi(\mathcal{S}^{(m)}))\,, \qquad m \geq 1\,. \tag{4.1}$$

If (i) holds, then, by (4.1), we have $I_{\mathcal{K}} = Q_2 = J_1 \in W^*(\Pi(\mathcal{S}))$. Thus (ii) is fulfilled. Conversely, if (ii) holds, then $\mathcal{N}_{\Pi(\mathcal{S})} = \{0\}$. Therefore $Q_2 = J_1 = I_{\mathcal{K}}$, which implies (i).

Assume now that Φ is nondegenerate. We prove (iii) by induction. The case $n = 2$ follows from Lemma 2.1. Suppose that (iii) holds for some $n \geq 2$. This means that $Q_n = I_{\mathcal{K}}$. By Lemma 2.1, we get $\mathcal{K} \ominus Q_{n+1}\mathcal{K} = \mathcal{N}_{Q_n\Pi(\mathcal{S})} = \mathcal{N}_{\Pi(\mathcal{S})} = \{0\}$, which proves (iii) for $n + 1$.

To prove (iv), notice that Lemma 2.1 and (iii) imply $\mathcal{N}_{\Pi(\mathcal{S}^{(n)})} = \mathcal{K} \ominus Q_{n+1}\mathcal{K} = \{0\}$. Thus, by (4.1), $I_{\mathcal{K}} = J_n \in W^*(\Pi(\mathcal{S}^{(n)}))$. ∎

The following is a consequence of Lemma 4.3: if $\mathcal{S}$ is a topological space such that the closure of $[\mathcal{S}^{(2)}]$ is equal to $\mathcal{S}$ and $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is a predilatable kernel which is jointly weakly continuous, then Φ is nondegenerate. Indeed, any minimal factorization $(\mathcal{K}, X)$ of Φ is then continuous in the strong operator topology and consequently $\bigvee\{X(s)\mathcal{H} : s \in \mathcal{S}\} = \bigvee\{X(s)\mathcal{H} : s \in [\mathcal{S}^{(2)}]\}$, which implies the condition (i) of Lemma 4.3.

If $\mathcal{S} = [\mathcal{S}^{(2)}]$, then the condition (i) of Lemma 4.3 is satisfied and consequently each predilatable kernel on $\mathcal{S}$ is automatically nondegenerate. This occurs when $\mathcal{S}$ is a complex Banach $*$-algebra with a bounded left approximate identity (use the Cohen factorization theorem, cf. [9], Theorem 11.10). In particular, each C^*-algebra $\mathcal{S}$ factors, i.e. $\mathcal{S} = \mathcal{S}^{(2)}$. The following is a consequence of Proposition 3.2 and Lemma 4.3.

COROLLARY 4.4. *If $\mathcal{S}$ is a complex Banach $*$-algebra such that $\mathcal{S} = [\mathcal{S}^{(2)}]$, then each positive definite bi-affine kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is predilatable and nondegenerate.*

Notice that there exist commutative complex Banach or Fréchet $*$-algebras which factor and do not have bounded approximate identities (cf. [15] and [29]). On the other hand, Ouzomgi [30] has determined a class of commutative convolution Banach $*$-algebras $\mathcal{S}$ having the property: $\mathcal{S} = \mathcal{S}^{(2)} \Leftrightarrow \mathcal{S} = [\mathcal{S}^{(2)}] \Leftrightarrow \mathcal{S}$ has a bounded approximate identity.

We are now in a position to prove the aforesaid characterization of nondegenerate kernels. Below by an ending of a net $\{x_\omega : \omega \in \Omega\}$ we mean a set of the form $\{x_\omega : \omega \geq \omega_0\}$ with some $\omega_0 \in \Omega$.

THEOREM 4.5. *A predilatable kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is nondegenerate if and only if for every integer $k \geq 1$, there are nets $\{e_{k,\omega} : \omega \in \Omega\} \subset \mathcal{S}$ and $\{\beta_{k,\omega} : \omega \in \Omega\} \subset \mathbb{C}$ such that*

(i) *$\beta_{k,\omega} = 0$ for sufficiently large k (depending on ω),*

(ii) *$\lim_\omega \sum_k \beta_{k,\omega} \langle \Phi(s, e_{k,\omega} t) f, f\rangle = \langle \Phi(s,t)f,f\rangle$ for all $s,t \in \mathcal{S}$ and $f \in \mathcal{H}$,*

(iii) *any net of the form $\{\sum_{m,n} \overline{\beta}_{m,\omega}\beta_{n,\omega}\langle \Phi(e_{m,\omega}t, e_{n,\omega}t)f,f\rangle : \omega \in \Omega\}$ with $t \in \mathcal{S}$ and $f \in \mathcal{H}$ has a bounded ending.*

Proof. Assume Φ is nondegenerate. Then Lemma 4.3 yields $I_{\mathcal{K}} \in W^*(\Pi(\mathcal{S}))$, which implies that there are nets $\{e_{k,\omega} : \omega \in \Omega\} \subset \mathcal{S}$ and $\{\beta_{k,\omega} : \omega \in \Omega\} \subset \mathbb{C}$ $(k = 1, 2, \ldots)$ such that (i) holds and $T_\omega := \sum_k \beta_{k,\omega} \Pi(e_{k,\omega})$ converges in the strong operator topology to $I_{\mathcal{K}}$. Thus $\sum_k \beta_{k,\omega} X(e_{k,\omega}t) f = T_\omega X(t) f$ converges to $X(t)f$. This, when combined with (2.2), implies (ii) and (iii).

Assume now that nets $\{e_{k,\omega} : \omega \in \Omega\} \subset \mathcal{S}$ and $\{\beta_{k,\omega} : \omega \in \Omega\} \subset \mathbb{C}$ satisfy (i)–(iii). Fixing $t \in \mathcal{S}$ and $f \in \mathcal{H}$, we set $g_\omega = \sum_k \beta_{k,\omega} X(e_{k,\omega}t) f$. It follows from (ii) and (iii) that $\lim_\omega \langle g_\omega, h\rangle = \langle X(t)f, h\rangle$ for $h \in \mathcal{E}_X$ and $\sup\{\|g_\omega\| : \omega \geq \omega_0\} < \infty$ for some $\omega_0 \in \Omega$. This and $\mathcal{K} = \overline{\mathcal{E}}_X$ imply that the net $\{g_\omega\} \subset \bigvee\{X(s)\mathcal{H} : s \in [\mathcal{S}^{(2)}]\}$ converges weakly to $X(t)f$. Thus, by Theorem 3.12 of [34], we have $X(t)f \in \bigvee\{X(s)\mathcal{H} : s \in [\mathcal{S}^{(2)}]\}$ for all $t \in \mathcal{S}$ and $f \in \mathcal{H}$. In virtue of (2.3), the condition (i) of Lemma 4.3 holds. ∎

Theorem 4.5 asserts, in particular, that if $\mathcal{S}$ is a complex $*$-algebra, then a predilatable kernel Φ on $\mathcal{S}$ is nondegenerate if and only if there exists a net $\{e_\omega\} \subseteq \mathcal{S}$ such that

$$\lim_\omega \langle \Phi(s, e_\omega t) f, f\rangle = \langle \Phi(s,t)f,f\rangle\,, \quad s,t \in \mathcal{S}\,,\ f \in \mathcal{H}\,, \tag{4.2}$$

$$\sup\{\langle \Phi(e_\omega t, e_\omega t) f, f\rangle : \omega \geq \omega_0\} < \infty\,, \quad t \in \mathcal{S}\,,\ f \in \mathcal{H}\,, \tag{4.3}$$

with ω_0 depending on $t \in \mathcal{S}$ and $f \in \mathcal{H}$.

5. Canonical decomposition of predilatable kernels. Repeating the arguments used in the proof of Theorem 2 in [44], we get the following decomposition theorem.

THEOREM 5.1. *Let $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ be a predilatable kernel. Then there exists a unique pair $(\Phi_{\mathrm{D}}, \Phi_{\mathrm{N}})$ of predilatable kernels on $\mathcal{S}$ such that $\Phi = \Phi_{\mathrm{D}} + \Phi_{\mathrm{N}}$, Φ_{D} is degenerate and Φ_{N} is nondegenerate.*

The kernels Φ_{D} and Φ_{N} will be called the *degenerate* and *nondegenerate parts* of Φ, respectively. They have the following properties.

LEMMA 5.2. *If $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1, \Phi_2 : \mathcal{S}\times\mathcal{S} \to B(\mathcal{H})$ are predilatable kernels, then*

(i) *$\Phi_1 \ll \Phi_{\mathrm{D}}$ and $\Phi_{\mathrm{N}} \ll \Phi_2$, provided Φ_1 is degenerate,*
(ii) *$\Phi_1 \ll \Phi_{\mathrm{N}}$ and $\Phi_{\mathrm{D}} \ll \Phi_2$, provided Φ_1 is nondegenerate.*

Proof. Assume that Φ_1 is degenerate (resp. nondegenerate). Then, by Proposition 4.1, $\Phi_1 + (\Phi_2)_{\mathrm{D}}$ (resp. $\Phi_1 + (\Phi_2)_{\mathrm{N}}$) is degenerate (resp. nondegenerate), so we can apply Theorem 5.1 to $\Phi = \Phi_1 + (\Phi_2)_{\mathrm{D}} + (\Phi_2)_{\mathrm{N}}$. In consequence, $\Phi_1 \ll \Phi_1 + (\Phi_2)_{\mathrm{D}} = \Phi_{\mathrm{D}}$ and $\Phi_{\mathrm{N}} = (\Phi_2)_{\mathrm{N}} \ll \Phi_2$ (resp. $\Phi_1 \ll \Phi_1 + (\Phi_2)_{\mathrm{N}} = \Phi_{\mathrm{N}}$ and $\Phi_{\mathrm{D}} = (\Phi_2)_{\mathrm{D}} \ll \Phi_2$). ∎

Now we show that Φ_{D} and Φ_{N} are the greatest elements of suitable classes of predilatable kernels.

PROPOSITION 5.3. *Let $\Phi : \mathcal{S}\times\mathcal{S} \to B(\mathcal{H})$ be a predilatable kernel. Then $\Phi_{\mathrm{D}} = \max\{\Psi : \Psi \ll \Phi,\ \Psi$ is a degenerate predilatable kernel$\}$ and $\Phi_{\mathrm{N}} = \max\{\Psi : \Psi \ll \Phi,\ \Psi$ is a nondegenerate predilatable kernel$\}$.*

Proof. Take a predilatable kernel $\Psi \ll \Phi$. Since $\Phi - \Psi$ is a positive definite bi-affine kernel on $\mathcal{S}$ which satisfies the condition (ii) of Theorem 3.1, it is predilatable. Thus Φ is the sum of two predilatable kernels Ψ and $\Phi - \Psi$, so we can apply Lemma 5.2. If Ψ is degenerate, then, by Lemma 5.2(i), we have $\Phi_{\mathrm{N}} \ll \Phi - \Psi = \Phi_{\mathrm{D}} + \Phi_{\mathrm{N}} - \Psi$. This implies that $\Psi \ll \Phi_{\mathrm{D}}$. Similarly we show that if Ψ is nondegenerate, then $\Psi \ll \Phi_{\mathrm{N}}$. ∎

We end this section with a result which relates the decomposition of a predilatable kernel Φ to that of Φ_f, $f \in \mathcal{H}$.

PROPOSITION 5.4. *Let $\Phi : \mathcal{S}\times\mathcal{S} \to B(\mathcal{H})$ be a predilatable kernel. Then Φ is degenerate (resp. nondegenerate) if and only if so is Φ_f for every $f \in \mathcal{H}$. Moreover, $(\Phi_f)_{\mathrm{D}} = (\Phi_{\mathrm{D}})_f$ and $(\Phi_f)_{\mathrm{N}} = (\Phi_{\mathrm{N}})_f$ for every $f \in \mathcal{H}$.*

Proof. It follows from Theorem 4.2 and the polarization formula for sesquilinear forms that Φ is degenerate if and only if so is Φ_f for every $f \in \mathcal{H}$.

Take a minimal propagator $(\mathcal{K}, X, \Pi)$ of Φ. Let $(\mathcal{K}_f, X_f, \Pi_f)$ be the restriction of $(\mathcal{K}, X, \Pi)$ to $\mathcal{K}_f$, $f \in \mathcal{H}$ (see Section 2). If Φ is nondegenerate, then $\mathcal{N}_{\Pi_f(\mathcal{S})} = \mathcal{N}_{\Pi(\mathcal{S})} \cap \mathcal{K}_f = \{0\}$ for every $f \in \mathcal{H}$, which means that all Φ_f are nondegenerate. Conversely, if all Φ_f are nondegenerate, then, by Lemma 4.3,

$$X(t)f \in \bigvee\{X_f(s) : s \in [\mathcal{S}^{(2)}]\} \subset \bigvee\{X(s)\mathcal{H} : s \in [\mathcal{S}^{(2)}]\}, \quad t \in \mathcal{S},\ f \in \mathcal{H}.$$

Therefore, again by Lemma 4.3, Φ is nondegenerate.

It follows from the previous two paragraphs that $(\Phi_{\mathrm{D}})_f$ is degenerate and $(\Phi_{\mathrm{N}})_f$ is nondegenerate. Since $\Phi_f = (\Phi_{\mathrm{D}})_f + (\Phi_{\mathrm{N}})_f$, the uniqueness of the decomposition implies that $(\Phi_f)_{\mathrm{D}} = (\Phi_{\mathrm{D}})_f$ and $(\Phi_f)_{\mathrm{N}} = (\Phi_{\mathrm{N}})_f$. ■

6. Weakly predilatable kernels. Here we generalize Theorem 5 of [44] (and also Theorem 3.2 of [26]) to the context of nonunital commutative algebraic structures mentioned in Section 1.

THEOREM 6.1. *Assume $\mathcal{S}$ is commutative and $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is an arbitrary kernel. Then Φ is predilatable and nondegenerate if and only if so is Φ_f for every $f \in \mathcal{H}$.*

Proof. The "only if" part follows from Proposition 5.4. Assume that $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is a kernel such that each Φ_f, $f \in \mathcal{H}$, is predilatable and nondegenerate. Then (cf. [44], Theorem 3 and [32], Theorem 5; see also [13], Théorème 15.9.2) for any $f \in \mathcal{H}$, there is a unique $\nu(\cdot\,; f) \in M_+(\Sigma(\mathcal{S}))$, called a representing scalar measure of Φ_f, such that

$$\Phi_f(s,t) = \int\limits_{\Sigma(\mathcal{S})} x(s^*t)\,\nu(dx; f)\,, \qquad s,t \in \mathcal{S}\,. \tag{6.1}$$

Denote by $\mathfrak{D}$ the class of all Borel subsets A of $\Sigma(\mathcal{S})$ such that $\nu(A; f) < \infty$ for every $f \in \mathcal{H}$. Then $\mathfrak{C}(\Sigma(\mathcal{S})) \subset \mathfrak{D}$ (see the appendix). Given $f, g \in \mathcal{H}$ and $A \in \mathfrak{D}$, define $\mu(A; f, g)$ by

$$\mu(A; f, g) := 4^{-1} \sum_{k=1}^{4} i^k \nu(A; f + i^k g)\,. \tag{6.2}$$

Step 1. *For every $A \in \mathfrak{D}$, the function $\mathcal{H} \times \mathcal{H} \ni (f,g) \to \mu(A; f, g) \in \mathbb{C}$ is a semi-inner product on $\mathcal{H}$ such that $\mu(A; f, f) = \nu(A; f)$ for $f \in \mathcal{H}$.*

Indeed, since both measures $\nu(\cdot\,; zf)$ and $|z|^2\nu(\cdot\,; f)$ represent Φ_{zf} via (6.1), they must be equal. Thus

$$\nu(A; zf) = |z|^2 \nu(A; f)\,, \qquad A \in \mathfrak{D}\,,\ f \in \mathcal{H}\,,\ z \in \mathbb{C}\,,$$

which implies that

$$\mu(A; f, f) = \nu(A; f)\,, \qquad A \in \mathfrak{D}\,,\ f \in \mathcal{H}\,, \tag{6.3}$$

$$\overline{\mu(A; f, g)} = \mu(A; g, f)\,, \qquad A \in \mathfrak{D}\,,\ f, g \in \mathcal{H}\,. \tag{6.4}$$

Take $z \geq 0$. Then, applying the polarization formula to both sides of $\langle \Phi(\cdot, -)zf, g\rangle = z\langle \Phi(\cdot, -)f, g\rangle$, we get

$$\Phi_{zf+g} + z\Phi_{f-g} = \Phi_{zf-g} + z\Phi_{f+g}\,, \qquad f, g \in \mathcal{H}\,, \tag{6.5}$$

$$\Phi_{zf+ig} + z\Phi_{f-ig} = \Phi_{zf-ig} + z\Phi_{f+ig}\,, \qquad f, g \in \mathcal{H}\,. \tag{6.6}$$

It follows from (6.5) that the measures $\nu(\cdot\,; zf + g) + z\nu(\cdot\,; f - g)$ and $\nu(\cdot\,; zf - g) + z\nu(\cdot\,; f + g)$, both in $M_+(\Sigma(\mathcal{S}))$, represent the same nondegenerate predilatable kernel $\Phi_{zf+g} + z\Phi_{f-g}$ and consequently they are equal. Similarly (6.6) implies that the measures $\nu(\cdot\,; zf + ig) + z\nu(\cdot\,; f - ig)$ and $\nu(\cdot\,; zf - ig) + z\nu(\cdot\,; f + ig)$ coincide. Combining these two facts we get

$$\mu(A; zf, g) = z\mu(A; f, g)\,, \quad A \in \mathfrak{D}\,,\ f, g \in \mathcal{H}\,,\ z \geq 0\,. \tag{6.7}$$

Using similar arguments we can show that

$$\mu(A; f + g, h) = \mu(A; f, h) + \mu(A; g, h)\,, \quad A \in \mathfrak{D}\,,\ f, g, h \in \mathcal{H}\,, \tag{6.8}$$

$$\mu(A; -f, g) = -\mu(A; f, g)\,, \quad A \in \mathfrak{D}\,,\ f, g \in \mathcal{H}\,, \tag{6.9}$$

$$\mu(A; if, g) = i\mu(A; f, g)\,, \quad A \in \mathfrak{D}\,,\ f, g \in \mathcal{H}\,. \tag{6.10}$$

Now the conclusion of Step 1 can be easily derived from (6.3)–(6.10).

Step 2. *Φ is a positive definite bi-affine kernel.*

Since all Φ_f, $f \in \mathcal{H}$, are bi-affine, so is Φ. Fix $s_1, \ldots, s_m \in \mathcal{S}$ and $f_1, \ldots, f_m \in \mathcal{H}$ with $m \geq 1$. Define a complex Borel measure λ on $\Sigma(\mathcal{S})$ by

$$\lambda(A) = 4^{-1} \sum_{p,q=1}^{m} \sum_{k=1}^{4} i^k \int_A x(s_q^* s_p)\, \nu(dx; f_p + i^k f_q)\,.$$

Take $C \in \mathfrak{C}(\Sigma(\mathcal{S}))$. Then for each $p = 1, \ldots, m$, there exists a sequence of simple Borel functions $\{\varphi_{n,p}\}_{n=1}^{\infty}$ defined on C which converges uniformly on C to the bounded function $\widehat{s}_p|_C$. Moreover, for each $n \geq 1$, we can choose a Borel partition $\{C_{n,1}, \ldots, C_{n,l_n}\}$ of C and sequences $\{\beta_{n,p,1}, \ldots, \beta_{n,p,l_n}\} \subset \mathbb{C}$ $(p = 1, \ldots, m)$ such that

$$\varphi_{n,p} = \sum_{j=1}^{l_n} \beta_{n,p,j} \chi_{C_{n,j}}\,.$$

Set $g_{n,j} = \sum_{p=1}^{m} \beta_{n,p,j} f_p$. Then, using Step 1 and the fact that $\nu(C; f) < \infty$ for $f \in \mathcal{H}$, we get

$$\begin{aligned}\lambda(C) &= \lim_{n\to\infty} 4^{-1} \sum_{p,q=1}^{m} \sum_{k=1}^{4} i^k \int_C \varphi_{n,p} \overline{\varphi}_{n,q}\, d\nu(\cdot\,; f_p + i^k f_q) \\ &= \lim_{n\to\infty} \sum_{j=1}^{l_n} \sum_{p,q=1}^{m} \beta_{n,p,j} \overline{\beta}_{n,q,j} \mu(C_{n,j}; f_p, f_q) \\ &= \lim_{n\to\infty} \sum_{j=1}^{l_n} \mu(C_{n,j}; g_{n,j}, g_{n,j}) \geq 0\,.\end{aligned}$$

Since $|\widehat{t}\,| d\nu(\cdot\,; f)$ is a finite Radon measure on $\Sigma(\mathcal{S})$ for all $t \in \mathcal{S}^{(2)}$ and $f \in \mathcal{H}$ (use Proposition 2.1.7 of [5]), we must have $\lambda(\Sigma(\mathcal{S})) = \lim_{C \in \mathfrak{C}(\Sigma(\mathcal{S}))} \lambda(C)$

≥ 0. On the other hand, the polarization formula and (6.1) yield

$$\sum_{p,q=1}^{m} \langle \Phi(s_q, s_p) f_p, f_q \rangle = \lambda(\Sigma(\mathcal{S})) \geq 0\,,$$

which proves positive definiteness of Φ.

Now the "if" part of the conclusion follows from Step 2, Theorem 3.1 and Proposition 5.4. ■

7. Disintegration of nondegenerate predilatable kernels. In this section we present an integral representation for nondegenerate predilatable kernels defined on a commutative algebraic structure $\mathcal{S}$ (see the appendix for notation and definitions concerning integration).

Let $M : \mathfrak{R} \to B(\mathcal{H})$ be a maximal tight PO measure on $\Sigma(\mathcal{S})$ whose closed support is $\widehat{\mathcal{S}}$-bounded. We say that M is a *representing measure* of a kernel $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ if $\widehat{s} \in L^1(M)$ for every $s \in \mathcal{S}^{(2)}$ and

$$\Phi(s,t) = \int\limits_{\Sigma(\mathcal{S})} x(s^* t)\, M(dx)\,, \qquad s, t \in \mathcal{S}\,. \tag{7.1}$$

M is said to be a *representing measure* of a mapping $\Theta : \mathcal{S} \to B(\mathcal{H})$ if $\widehat{s} \in L^1(M)$ for every $s \in \mathcal{S}$ and

$$\Theta(s) = \int\limits_{\Sigma(\mathcal{S})} x(s)\, M(dx)\,, \qquad s \in \mathcal{S}\,. \tag{7.2}$$

THEOREM 7.1. *Assume $\mathcal{S}$ is commutative and $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is an arbitrary kernel. If for every $f \in \mathcal{H}$, the kernel Φ_f is predilatable and nondegenerate, then Φ has a unique representing measure. Conversely, if Φ has a representing measure, then Φ is a nondegenerate predilatable kernel.*

Proof. Suppose that Φ has a representing measure. Then, by Proposition 5 of [44], each scalar kernel Φ_f is predilatable and nondegenerate ($f \in \mathcal{H}$). In virtue of Theorem 6.1, Φ is also predilatable and nondegenerate.

Assume now that each Φ_f, $f \in \mathcal{H}$, is predilatable and nondegenerate. Then, by Theorem 6.1, so is Φ. Let $(\mathcal{K}, X, \Pi)$ be a minimal propagator of Φ and let $E : \mathfrak{B}(\Sigma(\mathcal{S})) \to B(\mathcal{K})$ be the spectral measure of Π (see Section 1). Since Φ is nondegenerate, we have $E(\Sigma(\mathcal{S})) = I_{\mathcal{K}}$. Let $(\mathcal{K}_f, X_f, \Pi_f)$ be the restriction of $(\mathcal{K}, X, \Pi)$ to $\mathcal{K}_f$ (see Section 2) and let P_f be the orthogonal projection of $\mathcal{K}$ onto $\mathcal{K}_f$ ($f \in \mathcal{H}$). Since $\mathcal{K}_f$ reduces Π to Π_f, the projection P_f, $f \in \mathcal{H}$, commutes with any $\Pi(s)$, $s \in \mathcal{S}$. This implies that

$$\int\limits_{\Sigma(\mathcal{S})} \widehat{s}(x)\, \langle E(dx) g, P_f h \rangle = \langle P_f \Pi(s) g, h \rangle = \langle \Pi(s) P_f g, h \rangle$$

$$= \int_{\Sigma(\mathcal{S})} \widehat{s}(x)\, \langle E(dx)P_f g, h\rangle\,, \qquad s \in \mathcal{S}\,,\ g,h \in \mathcal{K}\,,$$

which leads to $\langle E(\cdot)g, P_f h\rangle = \langle E(\cdot)P_f g, h\rangle$ for any $g, h \in \mathcal{K}$ (cf. [44], Proposition 1). Thus P_f commutes with every projection $E(A)$, $A \in \mathfrak{B}(\Sigma(\mathcal{S}))$. This in turn implies that $\mathcal{K}_f$ reduces E to the spectral measure E_f of Π_f.

Let $\nu(\cdot\,; f)$ be a representing scalar measure of Φ_f. By the previous paragraph, the measure $\mu_{s,s}(\cdot\,; f) := \langle E_f(\cdot)X_f(s), X_f(s)\rangle$ coincides with $\langle E(\cdot)X(s)f, X(s)f\rangle$ on $\mathfrak{B}(\Sigma(\mathcal{S}))$ for all $s \in \mathcal{S}$ and $f \in \mathcal{H}$. Moreover (cf. [44], p. 356), the measures $\nu(\cdot\,; f)$ and $\mu_{s,s}(\cdot\,; f)$ are related to each other as follows:

$$\nu(A; f) = \int_A |\widehat{s}\,|^{-2}\, d\mu_{s,s}(\cdot\,; f)\,, \qquad A \in \mathfrak{B}(D_s)\,,\ f \in \mathcal{H}\,,\ s \in \mathcal{S}\,, \tag{7.3}$$

where $D_s := \{x \in \Sigma(\mathcal{S}) : x(s) \neq 0\}$.

Denote by $\mathfrak{D}$ the class of all Borel subsets A of $\Sigma(\mathcal{S})$ such that $\nu(A; f) < \infty$ for every $f \in \mathcal{H}$. Then $\mathfrak{D}$ is a δ-ring which contains $\mathfrak{D}(\Sigma(\mathcal{S}))$. We show that the function $\mathcal{H} \ni f \to \nu(A; f) \in \mathbb{R}_+$ is continuous for every $A \in \mathfrak{D}$.

Consider first the case where $A \in \mathfrak{B}(D_s)$ with some $s \in \mathcal{S}$. Let $\{\varphi_n\}_{n=1}^{\infty}$ be a sequence of nonnegative simple Borel functions on A, which is increasing and pointwise convergent to $|\widehat{s}\,|^{-2}$ on A. It follows from the Lebesgue monotone convergence theorem and (7.3) that

$$\nu(A; f) = \lim_{n\to\infty} \int_A \varphi_n(x)\, \langle E(dx)X(s)f, X(s)f\rangle\,, \qquad f \in \mathcal{H}\,. \tag{7.4}$$

Since φ_n is of the form $\varphi_n = \sum_{j=1}^{l_n} \beta_{n,j}\chi_{A_{n,j}}$ with $\{A_{n,1}, \ldots, A_{n,l_n}\} \subset \mathfrak{B}(A)$ and $\{\beta_{n,1}, \ldots, \beta_{n,l_n}\} \subset \mathbb{R}_+$, we get

$$\int_A \varphi_n(x)\, \langle E(dx)X(s)f, X(s)f\rangle = \langle T_n f, f\rangle\,, \qquad f \in \mathcal{H}\,,\ n \geq 1\,, \tag{7.5}$$

where $T_n = X(s)^*(\sum_{j=1}^{l_n} \beta_{n,j}E(A_{n,j}))X(s) \in B(\mathcal{H})$ for every $n \geq 1$. Thus, by (7.4) and (7.5), the sequence of continuous seminorms $\mathcal{H} \ni f \to \langle T_n f, f\rangle^{1/2} \in \mathbb{R}_+$, $n = 1, 2, \ldots$, converges pointwise to $\nu(A; \cdot)^{1/2}$ on the Hilbert space $\mathcal{H}$. It follows from the Banach–Steinhaus theorem that $\nu(A; \cdot)^{1/2}$ is a continuous seminorm.

Assume now that $C \in \mathfrak{C}(\Sigma(\mathcal{S}))$. Since $\{D_s : s \in \mathcal{S}\}$ is an open cover of the compact set C, there exist $s_1, \ldots, s_m \in \mathcal{S}$ such that $C \subset \bigcup_{k=1}^m D_{s_k}$. Choose a partition $\{C_1, \ldots, C_m\} \subset \mathfrak{B}(\Sigma(\mathcal{S}))$ of C such that $C_k \subset D_{s_k}$ for $k = 1, \ldots, m$. Then, by the previous paragraph, each $\nu(C_k; \cdot)$ is continuous and consequently so is $\nu(C; \cdot) = \sum_{k=1}^m \nu(C_k; \cdot)$.

Let finally $A \in \mathfrak{D}$. Since each $\nu(\cdot\,; f)$ is a Radon measure on $\Sigma(\mathcal{S})$ ($f \in \mathcal{H}$), we have

$$\nu(A; f) = \sup\{\nu(C; f) : C \in \mathfrak{C}(\Sigma(\mathcal{S}))\,,\ C \subset A\}\,, \qquad f \in \mathcal{H}\,.$$

Applying again the Banach–Steinhaus theorem to the family $\{\nu(C;\cdot)^{1/2} : C \in \mathfrak{C}(\Sigma(\mathcal{S}))\}$ of continuous seminorms (3) on the Hilbert space $\mathcal{H}$, we get the continuity of $\nu(A;\cdot)$.

Let $\mu(A; f, g)$ be defined by (6.2) for $A \in \mathfrak{D}$, $f, g \in \mathcal{H}$. Step 1 of the proof of Theorem 6.1 states that for every $A \in \mathfrak{D}$, the function $\mathcal{H} \times \mathcal{H} \ni (f, g) \to \mu(A; f, g) \in \mathbb{C}$ is a semi-inner product on $\mathcal{H}$ such that $\mu(A; f, f) = \nu(A; f)$ for $f \in \mathcal{H}$. Since $\nu(A;\cdot)$ is continuous, there exists a unique positive operator $M(A) \in B(\mathcal{H})$ such that

$$\mu(A; f, g) = \langle M(A)f, g\rangle\,, \qquad A \in \mathfrak{D}\,,\ f, g \in \mathcal{H}\,.$$

Consequently, M is a maximal tight PO measure such that

$$\langle \Phi(s,t)f, f\rangle = \int\limits_{\Sigma(\mathcal{S})} x(s^*t)\, M_f^{\intercal}\,(dx)\,, \qquad s, t \in \mathcal{S}\,,\ f \in \mathcal{H}\,.$$

Thus M satisfies (7.1). Since the closed support of the spectral measure E is $\widehat{\mathcal{S}}$-bounded, one can use (7.3) to show that so is the closed support of M. Finally, the uniqueness of M follows from the polarization formula and Theorem 3 of [44]. ∎

R e m a r k. It is worthwhile to notice that a representing measure M of Φ can be explicitly described on compact subsets of $\Sigma(\mathcal{S})$ with the help of a minimal propagator $(\mathcal{K}, X, \Pi)$ of Φ as follows. Let E be the spectral measure of Π and let C be a compact subset of $\Sigma(\mathcal{S})$. Then there exist $s_1, \ldots, s_m \in \mathcal{S}$ such that $C \subset \bigcup_{k=1}^m D_{s_k}$. Basing on (7.3), one can show that

$$M(C) = \sum_{k=1}^m X(s_k)^* \int\limits_C \Big(\sum_{j=1}^m |\widehat{s}_j|^2\Big)^{-1} dE\, X(s_k)\,.$$

This, in turn, can be used to show that the closed support of M is contained in the closed support A of E. It follows from the proof of Theorem 1 in [44] that $A \subset \{x \in \Sigma(\mathcal{S}) : |x(s)| \le \|\Pi(s)\| \text{ for all } s \in \mathcal{S}\}$. Hence (use Theorem 3.1) the closed support of M is contained in the closed $\widehat{\mathcal{S}}$-bounded set $\{x \in \Sigma(\mathcal{S}) : |x(s)| \le \varkappa_\Phi(s) \text{ for all } s \in \mathcal{S}\}$.

Though not every predilatable kernel on $\mathcal{S}$ has a representing measure (cf. [44] and [32]), we can disintegrate predilatable kernels on commutative involutory algebraic structures in the way proposed by the following corollary (see [32], Theorem 4 and [18], Théorème 1 for the scalar case).

(3) That $\nu(C;\cdot)^{1/2}$ is a seminorm follows from Step 1 of the proof of Theorem 6.1.

COROLLARY 7.2. *Assume $\mathcal{S}$ is commutative. Let $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ be a kernel such that all Φ_f ($f \in \mathcal{H}$) are predilatable. Then there exists a unique maximal tight PO measure $M : \mathfrak{R} \to B(\mathcal{H})$ on $\Sigma(\mathcal{S})$ whose closed support is $\widehat{\mathcal{S}}$-bounded and such that*

(i) $\widehat{s} \in L^1(M)$, $s \in \mathcal{S}^{(3)}$,
(ii) $\Phi(s, ut) = \int_{\Sigma(\mathcal{S})} x(s^* ut)\, M(dx)$, $s, t, u \in \mathcal{S}$.

Proof. Let $M : \mathfrak{R} \to B(\mathcal{H})$ be a representing measure of Φ_{N}. Then the closed support of M is $\widehat{\mathcal{S}}$-bounded and M fulfils (i). Moreover, in virtue of Theorem 4.2,

$$\Phi(s, ut) = \Phi_{\mathrm{D}}(s, ut) + \Phi_{\mathrm{N}}(s, ut) = \int\limits_{\Sigma(\mathcal{S})} x(s^* ut)\, M(dx)\,, \qquad s, t, u \in \mathcal{S}\,.$$

The uniqueness of M is a consequence of that for scalar kernels (see [44], p. 357 or [32], Theorem 4). ∎

Notice that any maximal tight PO measure on $\Sigma(\mathcal{S})$ whose closed support is $\widehat{\mathcal{S}}$-bounded and which fulfils conditions (i) and (ii) of Corollary 7.2 is necessarily a representing measure of the nondegenerate part Φ_{N} of Φ.

Now we show that representing measures of dilatable kernels on a commutative $\mathcal{S}$ are defined on the whole σ-algebra of Borel subsets of $\Sigma(\mathcal{S})$.

COROLLARY 7.3. *Assume $\mathcal{S}$ is commutative and $\Phi : \mathcal{S} \times \mathcal{S} \to B(\mathcal{H})$ is an arbitrary kernel. Then the following conditions are equivalent*:

(i) *Φ is dilatable*,
(ii) *for every $f \in \mathcal{H}$, Φ_f is dilatable*,
(iii) *Φ is a nondegenerate predilatable kernel with a representing measure $M : \mathfrak{R} \to B(\mathcal{H})$ such that $\mathfrak{R} = \mathfrak{B}(\Sigma(\mathcal{S}))$* (*or equivalently* $\sup\{\|M(A)\| : A \in \mathfrak{R}\} < \infty$).

Proof. (i)⇒(ii) follows from what has been done in the second part of Section 2.

(ii)⇒(iii). Assume that all Φ_f ($f \in \mathcal{H}$) are dilatable. Since each dilatable kernel is automatically nondegenerate, Theorem 7.1 implies that Φ is a nondegenerate predilatable kernel with a representing measure $M : \mathfrak{R} \to B(\mathcal{H})$. In particular, $M_f^{\intercal}$ is a representing scalar measure of Φ_f. Let $(\mathcal{L}^f, \zeta^f, \Pi^f)$ be a minimal dilation of Φ_f and let G^f be the spectral measure of Π^f. Then

$$\Phi_f(s, t) = \langle \Pi^f(s^* t)\zeta^f, \zeta^f \rangle = \int\limits_{\Sigma(\mathcal{S})} x(s^* t)\, \langle G^f(dx)\zeta^f, \zeta^f \rangle, \qquad s, t \in \mathcal{S},\ f \in \mathcal{H}.$$

This means that $M_f^{\intercal}(\cdot)$ and $\langle G^f(\cdot)\zeta^f, \zeta^f \rangle$ are representing scalar measures of Φ_f ($f \in \mathcal{H}$). Consequently, $M_f^{\intercal}(\cdot) = \langle G^f(\cdot)\zeta^f, \zeta^f \rangle$ for all $f \in \mathcal{H}$, which implies that $\mathfrak{R} = \mathfrak{B}(\Sigma(\mathcal{S}))$ (use the fact that M is maximal).

(iii)⇒(i). Assume now that $M : \mathfrak{B}(\Sigma(\mathcal{S})) \to B(\mathcal{H})$ is a representing measure of Φ. Then C, the closed support of M, is $\widehat{\mathcal{S}}$-bounded. It follows from the Naimark dilation theorem (cf. [27], Theorem 6.4) that there exist a complex Hilbert space $\mathcal{K}$, a spectral measure $E : \mathfrak{B}(\Sigma(\mathcal{S})) \to B(\mathcal{K})$ and $R \in B(\mathcal{H}, \mathcal{K})$ such that $\mathcal{K} = \bigvee\{E(A)R\mathcal{H} : A \in \mathfrak{B}(\Sigma(\mathcal{S}))\}$ and $M(A) = R^*E(A)R$ for $A \in \mathfrak{B}(\Sigma(\mathcal{S}))$. The last equality can be used to show that E is also supported by C. Since $\widehat{s}$ is bounded on C, the operator $\Pi(s) := \int_{\Sigma(\mathcal{S})} x(s)\, E(dx)$ is bounded for every $s \in \mathcal{S}$ and

$$\Phi(s,t) = R^* \int\limits_{\Sigma(\mathcal{S})} x(s^*t)\, E(dx)R = (\Pi(s)R)^*\Pi(t)R\,, \qquad s,t \in \mathcal{S}\,.$$

This implies that Φ is dilatable (cf. [49], Proposition 5.6). ∎

8. Continuity of predilatable mappings on topological $*$-algebras. Theorem 7.1 can be used to settle the question of continuity of positive linear mappings defined on some commutative topological $*$-algebras $\mathcal{A}$. Notice first that the set $\mathcal{A} \setminus [\mathcal{A}^{(2)}]$ is not involved in the definition of positivity, so it is natural to restrict our attention to the case $\mathcal{A} = [\mathcal{A}^{(2)}]$. Let $\mathcal{A}$ be a complex $*$-algebra equipped with a linear topology. We distinguish the following four properties of $\mathcal{A}$:

(P.1) each character of $\mathcal{A}$ is continuous,
(P.2) each positive linear functional on $\mathcal{A}$ is predilatable ([4]),
(P.3) each positive linear functional on $\mathcal{A}$ is continuous,
(P.4) each positive linear operator-valued mapping on $\mathcal{A}$ is continuous.

It is obvious that (P.4)⇒(P.3) and (P.3)⇒(P.1). Below we show that under some additional assumptions on $\mathcal{A}$, (P.3)⇒(P.2). The property (P.1) is connected with the Mazur–Michael problem (cf. [20]) whether each multiplicative linear functional on a Fréchet algebra is continuous.

Denote by $\mathcal{A}_1 = \mathcal{A} \times \mathbb{C}$ the unitization of $\mathcal{A}$. It is obvious that $\mathcal{A}$ has the property (P.1) if and only if so does $\mathcal{A}_1$. Below we show that the same is true for (P.2).

Proposition 8.1. *A commutative complex $*$-algebra $\mathcal{A}$ has the property* (P.2) *if and only if so does $\mathcal{A}_1$.*

Proof. The "only if" part follows from Corollary 2 in [43]. To prove the "if" part, take a positive linear functional φ on $\mathcal{A}$. It follows from the Cauchy–Schwarz inequality that

$$|{}_b\varphi(a)|^2 = |\varphi(b^*(ab))|^2 \le \varphi(b^*b) \cdot {}_b\varphi(a^*a)\,, \qquad a,b \in \mathcal{A}\,.$$

([4]) Or unitary in the terminology of [18].

This implies that for any $b \in \mathcal{A}$, ${}_b\varphi$ can be extended to a positive linear functional ${}_b\widetilde{\varphi}$ on $\mathcal{A}_1$ (cf. [9], Theorem 37.11). It follows from our assumptions that ${}_b\widetilde{\varphi}$ and consequently ${}_b\varphi$ are predilatable for all $b \in \mathcal{A}$. Thus, by Theorem 3.1(vi), φ is predilatable. ∎

The following proposition will help us to establish some other relationships between conditions just introduced.

PROPOSITION 8.2. *Let $\mathcal{A}$ be a commutative complex $*$-algebra such that $\mathcal{A} = [\mathcal{A}^{(2)}]$ and let $\Theta : \mathcal{A} \to B(\mathcal{H})$ be a positive linear mapping. If $\mathcal{A}$ is an F-space with property* (P.1) *and each functional $\langle {}_b\Theta(\cdot)f, f\rangle$ ($f \in \mathcal{H}$, $b \in \mathcal{A}$) is predilatable, then Θ is continuous. If $\mathcal{A}$ is locally multiplicatively-convex and each functional $\langle {}_b\Theta(\cdot)f, f\rangle$ ($f \in \mathcal{H}$, $b \in \mathcal{A}$) is continuous, then Θ is predilatable.*

Proof. Assume $\mathcal{A}$ is an F-space with property (P.1) and $\langle {}_b\Theta(\cdot)f, f\rangle$ is predilatable for all $f \in \mathcal{H}$ and $b \in \mathcal{A}$. Then, by Theorem 3.1(vi), each functional $\langle \Theta(\cdot)f, f\rangle$ is predilatable. Fix $f \in \mathcal{H}$ and set $\varphi(\cdot) := \langle \Theta(\cdot)f, f\rangle$. It follows from Theorem 7.1 (see also [18] and [32]) that φ has a representing scalar measure $\mu \in M_+(\Sigma(\mathcal{A}))$. Take $C \in \mathfrak{C}(\Sigma(\mathcal{A}))$ and define a linear functional φ_C on $\mathcal{A}$ via $\varphi_C(a) := \int_C x(a)\,\mu(dx)$ for $a \in \mathcal{A}$. Notice that φ_C is continuous. Indeed, by (P.1), each seminorm $\mathcal{A} \ni a \to |x(a)| \in \mathbb{R}_+$ ($x \in \Sigma(\mathcal{A})$) is continuous on $\mathcal{A}$. Consequently, the seminorm ϱ_C defined by $\varrho_C(a) := \sup\{|x(a)| : x \in C\}$ for $a \in \mathcal{A}$ is lower semicontinuous. Since $\mathcal{A}$ is an F-space, ϱ_C is continuous. However, $|\varphi_C| \le \mu(C)\varrho_C$, so φ_C is continuous for every $C \in \mathfrak{C}(\Sigma(\mathcal{A}))$. Moreover, one can show that the net $\{\varphi_C : C \in \mathfrak{C}(\Sigma(\mathcal{A}))\}$ is pointwise bounded and pointwise convergent to φ. In virtue of the Banach–Steinhaus theorem (cf. [34], Theorem 2.5), the linear functional $\varphi(\cdot) = \langle \Theta(\cdot)f, f\rangle$ is continuous for every $f \in \mathcal{H}$. Since $\mathcal{A}$ is an F-space, the continuity of Θ follows again from the Banach–Steinhaus theorem.

Assume $\mathcal{A}$ is a locally multiplicatively-convex algebra and each functional $\langle {}_b\Theta(\cdot)f, f\rangle$ ($f \in \mathcal{H}$, $b \in \mathcal{A}$) is continuous. Then, by Theorem 3.1(iii), $\langle {}_b\Theta(\cdot)f, f\rangle$ is predilatable for all $b \in \mathcal{A}$ and $f \in \mathcal{H}$. Using Theorem 3.1(vi) we see that $\langle \Theta(\cdot)f, f\rangle$ is predilatable for every $f \in \mathcal{H}$. Consequently, by Theorem 7.1, Θ is predilatable. ∎

THEOREM 8.3. *Let $\mathcal{A}$ be a commutative complex $*$-algebra. If $\mathcal{A}$ is an F-space, $\mathcal{A} = [\mathcal{A}^{(2)}]$ and $\mathcal{A}_1$ has the properties* (P.1) *and* (P.2), *then $\mathcal{A}$ has the property* (P.4). *If $\mathcal{A}$ is a locally multiplicatively-convex algebra and $\mathcal{A}_1$ has the property* (P.3), *then $\mathcal{A}$ has the properties* (P.1) *and* (P.2).

Proof. The first assertion follows from Propositions 8.1 and 8.2. Assume that $\mathcal{A}$ is a locally multiplicatively-convex algebra and $\mathcal{A}_1$ has the property (P.3). It is enough to show that $\mathcal{A}$ has the property (P.2). Take a positive linear functional φ on $\mathcal{A}$. Then, similarly to the proof of Proposition 8.1, we show that each ${}_b\varphi$ ($b \in \mathcal{A}$) extends to a positive linear functional ${}_b\widetilde{\varphi}$ on $\mathcal{A}_1$. It follows from our assumptions that all ${}_b\widetilde{\varphi}$ are continuous. Consequently, by Theorem 3.1(iii), every ${}_b\varphi$ is predilatable. This in turn implies that φ is predilatable. ∎

It is well known (cf. [9], Theorem 37.3 and Lemma 37.6(iv)) that any Banach $*$-algebra has the properties (P.1) and (P.2). Thus Theorem 8.3 can be regarded as a generalization of the Murphy theorem (cf. [9], Theorem 37.14) to the case of nonnormed topological $*$-algebras. Other examples of locally multiplicatively-convex algebras with the properties (P.1) and (P.2) can be found in [20].

Now we show that positive bi-affine kernels on commutative Banach $*$-algebras which factorize induce continuous positive linear mappings.

PROPOSITION 8.4. *Let $\mathcal{A}$ be a commutative complex Banach $*$-algebra such that $\mathcal{A} = [\mathcal{A}^{(2)}]$ and let $\Phi : \mathcal{A} \times \mathcal{A} \to B(\mathcal{H})$ be a positive bi-affine kernel. Then there exists a continuous positive linear mapping $\Theta : \mathcal{A} \to B(\mathcal{H})$ having a representing measure on $\Sigma^{\mathrm{c}}(\mathcal{A})$ such that*

$$\Phi(a, b) = \Theta(a^* b)\,, \qquad a, b \in \mathcal{A}\,.$$

Proof. It follows from Proposition 3.2 that all Φ_f ($f \in \mathcal{H}$) are predilatable. Thus, by Theorem 7.1, Φ has a representing measure M on $\Sigma^{\mathrm{c}}(\mathcal{A})$. Since $\mathcal{A} = [\mathcal{A}^{(2)}]$, we can define $\Theta : \mathcal{A} \to B(\mathcal{H})$ by (7.2). Then Θ is a positive linear mapping and M is its representing measure. Since any Banach $*$-algebra has the properties (P.1) and (P.2), the continuity of Θ follows from Theorem 8.3. ∎

Proposition 8.4 suggests the possibility of localizing representing measures of continuous predilatable kernels on the continuous characters of the considered topological $*$-algebra.

PROPOSITION 8.5. *Let $\mathcal{A} = [\mathcal{A}^{(2)}]$ be a commutative locally multiplicatively-convex $*$-algebra with continuous involution and let $\Phi : \mathcal{A} \times \mathcal{A} \to B(\mathcal{H})$ be a jointly continuous positive bi-affine kernel. Then there is a positive linear mapping $\Theta : \mathcal{A} \to B(\mathcal{H})$ having a representing measure supported by $\Sigma^{\mathrm{c}}(\mathcal{A})$ and such that $\Phi = \Phi^{\Theta}$.*

Proof. Applying Proposition 3.3 to Φ_f we deduce that each Φ_f is predilatable. Consequently, by Theorem 7.1, Φ is predilatable and it has a representing measure M defined on $\Sigma(\mathcal{A})$. Let $(\mathcal{K}, X, \Pi)$ be a minimal propagator of Φ and let $\mathcal{W}$ be the C^*-algebra generated by $\Pi(\mathcal{A}) \cup \{I_{\mathcal{K}}\}$. Then $C_\Pi := \{x \circ \Pi : x \in \Sigma(\mathcal{W})\}$ is a compact subset of $\Sigma(\mathcal{A}) \cup \{\mathbf{0}\}$. Put $A_\Pi := C_\Pi \setminus \{\mathbf{0}\}$. Since $\Sigma(\mathcal{W}) = \Sigma^{\mathrm{c}}(\mathcal{W})$ and Π is continuous (use Proposition 3.3), we get $A_\Pi \subset \Sigma^{\mathrm{c}}(\mathcal{A})$. It follows from the proof of Theorem 1 in

[44] that the spectral measure E of Π is supported by A_Π. Consequently, so is the spectral measure E_f of Π_f for every $f \in \mathcal{H}$ (see the proof of Theorem 7.1). This, in turn, implies (see the proof of Theorem 3 in [44]) that for any $f \in \mathcal{H}$, the closed support of M_f^{T} is contained in A_Π. Now an easy verification shows that M is also supported by A_Π. The mapping Θ can be defined by (7.2). ∎

We end this section with an open question. It is well known (cf. [27], Theorem 9.3) that if $\mathcal{A}$ is a Banach $*$-algebra with continuous involution and with a bounded two-sided approximate identity, then any linear positive definite mapping Θ on $\mathcal{A}$ is dilatable. If in addition $\mathcal{A}$ is commutative, then to have dilatability we can assume less, namely that Θ is a linear positive mapping. Since in this case $\mathcal{A} = [\mathcal{A}^{(2)}]$ (by the Cohen factorization theorem), the question is:

Question 1. Let $\mathcal{A}$ be a commutative Banach $*$-algebra such that $\mathcal{A} = [\mathcal{A}^{(2)}]$. Does there exist a positive linear mapping on $\mathcal{A}$ which is not dilatable?

9. Disintegration of holomorphic positive definite mappings on commutative Banach $*$-algebras. In this and subsequent sections we investigate holomorphic positive definite mappings acting either on a Banach $*$-algebra $\mathcal{A}$ or on the open unit ball $\mathcal{A}_\bullet$ of $\mathcal{A}$. To avoid any confusion, we denote by $\mathcal{A}_\odot$ the $*$-semigroup $(\mathcal{A}, \cdot, *)$ with involution and multiplication inherited from $\mathcal{A}$. Notice that $(\mathcal{A}_\bullet, \cdot, *)$ is a $*$-semigroup if and only if the involution of $\mathcal{A}$ is isometric, i.e. $\|a^*\| = \|a\|$ for $a \in \mathcal{A}$. It turns out that any positive definite holomorphic mapping on $\mathcal{A}_\bullet$ is predilatable (and consequently representable by measures, provided $\mathcal{A}$ is commutative). Since this is not the case for $\mathcal{A}_\odot$, our main interest is directed at the $*$-semigroup $\mathcal{A}_\bullet$.

We refer the reader to [7], [8] and [12] for holomorphy in topological vector spaces. Given a complex Hilbert space $\mathcal{H}$ and $k \geq 0$, denote by $L_k^{\mathrm{s}}(\mathcal{A}, \mathcal{H})$ and $P_k(\mathcal{A}, \mathcal{H})$ the Banach spaces of continuous k-linear symmetric mappings from $\mathcal{A}^k$ into $B(\mathcal{H})$ and continuous k-homogeneous polynomials from $\mathcal{A}$ into $B(\mathcal{H})$, respectively, with the corresponding norms

$$\|\Psi\| = \sup\{\|\Psi(a_1, \ldots, a_k)\| : \|a_j\| \leq 1,\ j = 1, \ldots, k\}, \qquad \Psi \in L_k^{\mathrm{s}}(\mathcal{A}, \mathcal{H}),$$
$$\|\Theta\| = \sup\{\|\Theta(a)\| : \|a\| \leq 1\}, \qquad \Theta \in P_k(\mathcal{A}, \mathcal{H}).$$

It follows from Proposition 1 of [7] that the mapping $P_k(\mathcal{A}, \mathcal{H}) \ni \Theta \to \Theta^\# \in L_k^{\mathrm{s}}(\mathcal{A}, \mathcal{H})$ defined by

$$\Theta^\#(a_1, \ldots, a_k) = \frac{1}{k!} \sum_{i_1=0}^{1} \ldots \sum_{i_k=0}^{1} (-1)^{k-(i_1+\ldots+i_k)} \Theta(i_1 a_1 + \ldots + i_k a_k)$$

is a linear homeomorphism. The last equality is called the *polarization formula*.

The following lemma states necessary (and also sufficient) conditions for an operator-valued mapping to be a holomorphic $*$-representation of $\mathcal{A}_{\odot}$ (resp. $\mathcal{A}_{\bullet}$).

LEMMA 9.1. *Let $\mathcal{A}$ be a Banach $*$-algebra (resp. a Banach $*$-algebra with isometric involution) and let Π be a holomorphic $*$-representation of $\mathcal{A}_{\odot}$ (resp. $\mathcal{A}_{\bullet}$) in a complex Hilbert space $\mathcal{K}$. Then for any $k \geq 0$, there is a $*$-representation $\Pi_k \in P_k(\mathcal{A}, \mathcal{K})$ of $\mathcal{A}_{\odot}$ such that*

(i) $\Pi_k(a)\Pi_l(b) = 0$, $k \neq l$, $a, b \in \mathcal{A}$,

(ii) $\Pi(a) = \sum_{k=0}^{\infty} \Pi_k(a)$ (*norm convergence*), $a \in \mathcal{A}$ (*resp.* $a \in \mathcal{A}_{\bullet}$).

If $\mathcal{A}$ has a unit e, then for every $k \geq 0$, $\Pi_k(e)$ is an orthogonal projection reducing Π to Π_k and $\Pi_k(e)\Pi_l(e) = 0$ for all $k \neq l$.

Proof. Because of similarity, we consider here only the case of the $*$-semigroup $\mathcal{A}_{\bullet}$. The mapping Π, being holomorphic, has the representation (cf. [8], Proposition 5.5)

$$\Pi(a) = \sum_{k=0}^{\infty} \Pi_k(a), \qquad a \in \mathcal{A}_{\bullet},$$

where $\Pi_k \in P_k(\mathcal{A}, \mathcal{K})$ and $\Pi_k(a) = (k!)^{-1}(d^k/dz^k)\Pi(za)|_{z=0}$ for $a \in \mathcal{A}$. Since Π preserves multiplication, the following equalities hold:

$$\begin{aligned}\Pi_k(ab) &= (k!)^{-1}\frac{d^k}{dz^k}\Pi(zab)\Big|_{z=0} = \Big((k!)^{-1}\frac{d^k}{dz^k}\Pi(za)\Big|_{z=0}\Big)\Pi(b)\\ &= \Pi_k(a)\Pi(b), \qquad a \in \mathcal{A}, \ b \in \mathcal{A}_{\bullet}, \ k \geq 0.\end{aligned}$$

Thus

$$\Pi_k(ab) = \Pi_k(a)\Pi(b), \qquad a \in \mathcal{A}, \ b \in \mathcal{A}_{\bullet}, \ k \geq 0. \tag{9.1}$$

By a similar argument we show that

$$\Pi_k(ab) = \Pi(a)\Pi_k(b), \qquad a \in \mathcal{A}_{\bullet}, \ b \in \mathcal{A}, \ k \geq 0. \tag{9.2}$$

Since Π_k is k-homogeneous and (9.1) holds, we get

$$\begin{aligned}\Pi_k(a)\Pi_l(b) &= (l!)^{-1}\frac{d^l}{dz^l}\Pi_k(a)\Pi(zb)\Big|_{z=0} = (l!)^{-1}\frac{d^l}{dz^l}\Pi_k(zab)\Big|_{z=0}\\ &= (l!)^{-1}\Big(\frac{d^l}{dz^l}z^k\Big|_{z=0}\Big)\Pi_k(ab), \qquad a, b \in \mathcal{A}, \ k, l \geq 0.\end{aligned}$$

Thus

$$\Pi_k(a)\Pi_l(b) = \begin{cases}\Pi_k(ab) & \text{if } k = l,\\ 0 & \text{if } k \neq l,\end{cases} \qquad a, b \in \mathcal{A}, \tag{9.3}$$

which shows (i). To prove that Π_k preserves involution notice that if Θ is a holomorphic $B(\mathcal{K})$-valued function on an open disc $D \subset \mathbb{C}$ centered at zero, then

$$\frac{d^k \Theta^\dagger}{dz^k} = \left(\frac{d^k \Theta}{dz^k}\right)^\dagger, \quad k \geq 0,$$

where $\Theta^\dagger(z) := \Theta(\bar{z})^*$ for $z \in D$. In particular, setting $\Theta(z) := \Pi(za)$ for $|z| < \|a\|^{-1}$, a being a nonzero element of $\mathcal{A}$, we get

$$\Pi_k(a)^* = (k!)^{-1}\left(\frac{d^k\Theta}{dz^k}\right)^\dagger(0) = (k!)^{-1}\frac{d^k\Theta^\dagger}{dz^k}(0) = \Pi_k(a^*), \quad a \in \mathcal{A}.$$

The last equality is a consequence of

$$\Theta^\dagger(z) = \Pi(\bar{z}a)^* = \Pi(za^*), \quad |z| < \|a\|^{-1}.$$

This completes the proof of the first part of the conclusion.

Assume now that $\mathcal{A}$ has a unit e. Since any Π_k is a ∗-representation of $\mathcal{A}_\odot$, $\Pi_k(e)$ is an orthogonal projection. Moreover, by (9.1) and (9.2), we have

$$\Pi_k(e)\Pi(a) = \Pi_k(ea) = \Pi_k(a) = \Pi_k(ae) = \Pi(a)\Pi_k(e), \quad a \in \mathcal{A}_\bullet, \ k \geq 0,$$

which implies that for every $k \geq 0$, the orthogonal projection $\Pi_k(e)$ reduces Π to Π_k. This and (9.3) complete the proof. ■

Let $\mathcal{A}$ be a commutative Banach ∗-algebra with isometric involution and a unit e. Given $k \geq 0$, denote by $\Sigma^{\mathrm{h}}(\mathcal{A}_\odot)$ (resp. $\Sigma^{\mathrm{h}}(\mathcal{A}_\bullet)$) the set of all holomorphic characters of $\mathcal{A}_\odot$ (resp. $\mathcal{A}_\bullet$). Set $\Sigma^{\mathrm{h}}_k(\mathcal{A}_\odot) := \Sigma^{\mathrm{h}}(\mathcal{A}_\odot) \cap P_k(\mathcal{A})$ with $P_k(\mathcal{A}) := P_k(\mathcal{A}, \mathbb{C})$ and $\Sigma^{\mathrm{h}}_k(\mathcal{A}_\bullet) := \Sigma^{\mathrm{h}}(\mathcal{A}_\bullet) \cap P_k(\mathcal{A}_\bullet)$ with $P_k(\mathcal{A}_\bullet) := P_k(\mathcal{A})|_{\mathcal{A}_\bullet}$. Then the topological spaces $\Sigma^{\mathrm{h}}_k(\mathcal{A}_\odot)$ and $\Sigma^{\mathrm{h}}_k(\mathcal{A}_\bullet)$ are homeomorphic via the usual restriction mapping. Moreover, $\Sigma^{\mathrm{h}}(\mathcal{A}_\bullet)$ can be described as follows (the case of $\mathcal{A}_\odot$ can be treated similarly).

PROPOSITION 9.2. *If $\mathcal{A}$ is a unital commutative Banach ∗-algebra with isometric involution, then*

(i) *for every $k \geq 0$, $\Sigma^{\mathrm{h}}_k(\mathcal{A}_\bullet)$ is a compact subset of $\Sigma(\mathcal{A}_\bullet)$*,
(ii) $\Sigma^{\mathrm{h}}(\mathcal{A}_\bullet) = \bigcup_{k=0}^\infty \Sigma^{\mathrm{h}}_k(\mathcal{A}_\bullet)$,
(iii) $\Sigma^{\mathrm{h}}_k(\mathcal{A}_\bullet) \cap \Sigma^{\mathrm{h}}_j(\mathcal{A}_\bullet) = \emptyset$ *for* $k \neq j$,
(iv) $\Sigma^{\mathrm{h}}(\mathcal{A}_\bullet) \cup \{\mathbf{0}\}$ *is compact.*

Proof. Let e be the unit of $\mathcal{A}$. Notice that

$$x(e) = 1 = \|x\|, \quad x \in \Sigma^{\mathrm{h}}_k(\mathcal{A}_\odot), \ k \geq 1. \tag{9.4}$$

Indeed, since x is multiplicative and bounded on $\mathcal{A}_\bullet$, we must have $\|x\| \leq 1$. On the other hand, $x(e)^2 = x(e)$, so $x(e) = 1$ (because otherwise $x = \mathbf{0} \notin \Sigma^{\mathrm{h}}_k(\mathcal{A}_\odot)$).

(i) Since $\Sigma_0^{\mathrm{h}}(\mathcal{A}_\bullet) = \{\mathbf{1}\}$, we can assume that $k \geq 1$. It follows from (9.4) that $\Sigma_k^{\mathrm{h}}(\mathcal{A}_\bullet) \subset \bigtimes\{C_a : a \in \mathcal{A}_\bullet\}$, where $C_a = \{z \in \mathbb{C} : |z| \leq 1\}$ for $a \in \mathcal{A}_\bullet$. By the Tikhonov theorem, $\bigtimes\{C_a : a \in \mathcal{A}_\bullet\}$ is compact in the topology of pointwise convergence on $\mathcal{A}_\bullet$. Thus we only have to prove that $\Sigma_k^{\mathrm{h}}(\mathcal{A}_\bullet)$ is closed in $\bigtimes\{C_a : a \in \mathcal{A}_\bullet\}$. Take a net $\{x_\omega\} \subset \Sigma_k^{\mathrm{h}}(\mathcal{A}_\odot)$ converging pointwise to a function $x : \mathcal{A} \to \mathbb{C}$. By the polarization formula, the net $\{x_\omega^{\#}\}$ converges pointwise on $\mathcal{A}^k$ to a function y, which is k-linear, symmetric and $x(a) = y(a, \ldots, a)$ for $a \in \mathcal{A}$. This means that x is a k-homogeneous polynomial. Moreover, by (9.4), we have $\|x\| \leq 1$. Thus $x \in P_k(\mathcal{A})$. Using again (9.4), we show that $x \neq \mathbf{0}$, which yields $x \in \Sigma_k^{\mathrm{h}}(\mathcal{A}_\odot)$.

(ii) and (iii) can be easily deduced from Lemma 9.1. So we only have to show that $\Sigma^{\mathrm{h}}(\mathcal{A}_\bullet) \cup \{\mathbf{0}\}$ is compact. Take a net $\{x_\omega : \omega \in \Omega\} \subset \Sigma^{\mathrm{h}}(\mathcal{A}_\bullet) \cup \{\mathbf{0}\}$. Set $\Omega_k := \{\omega \in \Omega : x_\omega \in \Sigma_k^{\mathrm{h}}(\mathcal{A}_\bullet)\}$ $(k \geq 0)$, $\Omega_\infty := \{\omega \in \Omega : x_\omega = \mathbf{0}\}$. Assume first that there exists $k \in \mathbb{N} \cup \{\infty\}$ ($\mathbb{N} := \{0, 1, \ldots\}$) such that Ω_k is cofinal with Ω (i.e. for every $\omega \in \Omega$ there exists $\omega' \in \Omega_k$ such that $\omega' \geq \omega$). Then $\{x_\omega : \omega \in \Omega_k\}$ is a subnet of $\{x_\omega : \omega \in \Omega\}$ which consists of elements of the compact set $\Sigma_k^{\mathrm{h}}(\mathcal{A}_\bullet)$, where $\Sigma_\infty^{\mathrm{h}}(\mathcal{A}_\bullet) := \{\mathbf{0}\}$. Consequently, there exists a subnet of $\{x_\omega\}$ which converges to an element of $\Sigma_k^{\mathrm{h}}(\mathcal{A}_\bullet)$.

Consider now the other possibility, where for every $k \in \mathbb{N} \cup \{\infty\}$ there exists $\omega_k \in \Omega$ such that $\{\omega \in \Omega : \omega \geq \omega_k\} \cap \Omega_k = \emptyset$. We show that $\{x_\omega\}$ converges to $\mathbf{0}$. Without loss of generality we can assume that $\{\omega_k\}$ is increasing. Take $a \in \mathcal{A}_\bullet$ and $\varepsilon > 0$. Then there exists $k \in \mathbb{N}$ such that $\|a\|^k \leq \varepsilon$. If $\omega \geq \omega_k$, then either $\omega \in \Omega_\infty$ and then $|x_\omega(a)| = 0 \leq \varepsilon$, or $\omega \in \Omega_j$ with $j \geq k+1$ and then $|x_\omega(a)| \leq \|a\|^j \leq \|a\|^k \leq \varepsilon$ because of (9.4). In both cases $\{x_\omega(a)\}$ converges to 0 for every $a \in \mathcal{A}_\bullet$. ■

It is worthwhile to notice that if a Banach $*$-algebra $\mathcal{A}$ has an isometric involution and a bounded approximate identity bounded by 1, then $\mathcal{A}_\bullet = \mathcal{A}_\bullet^{(2)}$. Indeed, if $a \in \mathcal{A}_\bullet$, then there exists $\alpha > 0$ such that $\|a\| < \alpha^2 < 1$. Applying Theorem 4.3 of [16] to $\alpha^{-2}a$, we get $b, c \in \mathcal{A}$ such that $\alpha^{-2}a = bc$, $\|b\| \leq 1$ and $\|c\| \leq 1$. Thus $a = (\alpha b)(\alpha c) \in \mathcal{A}_\bullet^{(2)}$. In particular, if $\mathcal{A}$ has a unit, then $\mathcal{A}_\bullet = \mathcal{A}_\bullet^{(2)}$; hence Theorem 7.1 can be applied to any predilatable mapping on $\mathcal{A}_\bullet$.

THEOREM 9.3. *Let $\mathcal{A}$ be a unital commutative Banach $*$-algebra with isometric involution. Then for a mapping $\Theta : \mathcal{A}_\bullet \to B(\mathcal{H})$, the following conditions are equivalent*:

(i) *for every $f \in \mathcal{H}$, $\langle \Theta(\cdot)f, f\rangle$ is a positive definite holomorphic function*,

(ii) *Θ is a predilatable holomorphic mapping*,

(iii) *Θ has a representing measure supported by $\Sigma^{\mathrm{h}}(\mathcal{A}_\bullet)$*,

(iv) *there exists a* (*unique*) *sequence of polynomials $\Theta_k \in P_k(\mathcal{A}, \mathcal{H})$*,

$k \geq 0$, *such that each* Θ_k *is positive definite and*

$$\Theta(a) = \sum_{k=0}^{\infty} \Theta_k(a) \quad (\textit{norm convergence}), \quad a \in \mathcal{A}_\bullet .$$

A mapping $\Theta : \mathcal{A} \to B(\mathcal{H})$ *is positive definite continuous and* k*-homogeneous if and only if it has a representing measure supported by* $\Sigma^{\mathrm{h}}_k(\mathcal{A}_\odot)$, $k \geq 0$.

P r o o f. (i)⇒(ii). Since Θ is a weakly holomorphic operator-valued mapping, it is holomorphic (cf. [12], Exercise 14C(b)). In particular, it is continuous, so there exist $\alpha \in (0,1)$ and $\beta > 0$ such that $\|\Theta(a)\| \leq \beta$ for every $a \in \alpha \cdot \mathcal{A}_\bullet$. Take $a \in \mathcal{A}_\bullet$. Then $(a^*a)^n \in \alpha \cdot \mathcal{A}_\bullet$ for n large enough and consequently

$$\lim_{n\to\infty} \langle \Theta((a^*a)^n)f, f\rangle^{1/2n} \leq \lim_{n\to\infty} (\beta\|f\|^2)^{1/2n} \leq 1, \quad f \in \mathcal{H}. \tag{9.5}$$

Thus, by Theorems 3.1 and 6.1, Θ is predilatable.

(ii)⇒(iii). Assume that Θ is a predilatable holomorphic mapping. Take a minimal propagator $(\mathcal{K}, X, \Pi)$ of Φ^Θ. Since (9.5) holds for all $a \in \mathcal{A}_\bullet$, Theorem 3.1 gives

$$\|\Pi(a)\| \leq 1, \quad a \in \mathcal{A}_\bullet . \tag{9.6}$$

Now we show that Π is holomorphic. If $f = X(b)h$ with $b \in \mathcal{A}_\bullet$ and $h \in \mathcal{H}$, then the function $\langle \Pi(\cdot)f, f\rangle = \langle \Theta(b^*(\cdot)b)h, h\rangle$ is holomorphic. This implies that $\langle \Pi(\cdot)f, f\rangle$ is holomorphic for every $f \in \mathcal{E}_X$. If $f \in \mathcal{K}$, then there exists a sequence $\{f_n\}_{n=1}^\infty \subset \mathcal{E}_X$ convergent to f. Since, in virtue of (9.6), we have

$$\begin{aligned} &|\langle \Pi(a)f, f\rangle - \langle \Pi(a)f_n, f_n\rangle| \\ &\quad \leq |\langle \Pi(a)f, f\rangle - \langle \Pi(a)f, f_n\rangle| + |\langle \Pi(a)f, f_n\rangle - \langle \Pi(a)f_n, f_n\rangle| \\ &\quad \leq (\|f\| + \|f_n\|)\|f - f_n\|, \quad a \in \mathcal{A}_\bullet, \end{aligned}$$

$\langle \Pi(\cdot)f_n, f_n\rangle$ converges uniformly on $\mathcal{A}_\bullet$ to $\langle \Pi(\cdot)f, f\rangle$. Thus, by Proposition 6.5 of [8], the function $\langle \Pi(\cdot)f, f\rangle$ is holomorphic for all $f \in \mathcal{K}$. This in turn implies that Π is holomorphic itself. Let $\mathcal{W}$ be the C^*-algebra generated by $\Pi(\mathcal{A}_\bullet) \cup \{I_\mathcal{K}\}$. As we know (see the proof of Proposition 8.5) the representing measure M of Θ is supported by a locally compact set $A_\Pi := \{x \circ \Pi : x \in \Sigma(\mathcal{W})\} \setminus \{\mathbf{0}\}$. Since the superposition of holomorphic mappings is again holomorphic (cf. [8], Theorem 6.4), we get $A_\Pi \subset \Sigma^{\mathrm{h}}(\mathcal{A}_\bullet)$, which proves (iii).

(iii)⇒(iv). Assume now that Θ has a representing measure M supported by $\Sigma^{\mathrm{h}}(\mathcal{A}_\bullet)$. It follows from Proposition 9.2(i) that the restriction of the measure M to $\mathfrak{B}(\Sigma^{\mathrm{h}}_k(\mathcal{A}_\bullet))$ is bounded. Consequently, so is its homeomorphic image M_k on $\mathfrak{B}(\Sigma^{\mathrm{h}}_k(\mathcal{A}_\odot))$. Using again Proposition 9.2 and Theorem A.3 of the appendix, we get

$$\Theta(a)f = \sum_{k=0}^{\infty} \Theta_k(a)f \quad \text{(norm convergence)}, \qquad a \in \mathcal{A}_\bullet,\ f \in \mathcal{H}, \tag{9.7}$$

where

$$\Theta_k(a) = \int\limits_{\Sigma_k^{\mathrm{h}}(\mathcal{A}_\odot)} x(a)\, M_k\,(dx), \qquad a \in \mathcal{A},\ k \geq 0. \tag{9.8}$$

In virtue of Theorem 7.1, each Θ_k is positive definite. Applying the polarization formula to members of $\Sigma_k^{\mathrm{h}}(\mathcal{A}_\odot)$ and using the integral formula (9.8), one can show that Θ_k is a k-homogeneous polynomial. Since M_k is bounded, (9.4) yields $\sup\{\|\Theta_k(a)\| : \|a\| \leq 1\} < \infty$, which implies the continuity of Θ_k ($k \geq 0$). It follows from Theorem 5.2 of [8] and (9.7) that the vector-valued mapping $\Theta(\cdot)f$ is holomorphic on $\mathcal{A}_\bullet$ for all $f \in \mathcal{H}$. Thus Θ is holomorphic and consequently $\sum_{k=0}^{\infty} \Theta_k(a)$ converges in the operator norm topology for every $a \in \mathcal{A}_\bullet$.

(iv)$\Rightarrow$(i) is obvious. The last part of the conclusion can be deduced from the above discussion. ∎

10. Holomorphic positive definite mappings on noncommutative Banach $*$-algebras. Theorem 9.3 states, among other things, that any holomorphic positive definite mapping on the open unit ball of a commutative Banach $*$-algebra with isometric involution can be represented as a series of holomorphic homogeneous polynomials which are positive definite. Below we show that the same is true for the noncommutative case.

To begin with consider the C^*-algebra $\mathbb{C}^m$ of all complex m-tuples with coordinatewise defined algebraic operations, equipped with the supremum norm (note that $(\mathbb{C}^m)_\bullet = \mathbb{D}^m$, where $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$). Then Theorem 9.3 leads to the following result (its one-dimensional version has been announced in [46] without proof; see also [11] and [50] for scalar versions). Below $\mathbb{N} = \{0, 1, \ldots\}$.

Proposition 10.1. *Let Θ be a $B(\mathcal{H})$-valued mapping defined on $\mathbb{D}^m$ (resp. $\mathbb{C}^m$), $m \geq 1$. Assume that for every $f \in \mathcal{H}$, $\langle \Theta(\cdot)f, f\rangle$ is holomorphic and positive definite. Then there is a net $\{T_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^m\} \subset B(\mathcal{H})$ of positive operators such that*

$$\Theta(\mathbf{z}) = \sum_{\mathbf{n} \in \mathbb{N}^m} \mathbf{z}^{\mathbf{n}} T_{\mathbf{n}}, \qquad \mathbf{z} \in \mathbb{D}^m \ (\textit{resp. } \mathbf{z} \in \mathbb{C}^m), \tag{i}$$

where the series converges in the operator norm topology.

Proof. Consider first the case of $\mathbb{D}^m$. It is easy to see that $\Sigma_k^{\mathrm{h}}(\mathbb{C}^m_\odot)$ consists of monomials of the form $\mathbb{C}^m \ni \mathbf{z} \to \mathbf{z}^{\mathbf{n}} \in \mathbb{C}$, where $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{N}^m$ and $|\mathbf{n}| := \sum_{j=1}^m n_j = k$. Thus, by Theorem 9.3 and Theorem A.3 of the appendix, there exists a net $\{T_{\mathbf{n}} : \mathbf{n} \in \mathbb{N}^m\} \subset B(\mathcal{H})$ of positive operators such that for each $\mathbf{z} \in \mathbb{D}^m$, the series $\sum_{\mathbf{n}} \mathbf{z}^{\mathbf{n}} T_{\mathbf{n}}$ converges unconditionally to $\Theta(\mathbf{z})$ in the strong operator topology. We show that the series is in fact convergent in the norm topology (our proof is a modification of that of Lemma 1 of [1]). Notice first that since all the operators $T_{\mathbf{n}}$ are positive,

$$\|\mathbf{z}^{\mathbf{n}} T_{\mathbf{n}}\| \leq \|\Theta(|\mathbf{z}|)\|, \quad \mathbf{z} \in \mathbb{D}^m, \ \mathbf{n} \in \mathbb{N}^m, \tag{10.1}$$

where $|\mathbf{z}| := (|z_1|, \ldots, |z_m|)$ for $\mathbf{z} = (z_1, \ldots, z_m) \in \mathbb{D}^m$. If $\mathbf{z} \in \mathbb{D}^m$, then there exists $\alpha \in \mathbb{R}$ such that $|z_j| < \alpha < 1$ for every $j = 1, \ldots, m$. Thus, in view of (10.1),

$$\begin{aligned}\sum_{\mathbf{n}} \|\mathbf{z}^{\mathbf{n}} T_{\mathbf{n}}\| &= \sum_{\mathbf{n}} \alpha^{|\mathbf{n}|} \|(\alpha^{-1}\mathbf{z})^{\mathbf{n}} T_{\mathbf{n}}\| \leq \sum_{\mathbf{n}} \alpha^{|\mathbf{n}|} \|\Theta(\alpha^{-1}|\mathbf{z}|)\| \\ &= (1-\alpha)^{-m} \|\Theta(\alpha^{-1}|\mathbf{z}|)\| < \infty.\end{aligned}$$

If Θ is defined on $\mathbb{C}^m$, then the functions $\Theta(\alpha(\cdot))$, $\alpha > 0$, being holomorphic and positive definite on $\mathbb{D}^m$, are of the form (i). Hence the uniqueness of Taylor's expansion leads to the conclusion. ∎

We are now in a position to prove the aforementioned result (it resembles Theorem 2 of [1]).

THEOREM 10.2. *Let $\mathcal{A}$ be a Banach $*$-algebra with isometric involution* (*resp. a Banach $*$-algebra*). *Then a mapping $\Theta : \mathcal{A}_\bullet \to B(\mathcal{H})$* (*resp. $\Theta : \mathcal{A} \to B(\mathcal{H})$*) *is positive definite and holomorphic if and only if there exists a* (*unique*) *sequence of polynomials $\Theta_k \in P_k(\mathcal{A}, \mathcal{H})$* ($k \geq 0$) *such that*

(i) *for every $k \geq 0$, Θ_k is positive definite*,

(ii) $\Theta(a) = \sum_{k=0}^{\infty} \Theta_k(a)$ (*norm convergence*), $a \in \mathcal{A}_\bullet$ (*resp. $a \in \mathcal{A}$*).

Moreover, *if $\Theta : \mathcal{A}_\bullet \to B(\mathcal{H})$ is positive definite and holomorphic*, *then it is predilatable.*

Proof. Assume Θ is defined on $\mathcal{A}_\bullet$ (the other case can be proved similarly). The "if" part of the conclusion follows from Theorem 5.2 of [8]. To prove the "only if" part assume that Θ is positive definite and holomorphic. Then the continuous k-homogeneous polynomials $\Theta_k \in P_k(\mathcal{A}, \mathcal{H})$, $k \geq 0$, given by

$$\Theta_k(a) = (k!)^{-1} \frac{d^k}{dz^k} \Theta(za)\bigg|_{z=0}, \quad a \in \mathcal{A}, \ k \geq 0, \tag{10.2}$$

fulfil (ii) (use Proposition 5.5 of [8]). Thus all we have to prove is that each Θ_k is positive definite. For an n-tuple $\mathbf{a} = (a_1, \ldots, a_n)$ of elements of $\mathcal{A}_\bullet$ we define a holomorphic function $\Psi_{\mathbf{a}} : \mathbb{D} \to B(\mathcal{H}^n)$ by

$$\Psi_{\mathbf{a}}(z) := [\Theta(za_q^* a_p)]_{p,q=1}^n, \quad z \in \mathbb{D}.$$

Since Θ is positive definite, for all finite sequences $\{f_i\}_{i=1}^m \subset \mathcal{H}^n$ and $\{z_i\}_{i=1}^m \subset \mathbb{D}$,

$$\sum_{i,j=1}^{m} \langle \Psi_{\mathbf{a}}(z_j^* z_i) f_i, f_j\rangle = \sum_{i=1}^{m}\sum_{p=1}^{n}\sum_{j=1}^{m}\sum_{q=1}^{n} \langle \Theta((z_j a_q)^*(z_i a_p)) f_{i,p}, f_{j,q}\rangle \geq 0\,,$$

where $f_i = f_{i,1} \oplus \ldots \oplus f_{i,n}$. Hence $\Psi_{\mathbf{a}}$ is a positive definite holomorphic function on $\mathbb{D}$. By Proposition 10.1, there are positive operators $T_k \in B(\mathcal{H}^n)$, $k \geq 0$, (depending on $\mathbf{a}$) such that

$$\Psi_{\mathbf{a}}(z) = \sum_{k=0}^{\infty} T_k z^k\,, \qquad z \in \mathbb{D}\,.$$

Since $T_k = (k!)^{-1}(d^k/dz^k)\Psi_{\mathbf{a}}(z)|_{z=0}$, we can apply (10.2) to get

$$\begin{aligned}[\Theta_k(a_q^* a_p)]_{p,q=1}^n &= \left[(k!)^{-1}\frac{d^k}{dz^k}\Theta(z a_q^* a_p)\bigg|_{z=0}\right]_{p,q=1}^n \\ &= (k!)^{-1}\frac{d^k}{dz^k}[\Theta(z a_q^* a_p)]_{p,q=1}^n\bigg|_{z=0} = T_k \geq 0\,, \qquad k \geq 0\,.\end{aligned}$$

However, Θ_k is k-homogeneous, so the matrix $[\Theta_k(a_q^* a_p)]_{p,q=1}^n$ is positive for all $a_1, \ldots, a_n \in \mathcal{A}$ and $n \geq 1$. This is equivalent to the positive definiteness of Θ_k.

That Θ is predilatable can be proved similarly to the implication (i)⇒(ii) of Theorem 9.3, but now we must use the criterion (ii) of Theorem 3.1. ∎

Theorem 10.2 states that any positive definite holomorphic mapping defined on $\mathcal{A}_\bullet$ is automatically predilatable. On the other hand, basing on Theorem 3.1, one can show that if $\mathcal{A}$ is an arbitrary Banach $*$-algebra, then any positive definite k-homogeneous polynomial $\Theta \in P_k(\mathcal{A}, \mathcal{H})$ is predilatable. However, this is not the case for all positive definite holomorphic mappings on the $*$-semigroup $\mathcal{A}_\odot$.

Example 10.3. Consider $\mathcal{A} = \mathbb{C}$, $\mathcal{H} = \mathbb{C}$ and $\Theta : \mathcal{A} \to B(\mathcal{H})$ given by

$$\Theta(a) = \exp(a)\,, \qquad a \in \mathcal{A}\,.$$

Then Θ is a positive definite holomorphic function. However, it is not predilatable. Suppose, by contradiction, that Θ has a minimal propagator $(\mathcal{K}, X, \Pi)$. Then

$$\begin{aligned}\exp(|b|^2(|a|^2-1)) &= \|\Pi(a)X(b)\|^2 \exp(-|b|^2) \\ &\leq \|\Pi(a)\|^2\|X(b)\|^2\exp(-|b|^2) = \|\Pi(a)\|^2\,, \qquad a, b \in \mathcal{A}\,,\end{aligned}$$

which is impossible when $|a| > 1$. ∎

Following Szafraniec [57] we give examples of positive definite holomorphic mappings on $\mathcal{A}_\bullet$ which are not dilatable (though they are predilatable).

EXAMPLE 10.4. Suppose $\Psi : \mathcal{A} \to B(\mathcal{H})$ is a positive linear mapping of a C^*-algebra $\mathcal{A}$ with a unit e such that the range of Ψ is commutative and $\Psi(e) = I_{\mathcal{H}}$. Then, by Proposition 1.2.2 of [2] (see also [27], Proposition 9.5), Ψ is dilatable and

$$\|\Psi(a)\| \le \|a\|, \quad a \in \mathcal{A}.$$

Consequently, for any $a \in \mathcal{A}_\bullet$, the operator $\Theta(a) := (I_{\mathcal{H}} - \Psi(a))^{-1}$ exists and $\Theta(a) = \sum_{k=0}^{\infty} \Psi(a)^k$. It is clear that $\Theta : \mathcal{A}_\bullet \to B(\mathcal{H})$ is holomorphic. Moreover, according to Proposition 4 of [57], the polynomials $\Psi(\cdot)^k$ are positive definite and consequently so is Θ. By Theorem 10.2, Θ is predilatable. However, contrary to Corollary 2 in [57], Θ is not dilatable, because Θ is not bounded on $\mathcal{A}_\bullet$. ∎

11. Completely positive k-linear mappings. In this and the next two sections we discuss the relationship between the notions of positive definiteness and complete positivity. Let us recall the definitions. Given two C^*-algebras $\mathcal{A}$ and $\mathcal{B}$, we say (following [1]) that a mapping $\Theta : \mathcal{A} \to \mathcal{B}$ (resp. $\Theta : \mathcal{A}_\bullet \to \mathcal{B}$) is *completely positive* if for every $n \ge 1$, the n-square $\mathcal{B}$-valued matrix $[\Theta(a_{i,j})]_{ij}$ is positive whenever so is the n-square $\mathcal{A}$-valued (resp. $\mathcal{A}_\bullet$-valued) matrix $[a_{i,j}]_{ij}$. Similarly a mapping $\Theta : \mathcal{A}^k \to \mathcal{B}$ is said to be *completely positive* ($k \ge 1$) if the n-square $B(\mathcal{H})$-valued matrix $[\Theta(a_{1,ij}, \ldots, a_{k,ij})]_{ij}$ is positive whenever so are the n-square $\mathcal{A}$-valued matrices $[a_{p,ij}]_{ij}$ for $p = 1, \ldots, k$. It turns out that any completely positive mapping is automatically positive definite. The question is whether the reverse implication holds within the class of holomorphic mappings.

First we answer the question in the affirmative for k-linear mappings defined on C^*-algebras (recall that the positive definiteness of a mapping $\Theta : \mathcal{A}^k \to B(\mathcal{H})$ is understood with respect to the direct product $\mathcal{A}_\odot^k$ of k copies of the $*$-semigroup $\mathcal{A}_\odot$).

PROPOSITION 11.1. *Let $\mathcal{A}$ be a Banach $*$-algebra with continuous involution and with a bounded two-sided approximate identity, and let $\Theta : \mathcal{A}^k \to B(\mathcal{H})$ be a k-linear mapping, $k \ge 1$. Then the following conditions are equivalent*:

(i) *Θ is positive definite*,

(ii) *Θ is continuous and positive definite*,

(iii) *there are a complex Hilbert space $\mathcal{K}$, an operator $R \in B(\mathcal{H}, \mathcal{K})$ and linear $*$-representations $\Pi_j : \mathcal{A} \to B(\mathcal{K})$ $(1 \le j \le k)$ such that*

$$\Pi_i(a)\Pi_j(b) = \Pi_j(b)\Pi_i(a), \quad i \ne j, \ a, b \in \mathcal{A}, \tag{11.1}$$

$$\Theta(a_1, \ldots, a_k) = R^* \Pi_1(a_1) \ldots \Pi_k(a_k) R, \quad a_1, \ldots, a_k \in \mathcal{A}. \tag{11.2}$$

If $\mathcal{A}$ is a C^*-algebra, then (i) *holds if and only if Θ is completely positive.*

Proof. (i)⇒(ii). Suppose that Θ is positive definite. If $k = 1$, then the continuity of Θ follows from Theorem 9.3 of [27] and Theorem 37.3 of [9]. So assume $k > 1$. Since the linear mappings $\Theta(a_1^*a_1, \ldots, a_{k-1}^*a_{k-1}, \cdot)$ $(a_1, \ldots, a_{k-1} \in \mathcal{A})$ are positive definite, they are continuous. However,

$$a^*b = 2^{-1}[(a+b)^*(a+b)-a^*a-b^*b-i((a+ib)^*(a+ib)-a^*a-b^*b)]\,, \quad a, b \in \mathcal{A},$$

so the linear mappings $\Theta(a_1^*b_1, \ldots, a_{k-1}^*b_{k-1}, \cdot)$ $(a_1, \ldots, a_{k-1} \in \mathcal{A}, b_1, \ldots \ldots, b_{k-1} \in \mathcal{A})$ are continuous as well. Using the Cohen factorization theorem ([9], Theorem 11.10), we conclude that all linear mappings $\Theta(a_1, \ldots \ldots, a_{k-1}, \cdot)$ $(a_1, \ldots, a_{k-1} \in \mathcal{A})$ are continuous. In the same way we show that Θ is continuous with respect to any other variable with the remaining ones being fix. Thus (cf. [34], Theorem 2.17) the multilinear map Θ is jointly continuous.

(ii)⇒(iii). Suppose that Θ is continuous and positive definite. The proof of (ii)⇒(iii) will be divided into a few steps.

Step 1. *Θ is dilatable on the $*$-semigroup $\mathcal{A}^k_\odot$.*

It follows from the continuity of Θ that the condition (iii) of Theorem 3.1 holds for $\Phi = \Phi^\Theta$. In the terminology of [27], Θ satisfies the boundedness condition. Since $\mathcal{A}$ has a bounded two-sided approximate identity and Θ is jointly continuous, all the assumptions of Theorem 6.2 of [27] are satisfied. Therefore Θ is dilatable on $\mathcal{A}^k_\odot$.

Take a minimal dilation $(\mathcal{K}, R, \Pi)$ of Θ. It follows from Proposition 6.2(a) of [27] that Π is k-linear. Moreover, by (i)⇒(ii), Π is continuous.

Step 2. *If a bounded net $\{\mathbf{a}_\omega\} \subset \mathcal{A}^k$ is such that all nets of the form $\{\mathbf{a}_\omega\mathbf{b}\}$, $\mathbf{b} \in \mathcal{A}^k$, are convergent in $\mathcal{A}^k$, then the net $\{\Pi(\mathbf{a}_\omega)\}$ is convergent in the strong operator topology. Moreover, if $\{\mathbf{a}'_\omega\} \subset \mathcal{A}^k$ is another bounded net such that* $\lim_\omega \mathbf{a}'_\omega\mathbf{b} = \lim_\omega \mathbf{a}_\omega\mathbf{b}$ *for all* $\mathbf{b} \in \mathcal{A}^k$, *then* (SOT) $\lim_\omega \Pi(\mathbf{a}'_\omega) =$ (SOT) $\lim_\omega \Pi(\mathbf{a}_\omega)$.

Since Π is continuous and $\{\mathbf{a}_\omega\}$ is bounded, the net $\{\Pi(\mathbf{a}_\omega)\}$ is bounded too. Take a vector $g \in \mathcal{K}$ of the form $\Pi(\mathbf{b})Rf$ with $\mathbf{b} \in \mathcal{A}^k$ and $f \in \mathcal{H}$. Then $\|\Pi(\mathbf{a}_\omega)g - \Pi(\mathbf{a}_\tau)g\| = \|\Pi(\mathbf{a}_\omega\mathbf{b})Rf - \Pi(\mathbf{a}_\tau\mathbf{b})Rf\|$ for $\omega, \tau \in \Omega$. This implies that $\{\Pi(\mathbf{a}_\omega)g\}$ is a Cauchy net in $\mathcal{K}$. Since the set of all g's is total in $\mathcal{K}$, the net $\{\Pi(\mathbf{a}_\omega)\}$ is convergent in the strong operator topology. Using the same kind of argument, one can show the other part of Step 2.

Arguing similarly to the previous paragraph we show

Step 3. *If $\{e_\omega\}$ is a bounded two-sided approximate identity of $\mathcal{A}$, then* (SOT) $\lim_{(\omega_1,\ldots,\omega_k)} \Pi(e_{\omega_1}, \ldots, e_{\omega_k}) = I_\mathcal{K}$.

Let $\{e_\omega\}$ be a fixed bounded two-sided approximate identity of $\mathcal{A}$. Set $\Pi_{j,\omega}(a) := \Pi(e_\omega, \ldots, e_\omega, a, e_\omega, \ldots, e_\omega)$ with $a \in \mathcal{A}$ in the jth position, $1 \le j \le k$. In virtue of Step 2, the net $\{\Pi_{j,\omega}(a)\}$ converges in the strong operator topology to an operator $\Pi_j(a)$, $a \in \mathcal{A}$. It is obvious that all the mappings $\Pi_j : \mathcal{A} \to B(\mathcal{K})$ are linear. Moreover, again by Step 2, the definition of $\Pi_j(a)$ does not depend on the choice of a bounded two-sided approximate identity of $\mathcal{A}$.

Step 4. *For every $j = 1, \ldots, k$, Π_j preserves involution.*

To show this, notice that $\{e_\omega^*\}$ is a bounded two-sided approximate identity of $\mathcal{A}$ and $\Pi_{j,\omega}(a)^* = \Pi(e_\omega^*, \ldots, e_\omega^*, a^*, e_\omega^*, \ldots, e_\omega^*)$ for $a \in \mathcal{A}$. Thus for every $a \in \mathcal{A}$, the net $\{\Pi_{j,\omega}(a)^*\}$ converges in the strong operator topology to $\Pi_j(a^*)$. On the other hand, $\{\Pi_{j,\omega}(a)^*\}$ converges in the weak operator topology to $\Pi_j(a)^*$. Thus $\Pi_j(a^*) = \Pi_j(a)^*$ for every $a \in \mathcal{A}$.

Step 5. $\Pi_i(a)\Pi_j(b) = \Pi_j(b)\Pi_i(a)$, $i \ne j$, $a, b \in \mathcal{A}$.

Without loss of generality we can assume that $i = 1$ and $j = 2$. It follows from Step 4 that

$$\begin{aligned}
\langle \Pi_1(a)\Pi_2(b)g, h\rangle &= \langle \Pi_2(b)g, \Pi_1(a^*)h\rangle \\
&= \lim_\omega \langle \Pi(e_\omega, b, e_\omega, \ldots, e_\omega)g, \Pi(a^*, e_\omega^*, e_\omega^*, \ldots, e_\omega^*)h\rangle \\
&= \lim_\omega \langle \Pi(ae_\omega, e_\omega b, e_\omega^2, \ldots, e_\omega^2)g, h\rangle\,, \qquad g, h \in \mathcal{K}\,.
\end{aligned} \tag{11.3}$$

In the same manner we show that

$$\langle \Pi_2(b)\Pi_1(a)g, h\rangle = \lim_\omega \langle \Pi(e_\omega a, be_\omega, e_\omega^2, \ldots, e_\omega^2)g, h\rangle\,, \qquad g, h \in \mathcal{K}\,. \tag{11.4}$$

Since $\{e_\omega^2\}$ is a bounded two-sided approximate identity, we can apply Step 2 to deduce that the nets $\{\Pi(ae_\omega, e_\omega b, e_\omega^2, \ldots, e_\omega^2)\}$ and $\{\Pi(e_\omega a, be_\omega, e_\omega^2, \ldots, e_\omega^2)\}$ are convergent in the strong operator topology to the same limit. This, when combined with (11.3) and (11.4), leads to the desired equality.

The next step can be proved similarly:

Step 6. *For every $j = 1, \ldots, k$, Π_j preserves multiplication.*

To complete the proof of (ii)$\Rightarrow$(iii), it is enough to show that for all $a_1, \ldots, a_k \in \mathcal{A}$, $\Pi(a_1, \ldots, a_k) = \Pi_1(a_1) \ldots \Pi_k(a_k)$. The latter can be proved with the help of Steps 2–4 as follows:

$$\begin{aligned}
\langle \Pi(a_1, \ldots, a_k)g, h\rangle &= \lim_{\omega_1} \langle \Pi(a_1 e_{\omega_1}, e_{\omega_1} a_2, \ldots, e_{\omega_1} a_k)g, h\rangle \\
&= \langle \lim_{\omega_1} \Pi(e_{\omega_1}, a_2, \ldots, a_k)g, \lim_{\omega_1} \Pi(a_1^*, e_{\omega_1}^*, \ldots, e_{\omega_1}^*)h\rangle \\
&= \lim_{\omega_1} \langle \Pi(e_{\omega_1}, a_2, \ldots, a_k)g, \Pi_1(a_1)^* h\rangle \\
&= \lim_{\omega_1} \lim_{\omega_2} \langle \Pi(e_{\omega_2} e_{\omega_1}, a_2 e_{\omega_2}, e_{\omega_2} a_3, \ldots, e_{\omega_2} a_k)g, \Pi_1(a_1)^* h\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \lim_{\omega_1}\lim_{\omega_2}\langle \Pi(e_{\omega_1}, e_{\omega_2}, a_3, \ldots, a_k)g, \Pi(e^*_{\omega_2}, a^*_2, e^*_{\omega_2}, \ldots, e^*_{\omega_2})\Pi_1(a_1)^*h\rangle \\
&= \lim_{\omega_1}\lim_{\omega_2}\langle \Pi(e_{\omega_1}, e_{\omega_2}, a_3, \ldots, a_k)g, \Pi_2(a_2)^*\Pi_1(a_1)^*h\rangle = \ldots \\
&= \lim_{\omega_1}\ldots\lim_{\omega_k}\langle \Pi(e_{\omega_1}, \ldots, e_{\omega_k})g, \Pi_k(a_k)^*\ldots\Pi_1(a_1)^*h\rangle \\
&= \langle \Pi_1(a_1)\ldots\Pi_k(a_k)g, h\rangle\,, \qquad g, h \in \mathcal{K}\,.
\end{aligned}
$$

The implication (iii)⇒(i) is obvious.

Assume that $\mathcal{A}$ is a C^*-algebra. If Θ is positive definite, then it is of the form (11.2) with Π_j satisfying (11.1). Thus, by Proposition 4 of [57], the k-linear mapping $\mathcal{A}^k \ni (a_1, \ldots, a_k) \to \Pi_1(a_1)\ldots\Pi_k(a_k) \in B(\mathcal{K})$ is completely positive. Hence so is Θ. The converse implication can be easily verified with the help of the Gelfand–Naimark theorem. This completes the proof of Proposition 11.1. ∎

In case $\mathcal{A}$ is commutative, positive definite k-linear mappings can be described with the aid of spectral measures. Below $\xi_1 \otimes \ldots \otimes \xi_k$ stands for the k-fold tensor product of functions $\xi_1, \ldots, \xi_k : \mathcal{X} \to \mathbb{C}$ defined on a set $\mathcal{X}$, i.e. $\xi_1 \otimes \ldots \otimes \xi_k(x_1, \ldots, x_k) := \xi_1(x_1)\ldots\xi_k(x_k)\,,\ x_1, \ldots, x_k \in \mathcal{X}\,.$

PROPOSITION 11.2. *Let $\mathcal{A}$ be a commutative Banach $*$-algebra with continuous involution and with a bounded two-sided approximate identity, and let $\Theta : \mathcal{A}^k \to B(\mathcal{H})$ be a k-linear positive definite mapping, $k \geq 1$. Then there exist a complex Hilbert space $\mathcal{K}$, an operator $R \in B(\mathcal{H}, \mathcal{K})$ and a regular spectral measure $E : \mathfrak{B}(\Sigma(\mathcal{A})^k) \to B(\mathcal{K})$ such that*

$$
\Theta(a_1, \ldots, a_k) = R^*\Big(\int_{\Sigma(\mathcal{A})^k} \widehat{a}_1 \otimes \ldots \otimes \widehat{a}_k\, dE\Big)R\,, \qquad a_1, \ldots, a_k \in \mathcal{A}\,.
$$

P r o o f. It follows from Proposition 11.1 that there are a complex Hilbert space $\mathcal{K}$, an operator $R \in B(\mathcal{H}, \mathcal{K})$ and linear $*$-representations $\Pi_j : \mathcal{A} \to B(\mathcal{K})$, $1 \leq j \leq k$, which satisfy (11.1) and (11.2). Let E_j be the spectral measure of Π_j, $1 \leq j \leq k$. The condition (11.1) yields that any two measures E_i and E_j commute. Since $\Sigma(\mathcal{A})$ is a locally compact Hausdorff space and the measures E_j are regular, there exists a (unique) regular spectral measure E on $\mathfrak{B}(\Sigma(\mathcal{A})^k)$ such that (5)

$$
E(A_1 \times \ldots \times A_k) = E_1(A_1)\ldots E_k(A_k)\,, \qquad A_1, \ldots, A_k \in \mathfrak{B}(\Sigma(\mathcal{A}))\,.
$$

This in turn implies that

$$
\Pi_1(a_1)\ldots\Pi_k(a_k) = \int_{\Sigma(\mathcal{A})^k} \widehat{a}_1 \otimes \ldots \otimes \widehat{a}_k\, dE\,, \qquad a_1, \ldots, a_k \in \mathcal{A}\,,
$$

so, by (11.2), the proof is complete. ∎

(5) This can be deduced from Proposition 3 of [48] via Lemmas 11.2 and 11.3 of [4] (see the proof of Proposition 4 of [48]).

12. Multiplicative k-homogeneous polynomials. It is of special interest to know whether k-homogeneous characters of $\mathcal{A}_\odot$ come from k-fold tensor products of (linear) characters of the $*$-algebra $\mathcal{A}$. A partial answer to the question is given in Proposition 12.2. In order to prove it, we need the notion of kth symmetric power of a topological space (see [19] for symmetrization of measurable spaces).

In the sequel $G(\mathcal{X})$ stands for the multiplicative group of all topological automorphisms of a topological space $\mathcal{X}$. Given an integer $k \geq 1$, set $\mathcal{G}_k := G(\{1,\dots,k\})$ and define the group monomorphism $\Delta_k : \mathcal{G}_k \to G(\mathcal{X}^k)$ by

$$\Delta_k(\sigma)(x_1,\dots,x_k) := (x_{\sigma(1)},\dots,x_{\sigma(k)}), \qquad x_1,\dots,x_k \in \mathcal{X}.$$

The mapping Δ_k induces an equivalence relation $\cong$ on $\mathcal{X}$ as follows:

$$\mathbf{x} \cong \mathbf{y} \;\Leftrightarrow\; \exists \sigma \in \mathcal{G}_k : \; \mathbf{y} = \Delta_k(\sigma)\mathbf{x} \qquad (\mathbf{x},\mathbf{y} \in \mathcal{X}^k).$$

In case $\mathcal{X}$ is a locally compact Hausdorff space, the relation $\cong$ can be described in another equivalent way (use Urysohn's lemma; cf. [35], Theorem 2.12):

$$\text{(P.5)}\quad \mathbf{x} \cong \mathbf{y} \Leftrightarrow \forall \xi \in C_0(\mathcal{X}) : \; \xi\otimes\ldots\otimes\xi(\mathbf{x}) = \xi\otimes\ldots\otimes\xi(\mathbf{y}) \qquad (\mathbf{x},\mathbf{y} \in \mathcal{X}^k),$$

where $C_0(\mathcal{X})$ stands for the space of all complex-valued continuous functions on $\mathcal{X}$ which vanish at infinity. Denote by $\Gamma_k : \mathcal{X}^k \to S(\mathcal{X}^k)$ the quotient mapping from $\mathcal{X}^k$ onto the quotient space $S(\mathcal{X}^k) := \mathcal{X}^k/\cong$. Then

$$\text{(P.6)}\qquad \Gamma_k^{-1}\Gamma_k(A) = \mathcal{A} \;\Leftrightarrow\; \forall\, \sigma \in \mathcal{G}_k : \; \Delta_k(\sigma)A = A \qquad (A \subset \mathcal{X}^k).$$

Equip $S(\mathcal{X}^k)$ with the quotient topology (i.e. D is open in $S(\mathcal{X}^k)$ if and only if $\Gamma_k^{-1}(D)$ is open in $\mathcal{X}^k$). It follows from (P.6) that

(P.7) if D is an open subset of $\mathcal{X}^k$ such that $\Delta_k(\sigma)D = D$ for all $\sigma \in \mathcal{G}_k$, then $\Gamma_k(D)$ is an open subset of $S(\mathcal{X}^k)$.

Though, in general, taking quotient spaces makes topology worse, this is not the case for the relation $\cong$ at least from the following point of view.

LEMMA 12.1. *If $\mathcal{X}$ is a compact* (*resp. a locally compact*) *Hausdorff space*, *then so is $S(\mathcal{X}^k)$*, $k \geq 1$.

Proof. To make the proof clearer we write $\mathcal{G}$, Δ and Γ instead of $\mathcal{G}_k$, Δ_k and Γ_k, respectively. Assume $\mathcal{X}$ is a locally compact Hausdorff space. First we show that $S(\mathcal{X}^k)$ is a Hausdorff space. Take $\mathbf{x}_1, \mathbf{x}_2 \in \mathcal{X}^k$ such that $\Gamma(\mathbf{x}_1) \neq \Gamma(\mathbf{x}_2)$. Since $\Gamma(\mathbf{x}_1)$ and $\Gamma(\mathbf{x}_2)$ are finite, there are two disjoint open subsets D_1 and D_2 of $\mathcal{X}^k$ such that $\Gamma(\mathbf{x}_j) \subset D_j$, $j = 1,2$. Define new open sets $A_j := \bigcap\{\Delta(\sigma)D_j : \sigma \in \mathcal{G}\} \subset D_j$, $j = 1,2$. Then $\Delta(\sigma)A_j = A_j$ for all $\sigma \in \mathcal{G}$ and consequently, by (P.7), $\Gamma(A_j)$ is an open subset of $S(\mathcal{X}^k)$, $j = 1,2$. Since for all $\sigma \in \mathcal{G}$, $\mathbf{x}_j \cong \Delta(\sigma)\mathbf{x}_j$, we get $\Delta(\sigma)\mathbf{x}_j \in \Gamma(\mathbf{x}_j) \subset D_j$, which in turn implies that $\mathbf{x}_j \in A_j$, $j = 1,2$. Thus $\Gamma(\mathbf{x}_j) \in \Gamma(A_j)$, $j = 1,2$. However, the sets D_1 and D_2 and consequently A_1 and A_2 are disjoint, so the same remains true for $\Gamma(A_1)$ and $\Gamma(A_2)$. Indeed, if $\Gamma(A_1)\cap\Gamma(A_2)\neq\emptyset$, then, by (P.6), we obtain

$$A_1 \cap A_2 = \Gamma^{-1}\Gamma(A_1)\cap\Gamma^{-1}\Gamma(A_2) = \Gamma^{-1}(\Gamma(A_1)\cap\Gamma(A_2)) \neq \emptyset\,,$$

which contradicts $A_1\cap A_2=\emptyset$.

To show that $S(\mathcal{X}^k)$ is locally compact take $\mathbf{x}\in S(\mathcal{X}^k)$. Then, because $\Gamma(\mathbf{x})$ is finite and $\mathcal{X}^k$ is a locally compact Hausdorff space, there exists an open set D, whose closure $\overline{D}$ is compact, such that $\Gamma(\mathbf{x})\subset D$. Set $A := \bigcap\{\Delta(\sigma)D : \sigma\in\mathcal{G}\}\subset D$. Then, just as in the previous paragraph, we show that $\Gamma(A)$ is an open neighbourhood of $\Gamma(\mathbf{x})$. Since $\Gamma(A)\subset\Gamma(\overline{D})$ and $\Gamma(\overline{D})$ is compact, so is $\overline{\Gamma(A)}$.

Finally, if $\mathcal{X}$ is a compact Hausdorff space, then so is $S(\mathcal{X}^k)$, the image of the compact Hausdorff space $\mathcal{X}^k$ via the continuous mapping Γ_k. ∎

We are now in a position to present a partial answer to the question mentioned at the beginning of this section. Below ι_k, $k\ge 1$, stands for the multiplicative mapping from $\mathcal{A}$ into $\mathcal{A}^k$ defined by

$$\iota_k(a) := (a,\ldots,a), \qquad a\in\mathcal{A}\,.$$

Proposition 12.2. *Let $\mathcal{A}$ be a commutative Banach $*$-algebra with continuous involution, having a bounded two-sided approximate identity, and let $x:\mathcal{A}\to\mathbb{C}$ be an arbitrary function. Then for any $k\ge1$, the following conditions are equivalent*:

(i) *x is a character of $\mathcal{A}_\odot$ of the form $x = y\circ\iota_k$, where $y:\mathcal{A}^k\to\mathbb{C}$ is k-linear and positive definite*,

(ii) *there exist* (*linear*) *characters $x_1,\ldots,x_k$ of the $*$-algebra $\mathcal{A}$ such that $x=(x_1\otimes\ldots\otimes x_k)\circ\iota_k$.*

(iii) *x is a k-homogeneous character of $\mathcal{A}_\odot$ and $x^\#$ is positive definite.*

Moreover, *if x satisfies* (i), *then $x\in\Sigma_k^{\mathrm{h}}(\mathcal{A}_\odot)$.*

Proof. Denote by $\mathcal{X}$ the locally compact Hausdorff space $\Sigma(\mathcal{A})$.

Step 1. *The set $\widehat{\mathcal{A}} := \{\widehat{a} : a\in\mathcal{A}\}$ is uniformly dense in $C_0(\mathcal{X})$.*

For if $a\in\mathcal{A}$ and $\varepsilon>0$, then the set $C_{a,\varepsilon} := \{x\in\mathcal{X} : |x(a)|\ge\varepsilon\}$ is compact in $\mathcal{X}$ (as a closed subset of the compact set $\mathcal{X}\cup\{\mathbf{0}\}$) and $|\widehat{a}(x)|<\varepsilon$ for all $x\in\mathcal{X}\setminus C_{a,\varepsilon}$. Thus $\widehat{a}\in C_0(\mathcal{X})$ for all $a\in\mathcal{A}$. Since the $*$-algebra $\widehat{\mathcal{A}}$ separates the points of $\mathcal{X}$ and for any $x\in\mathcal{X}$, there exists $\xi\in\widehat{\mathcal{A}}$ such that $\xi(x)\neq0$, the denseness of $\widehat{\mathcal{A}}$ in $C_0(\mathcal{X})$ follows from the Stone–Weierstrass theorem.

Notice that for every $\xi\in C_0(\mathcal{X})$, the function $\xi^{\otimes k} := \xi\otimes\ldots\otimes\xi$ is constant on any coset $\Gamma_k(\mathbf{x})$ with $\mathbf{x}\in\mathcal{X}^k$. Thus there exists a unique continuous function $\xi^{\odot k} : S(\mathcal{X}^k)\to\mathbb{C}$ such that $\xi^{\otimes k} = \xi^{\odot k}\circ\Gamma_k$.

S t e p 2. *The linear span $\mathcal{V}$ of the set $\{\widehat{a}^{\odot k} : a \in \mathcal{A}\}$ is uniformly dense in $C_0(S(\mathcal{X}^k))$.*

For if $\xi \in C_0(\mathcal{X})$, then $\xi^{\otimes k} \in C_0(\mathcal{X}^k)$ and consequently $\xi^{\odot k} \in C_0(S(\mathcal{X}^k))$. Thus $\mathcal{V}$ is a symmetric subalgebra of $C_0(S(\mathcal{X}^k))$, which separates points of $S(\mathcal{X}^k)$. The latter is a consequence of the following:

(P.8) $\mathbf{x} \cong \mathbf{y} \Leftrightarrow \forall\, a \in \mathcal{A}: (\widehat{a}\otimes\ldots\otimes\widehat{a})(\mathbf{x}) = (\widehat{a}\otimes\ldots\otimes\widehat{a})(\mathbf{y}) \quad (\mathbf{x}, \mathbf{y} \in \mathcal{X}^k)$.

((P.8) can be deduced from (P.5) via Step 1.) Suppose now that $\mathbf{x} \in \mathcal{X}^k$. Then, by Urysohn's lemma, there exists $\xi \in C_0(\mathcal{X})$ such that $\xi^{\otimes k}(\mathbf{x}) \neq 0$. This and Step 1 enable us to find $a \in \mathcal{A}$ such that $\widehat{a}^{\otimes k}(\mathbf{x}) \neq 0$. In other words, we have proved that for every $x \in S(\mathcal{X}^k)$, there exists $\varphi \in \mathcal{V}$ such that $\varphi(x) \neq 0$. Consequently, by the Stone–Weierstrass theorem, $\mathcal{V}$ is uniformly dense in $C_0(S(\mathcal{X}^k))$.

(i)$\Rightarrow$(ii). Assume that x is a character of $\mathcal{A}_\odot$ of the form $x = y \circ \iota_k$, where $y : \mathcal{A}^k \to \mathbb{C}$ is k-linear and positive definite. Applying Proposition 11.2 to y we get a finite regular positive Borel measure μ on $\mathcal{X}^k$ such that

$$x(a) = \int\limits_{\mathcal{X}^k} \widehat{a}^{\otimes k}\, d\mu\,, \qquad a \in \mathcal{A}\,.$$

Since x is a character of $\mathcal{A}_\odot$, we get

$$\int\limits_{S(\mathcal{X}^k)} \widehat{a}^{\odot k}\widehat{b}^{\odot k}\, d\widetilde{\mu} = \int\limits_{S(\mathcal{X}^k)} \widehat{a}^{\odot k}\, d\widetilde{\mu} \int\limits_{S(\mathcal{X}^k)} \widehat{b}^{\odot k}\, d\widetilde{\mu}\,, \qquad a, b \in \mathcal{A}\,,$$

where $\widetilde{\mu}$ is a regular measure on $S(\mathcal{X}^k)$ defined by $\widetilde{\mu}(A) := \mu(\mathit{\Gamma}_k^{-1}(A))$, $A \in \mathfrak{B}(S(\mathcal{X}^k))$. Applying Step 2 to the above equality we get

$$\int\limits_{S(\mathcal{X}^k)} \varphi\psi\, d\widetilde{\mu} = \int\limits_{S(\mathcal{X}^k)} \varphi\, d\widetilde{\mu} \int\limits_{S(\mathcal{X}^k)} \psi\, d\widetilde{\mu}\,, \qquad \varphi, \psi \in C_0(S(\mathcal{X}^k))\,. \tag{12.1}$$

Let C be the closed support of $\widetilde{\mu}$. Since x is nonzero, $C \neq \emptyset$. Suppose that there are $x_1, x_2 \in C$ such that $x_1 \neq x_2$. Then, by Urysohn's lemma, there are nonnegative functions $\varphi, \psi \in C_0(S(\mathcal{X}^k))$ such that $\varphi(x_1) = 1 = \psi(x_2)$ and $\varphi\psi = 0$. It follows from (12.1) that either $\int_{S(\mathcal{X}^k)} \varphi\, d\widetilde{\mu} = 0$ or $\int_{S(\mathcal{X}^k)} \psi\, d\widetilde{\mu} = 0$, which contradicts $x_1, x_2 \in C$. We have proved that there exists $\mathbf{x} = (x_1, \ldots, x_k) \in \mathcal{X}^k$ such that $C = \{\mathit{\Gamma}_k(\mathbf{x})\}$. This in turn implies that

$$x(a) = \int\limits_{C} \widehat{a}^{\odot k}\, d\widetilde{\mu} = \widetilde{\mu}(C)\widehat{a}^{\odot k}(\mathit{\Gamma}_k(\mathbf{x})) = \widetilde{\mu}(C) x_1(a)\ldots x_k(a)\,, \qquad a \in \mathcal{A}\,.$$

Since x is a nonzero character of $\mathcal{A}_\odot$ and $\mathcal{A} = \mathcal{A}^{(2)}$, we must have $\widetilde{\mu}(C) = 1$, which completes the proof of (i)$\Rightarrow$(ii).

(ii)⇒(iii). Assume that $x = (x_1 \otimes \ldots \otimes x_k) \circ \iota_k$, where $x_1, \ldots, x_k \in \Sigma(\mathcal{A})$. Then the function $x^{\sim} : \mathcal{A}^k \to \mathbb{C}$ defined by

$$x^{\sim}(a_1, \ldots, a_k) = \frac{1}{k!} \sum_{\sigma \in \mathcal{G}_k} x_{\sigma(1)}(a_1) \ldots x_{\sigma(k)}(a_k)$$

is a k-linear symmetric form such that $x = x^{\sim} \circ \iota_k$. Thus, by the uniqueness of symmetric extensions (cf. [12], Theorem 4.7), $x^{\#} = x^{\sim}$. Fix finite sequences $\{\alpha_i\}_{i=1}^l \subset \mathbb{C}$ and $\{\mathbf{a}_i\}_{i=1}^l \subset \mathcal{A}^k$ with $\mathbf{a}_i := (a_{i,1}, \ldots, a_{i,k})$. Then the equality $x^{\#} = x^{\sim}$ implies

$$\begin{aligned} k! &\sum_{p,q=1}^{l} x^{\#}(\mathbf{a}_q^* \mathbf{a}_p) \alpha_p \overline{\alpha}_q \\ &= \sum_{\sigma \in \mathcal{G}_k} \sum_{p,q=1}^{l} \overline{x_{\sigma(1)}(a_{q,1})} x_{\sigma(1)}(a_{p,1}) \ldots \overline{x_{\sigma(k)}(a_{q,k})} x_{\sigma(k)}(a_{p,k}) \alpha_p \overline{\alpha}_q \\ &= \sum_{\sigma \in \mathcal{G}_k} \Big| \sum_{p=1}^{l} \alpha_p x_{\sigma(1)}(a_{p,1}) \ldots x_{\sigma(k)}(a_{p,k}) \Big|^2 \geq 0 , \end{aligned}$$

which yields the positive definiteness of $x^{\#}$.

Since (iii)⇒(i) is obvious, the proof of Proposition 12.2 is complete. ∎

In case $\mathcal{A}$ is a commutative W^*-algebra (i.e. $\mathcal{A}$ is a commutative C^*-algebra which as a Banach space is dual to some other Banach space), the answer to the question mentioned at the beginning of this section is in the affirmative for continuous k-homogeneous characters of $\mathcal{A}_{\odot}$. Another question is whether any k-homogeneous character of $\mathcal{A}_{\odot}$ is continuous.

First we show that a symmetric k-linear extension of a continuous k-homogeneous character of $\mathcal{A}_{\odot}$ is positive definite. Notice that, in general, such an extension need not be multiplicative (e.g. if $\mathcal{A} = \mathbb{C}^2$, then the symmetric bilinear form defined by

$$\mathcal{A} \times \mathcal{A} \ni ((z_1, z_2), (w_1, w_2)) \to 2^{-1}(z_1 w_2 + w_1 z_2) \in \mathbb{C}$$

is not multiplicative on $\mathcal{A}^2$, though its restriction to the diagonal of $\mathcal{A}^2$ is multiplicative).

LEMMA 12.3. *Let $\mathcal{A}$ be a commutative W^*-algebra and let $x \in P_k(\mathcal{A})$ be a character of $\mathcal{A}_{\odot}$, $k \geq 1$. Then $x^{\#}$ is positive definite.*

Proof. It follows from Theorem of [36] and Theorem I.7.1 of [14] that $\mathcal{A}$ can be identified up to an isometric $*$-isomorphism of normed $*$-algebras with $L^{\infty}(\mathcal{X}, \nu)$, the W^*-algebra of all complex Borel functions on a locally compact Hausdorff space $\mathcal{X}$ which are essentially bounded with respect to a positive Borel measure ν on $\mathcal{X}$ (see also [37], Proposition 1.18.1). So without loss of generality we can assume that $\mathcal{A} = L^{\infty}(\mathcal{X}, \nu)$ and $k \geq 2$.

Fix finite sequences $\{\alpha_i\}_{i=1}^l \subset \mathbb{C}$ and $\{\mathbf{a}_i\}_{i=1}^l \subset \mathcal{A}^k$ with $\mathbf{a}_i := (a_{i,1}, \ldots \ldots, a_{i,k})$. All we have to show is that $\sum_{p,q=1}^l x^{\#}(\mathbf{a}_q^* \mathbf{a}_p)\alpha_p \overline{\alpha}_q \geq 0$. Denote by $\mathcal{P}$ the set of all nonzero idempotents in $\mathcal{A}$ and by $\operatorname{lin}\mathcal{P}$ the linear span of $\mathcal{P}$. Since $\operatorname{lin}\mathcal{P}$ is norm dense in $\mathcal{A}$ and $x^{\#}$ is continuous, we can assume without loss of generality that $\mathcal{F} := \{a_{i,j} : 1 \leq i \leq l,\, 1 \leq j \leq k\} \subset \operatorname{lin}\mathcal{P}$. This in turn implies that there exists a finite set $\mathcal{C} := \{c_1, \ldots, c_m\} \subset \mathcal{P}$ such that $c_p c_q = 0$ for all $p \neq q$ and $\mathcal{F} \subset \operatorname{lin}\mathcal{C}$. Notice that the $*$-algebras $\mathbb{C}^m$ and $\operatorname{lin}\mathcal{C}$ are $*$-isomorphic via $\mathbb{C}^m \ni (z_1, \ldots, z_m) \to z_1 c_1 + \ldots + z_m c_m \in \operatorname{lin}\mathcal{C}$. Thus we have reduced the general situation to the case where x is a multiplication preserving k-homogeneous polynomial on $\mathcal{A} = \mathbb{C}^m$. An easy verification shows that x must be of the form $x = (x_1 \otimes \ldots \otimes x_k) \circ \iota_k$, where $x_1, \ldots, x_k \in \Sigma(\mathbb{C}^m) = \{1, \ldots, m\}$. So the positive definiteness of $x^{\#}$ follows from Proposition 12.2. ■

We now pass on to the case of a compact Hausdorff space $\mathcal{X} = \Sigma(\mathcal{A})$, attached to a commutative W^*-algebra (6) $\mathcal{A}$. For an integer $k \geq 1$, define the continuous mapping $\vartheta_k : \mathcal{X}^k \to \Sigma_k^{\mathrm{h}}(\mathcal{A}_\odot)$ by

$$\vartheta_k(x_1, \ldots, x_k) = (x_1 \otimes \ldots \otimes x_k) \circ \iota_k\,, \quad x_1, \ldots, x_k \in \mathcal{X}\,.$$

Then (P.8) yields

$$\mathbf{x} \cong \mathbf{y} \;\Leftrightarrow\; \vartheta_k(\mathbf{x}) = \vartheta_k(\mathbf{y})\,, \quad \mathbf{x}, \mathbf{y} \in \mathcal{X}^k\,.$$

Thus there exists a unique (necessarily continuous) one-to-one mapping $\Xi_k : S(\mathcal{X}^k) \to \Sigma_k^{\mathrm{h}}(\mathcal{A}_\odot)$ such that $\vartheta_k = \Xi_k \circ \Gamma_k$. It turns out that Ξ_k is onto.

THEOREM 12.4. *If $\mathcal{A}$ is a commutative W^*-algebra, then for every $k \geq 1$, the mapping $\Xi_k : S(\Sigma(\mathcal{A})^k) \to \Sigma_k^{\mathrm{h}}(\mathcal{A}_\odot)$ is a homeomorphism.*

Proof. By Lemma 12.3 and Proposition 12.2, the mapping Ξ_k is one-to-one and onto. On the other hand, Lemma 12.1 implies that $S(\Sigma(\mathcal{A})^k)$ is a compact Hausdorff space, so Ξ_k is a closed mapping. Consequently, Ξ_k is a homeomorphism. ■

13. Positive definiteness versus complete positivity. In this section we show that the notions of positive definiteness and complete positivity coincide within the class of holomorphic mappings on commutative W^*-algebras. Notice that there are completely positive mappings which are not holomorphic.

To begin with we prove a version of Lemma 12.3 for positive definite k-homogeneous polynomials.

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(6) That $\mathcal{X}$ is a compact Hausdorff space follows from the fact that any W^*-algebra has a unit.

LEMMA 13.1. *Let $\mathcal{A}$ be a commutative W^*-algebra and let $\Theta \in P_k(\mathcal{A},\mathcal{H})$ be a positive definite polynomial*, $k \geq 1$. *Then $\Theta^{\#}$ is positive definite.*

Proof I. By Theorem 9.3, we have

$$\Theta^{\#}(a_1,\ldots,a_k) = \int\limits_{\Sigma_k^{\mathrm{h}}(\mathcal{A}_{\odot})} x^{\#}(a_1,\ldots,a_k)\, M(dx), \quad a_1,\ldots,a_k \in \mathcal{A}, \tag{13.1}$$

where M is a representing measure of Θ. Applying Lemma 12.3 to (13.1) we see that $\langle \Theta^{\#}(\cdot)f,f\rangle$ is positive definite for all $f \in \mathcal{H}$. This and Theorem 6.1 complete the proof. ■

Proof II (independent of Theorem 9.3). Similarly to the proof of Lemma 12.3, we reduce the general situation to the case of $\mathcal{A} = \mathbb{C}^m$. Then, by Proposition 10.1, we have

$$\Theta^{\#} = \sum_{|\mathbf{n}|=k} x_{\mathbf{n}}^{\#} T_{\mathbf{n}}\,,$$

where $T_{\mathbf{n}}$ are positive operators on $\mathcal{H}$ and $x_{\mathbf{n}}$ are monomials on $\mathbb{C}^m$ of the form $x_{\mathbf{n}}(\mathbf{z}) = \mathbf{z}^{\mathbf{n}}$, $\mathbf{z} \in \mathbb{C}^m$. It is enough to show that any $x_{\mathbf{n}}^{\#}T_{\mathbf{n}}$ is positive definite. It follows from Lemma 12.3 that $x_{\mathbf{n}}^{\#}$ is positive definite. Let $(\mathcal{K},X)$ be a minimal factorization of $x_{\mathbf{n}}^{\#}$. Then

$$\sum_{p,q=1}^{l} \langle x_{\mathbf{n}}^{\#}(\mathbf{a}_q^*\mathbf{a}_p)T_{\mathbf{n}}f_p, f_q\rangle_{\mathcal{H}} = \Big\| \sum_{p=1}^{l} X(\mathbf{a}_p)\otimes \sqrt{T_{\mathbf{n}}}f_p \Big\|_{\mathcal{K}\otimes\mathcal{H}}^2 \geq 0$$

for all finite sequences $\{f_i\}_{i=1}^{l} \subset \mathcal{H}$ and $\{\mathbf{a}_i\}_{i=1}^{l} \subset \mathcal{A}^k$, which completes the proof. ■

We are now in a position to show that positive definite continuous homogeneous polynomials on commutative W^*-algebras are completely positive.

PROPOSITION 13.2. *Let $\mathcal{A}$ be a commutative W^*-algebra and let $\Theta \in P_k(\mathcal{A},\mathcal{H})$ $(k \geq 1)$. Then the following conditions are equivalent*:

(i) *for every $f \in \mathcal{H}$, $\langle \Theta(\cdot)f,f\rangle$ is positive definite*,

(ii) *there are a complex Hilbert space $\mathcal{K}$, an operator $R \in B(\mathcal{H},\mathcal{K})$ and linear $*$-representations $\Pi_j : \mathcal{A} \to B(\mathcal{K})$ $(1 \leq j \leq k)$ such that*

$$\Pi_i(a)\Pi_j(b) = \Pi_j(b)\Pi_i(a)\,, \quad a,b \in \mathcal{A}\,,\ 1 \leq i,j \leq k\,,$$
$$\Theta(a) = R^*\Pi_1(a)\ldots\Pi_k(a)R\,, \quad a \in \mathcal{A}\,,$$

(iii) *Θ is completely positive*,

(iv) *there exists a* (*necessarily unique*) *regular semispectral measure $E : \mathfrak{B}(S(\mathcal{X}^k)) \to B(\mathcal{H})$ such that*

$$\Theta(a) = \int\limits_{S(\mathcal{X}^k)} \widehat{a}^{\odot k}\, dE\,, \quad a \in \mathcal{A} \quad (\mathcal{X} = \Sigma(\mathcal{A}))\,. \tag{13.2}$$

Moreover, Θ *is a $*$-representation of* $\mathcal{A}_{\odot}$ *if and only if* E *is a spectral measure.*

Proof. (i)$\Rightarrow$(ii) is a consequence of Theorem 9.3, Lemma 13.1 and Proposition 11.1. That (ii) implies (iii) follows from Proposition 4 of [57]. Since any completely positive mapping is automatically positive definite, (iii)$\Rightarrow$(iv) can be deduced from Lemma 13.1 and Proposition 11.2 (the uniqueness of E in (13.2) follows from Step 2 of the proof of Proposition 12.2). The proof of (iv)$\Rightarrow$(i) is straightforward. If the measure E in (13.2) is spectral, then obviously Θ is a $*$-representation of $\mathcal{A}_{\odot}$. Conversely, if Θ is a $*$-representation of $\mathcal{A}_{\odot}$ given by (13.2), then the uniform denseness of the linear span of $\{\widehat{a}^{\odot k} : a \in \mathcal{A}\}$ in $C_0(S(\mathcal{X}^k))$ (see Step 2 of the proof of Proposition 12.2) implies

$$\int\limits_{S(\mathcal{X}^k)} \varphi\psi\, dE = \int\limits_{S(\mathcal{X}^k)} \varphi\, dE \int\limits_{S(\mathcal{X}^k)} \psi\, dE\,, \qquad \varphi,\psi \in C_0(S(\mathcal{X}^k))\,.$$

Thus (cf. [26], Theorem 2.1), E must be a spectral measure. ∎

In virtue of Theorem 9.3, any holomorphic positive definite function on the open unit ball of a commutative W^*-algebra can be represented as a series of holomorphic homogeneous polynomials which are positive definite. Thus we can apply Proposition 13.2 (and also Theorem 9.3) to get

THEOREM 13.3. *Let* $\mathcal{A}$ *be a commutative* W^**-algebra and let* $\Theta : \mathcal{A}_{\bullet} \to B(\mathcal{H})$ (*resp.* $\Theta : \mathcal{A} \to B(\mathcal{H})$) *be a holomorphic mapping. Then the following conditions are equivalent*:

(i) *for every* $f \in \mathcal{H}$, $\langle \Theta(\cdot)f, f\rangle$ *is positive definite*,
(ii) Θ *is completely positive.*

Considering positive definite functions on commutative W^*-algebras leads to a result which resembles Theorem 7 of [1]. Its precise formulation is left to the reader. We end this section with two open questions.

QUESTION 2. Is any positive definite k-homogeneous polynomial on a (commutative) C^*-algebra $\mathcal{A}$ continuous ?

QUESTION 3. Is it true that $\Theta^{\#}$ is positive definite for any continuous positive definite k-homogeneous polynomial Θ on a (commutative) C^*-algebra $\mathcal{A}$?

Basing on the Gelfand–Naimark theorem, one can easily reduce Question 3 to the case $\mathcal{A} = C_0(A)$, where A is a closed subset of $\mathbb{C}^m$ with some $m \geq 1$. Moreover, if the answer to Question 3 is in the affirmative, then Theorem 12.4, Proposition 13.2 and Theorem 13.3 also hold for commutative C^*-algebras.

14. Appendix. For the convenience of the reader we collect in this section basic facts concerning integration of scalar functions with respect to positive operator-valued measures defined on δ-rings (see [23] for another approach).

Let $\mathcal{X}$ be a topological Hausdorff space and let $\mathfrak{B}(\mathcal{X})$ stand for the σ-algebra of Borel sets in $\mathcal{X}$ (i.e. the σ-algebra generated by open sets in $\mathcal{X}$). Denote by $\mathfrak{C}(\mathcal{X})$ and $\mathfrak{D}(\mathcal{X})$ the families of all compact subsets of $\mathcal{X}$ and of all relatively compact Borel subsets of $\mathcal{X}$, respectively. Let $\mathfrak{R}$ be a δ-ring of Borel subsets of $\mathcal{X}$ (i.e. $\mathfrak{R}$ is a ring closed under countable intersections) containing $\mathfrak{D}(\mathcal{X})$. We say that a ($\sigma$-additive) measure $\mu : \mathfrak{R} \to \mathbb{R}_+$ is *tight* if

$$\mu(A) = \sup\{\mu(C) : C \subset A,\ C \in \mathfrak{C}(\mathcal{X})\}, \qquad A \in \mathfrak{R}.$$

It is well known (cf. [21], Theorem 1.2) that every tight measure $\mu : \mathfrak{R} \to \mathbb{R}_+$ has a unique tight (=Radon) extension $\mu^{\intercal} : \mathfrak{B}(\mathcal{X}) \to \overline{\mathbb{R}}_+$.

Let $\mathcal{H}$ be a complex Hilbert space. A mapping $M : \mathfrak{R} \to B(\mathcal{H})$ is said to be a *tight positive operator-valued measure* (in short: a *tight PO measure*) on $\mathcal{X}$ if for every $f \in \mathcal{H}$, $M_f(\cdot) := \langle M(\cdot)f, f\rangle$ is a tight measure. Given a tight PO measure $M : \mathfrak{R} \to B(\mathcal{H})$, set

$$\mathfrak{D}_M := \{A \in \mathfrak{B}(\mathcal{X}) : M_f^{\intercal}(A) < \infty \text{ for all } f \in \mathcal{H}\}.$$

We say that$M$ is *maximal* if $\mathfrak{R} = \mathfrak{D}_M$. One can show (see the proof of Theorem 7.1) that each tight PO measure can be extended to a (unique) maximal one. By a *semispectral measure* we understand a positive-operator-valued mapping defined on a σ-algebra, which is σ-additive in the weak operator topology. A semispectral measure E is said to be *spectral* if for every $A \in \mathfrak{B}(\mathcal{X})$, $E(A)$ is an orthogonal projection. Notice that if M is a tight PO measure whose values are orthogonal projections, then its unique maximal extension is a regular spectral measure.

Let $M : \mathfrak{R} \to B(\mathcal{H})$ be a tight PO measure on $\mathcal{X}$ and let $p \geq 1$. Denote by $\mathcal{L}^p(M)$ the linear space of all complex Borel functions ξ on $\mathcal{X}$ such that $\int_{\mathcal{X}} |\xi|^p \, dM_f^{\intercal} < \infty$ for all $f \in \mathcal{H}$. Set

$$\|\xi\|_{p,M} := \sup\Big\{\Big(\int_{\mathcal{X}} |\xi|^p \, dM_f^{\intercal}\Big)^{1/p} : f \in \mathcal{H},\ \|f\| = 1\Big\}, \qquad \xi \in \mathcal{L}^p(M).$$

Below we show that $\|\xi\|_{p,M} < \infty$ for every $\xi \in \mathcal{L}^p(M)$. We say that a property $\mathcal{P}$ (concerning points of $\mathcal{X}$) holds a.e. $[M]$ if the set $\{x \in \mathcal{X} : x$ does not have the property $\mathcal{P}\}$ is in $\mathfrak{R}$ and has M-measure 0.

We are now in a position to show that for any $\xi \in \mathcal{L}^1(M)$, the integral $\int_{\mathcal{X}} \xi \, dM$ converges in the weak operator topology. The following result has been announced in [46].

THEOREM A.1. *Let $M : \mathfrak{R} \to B(\mathcal{H})$ be a maximal tight PO measure on a topological Hausdorff space $\mathcal{X}$. Then for every $\xi \in \mathcal{L}^1(M)$, there is a unique* *operator* $\int_{\mathcal{X}} \xi\, dM \in B(\mathcal{H})$ *such that*

$$\text{(i)} \qquad \Big\langle \int_{\mathcal{X}} \xi\, dM f, f \Big\rangle = \int_{\mathcal{X}} \xi\, dM^{\intercal}_{f}, \qquad f \in \mathcal{H}.$$

The integral $\int_{\mathcal{X}} \xi\, dM$ *has the following properties* :

$$\text{(ii)} \qquad \int_{\mathcal{X}} \chi_A\, dM = M(A), \qquad A \in \mathfrak{R},$$

$$\text{(iii)} \qquad \int_{\mathcal{X}} (\alpha\xi + \beta\eta)\, dM = \alpha \int_{\mathcal{X}} \xi\, dM + \beta \int_{\mathcal{X}} \eta\, dM, \qquad \alpha, \beta \in \mathbb{C},\ \xi, \eta \in \mathcal{L}^1(M),$$

$$\text{(iv)} \qquad \Big(\int_{\mathcal{X}} \xi\, dM \Big)^{*} = \int_{\mathcal{X}} \overline{\xi}\, dM, \qquad \xi \in \mathcal{L}^1(M),$$

(v) *if* $\xi \in \mathcal{L}^1(M)$ *and* $\xi \geq 0$ *a.e.* $[M]$, *then* $\int_{\mathcal{X}} \xi\, dM \geq 0$,

(vi) *if* $\xi, \eta \in \mathcal{L}^1(M)$ *and* $\xi = \eta$ *a.e.* $[M]$, *then* $\int_{\mathcal{X}} \xi\, dM = \int_{\mathcal{X}} \eta\, dM$,

$$\text{(vii)} \qquad \Big| \Big\langle \int_{\mathcal{X}} \xi\, dM f, g \Big\rangle \Big|^2 \leq \int_{\mathcal{X}} |\xi|\, dM^{\intercal}_{f} \int_{\mathcal{X}} |\xi|\, dM^{\intercal}_{g}, \qquad \xi \in \mathcal{L}^1(M),\ f, g \in \mathcal{H},$$

$$\text{(viii)} \qquad \|\xi\|_{1,M} = \Big\| \int_{\mathcal{X}} |\xi|\, dM \Big\|, \qquad \xi \in \mathcal{L}^1(M),$$

$$\text{(ix)} \qquad \Big\| \int_{\mathcal{X}} \xi\, dM \Big\| \leq \Big\| \int_{\mathcal{X}} |\xi|\, dM \Big\|, \qquad \xi \in \mathcal{L}^1(M).$$

P r o o f. The uniqueness of the operator $\int_{\mathcal{X}} \xi\, dM$ satisfying (i) follows from the polarization formula. So we only have to prove that such an operator exists. Denote by $\mathcal{T}$ the set of all complex functions ξ on $\mathcal{X}$ of the form

$$\text{(A.1)} \qquad \xi = \sum_{k=1}^{m} \alpha_k \chi_{A_k}$$

with $\alpha_1, \ldots, \alpha_m \in \mathbb{C}$, $A_1, \ldots, A_m \in \mathfrak{R}$ and $m \geq 1$. Then $\mathcal{T} \subset \mathcal{L}^1(M)$. If $\xi \in \mathcal{T}$ is of the form (A.1), then we set $\int_{\mathcal{X}} \xi\, dM := \sum_{k=1}^{m} \alpha_k M(A_k)$. Since $\langle \sum_{k=1}^{m} \alpha_k M(A_k) f, f \rangle = \int_{\mathcal{X}} \xi\, dM^{\intercal}_{f}$ for every $f \in \mathcal{H}$, the definition of $\int_{\mathcal{X}} \xi\, dM$ does not depend on α_k's and A_k's representing ξ via (A.1). Since $\mathfrak{R}$ is a ring of sets, we can assume, without loss of generality, that the sets A_k, $k = 1, \ldots, m$, are pairwise disjoint. Thus, by the Cauchy–

Schwarz inequality,

$$\text{(A.2)}\qquad \Big|\Big\langle \int\limits_{\mathcal{X}} \xi\, dMf, g\Big\rangle\Big| \le \sum_{k=1}^{m} |\alpha_k|\, |\langle M(A_k)^{1/2}f, M(A_k)^{1/2}g\rangle|$$
$$\le \sum_{k=1}^{m} (|\alpha_k|^{1/2}\|M(A_k)^{1/2}f\|)(|\alpha_k|^{1/2}\|M(A_k)^{1/2}g\|)$$
$$\le \Big(\sum_{k=1}^{m} |\alpha_k|\|M(A_k)^{1/2}f\|^2\Big)^{1/2}\Big(\sum_{k=1}^{m} |\alpha_k|\|M(A_k)^{1/2}g\|^2\Big)^{1/2}$$
$$= \Big(\sum_{k=1}^{m} |\alpha_k|\langle M(A_k)f, f\rangle\Big)^{1/2}\Big(\sum_{k=1}^{m} |\alpha_k|\langle M(A_k)g, g\rangle\Big)^{1/2}$$
$$= \Big(\int\limits_{\mathcal{X}} |\xi|\, dM_f^{\intercal}\Big)^{1/2}\Big(\int\limits_{\mathcal{X}} |\xi|\, dM_g^{\intercal}\Big)^{1/2}, \qquad f, g \in \mathcal{H},$$

which means that (vii) holds for any $\xi \in \mathcal{T}$.

Assume now that $\xi \in \mathcal{L}^1(M)$. Then there exists a sequence $\{\xi_n\}_{n=1}^{\infty}$ of simple Borel complex functions on $\mathcal{X}$ such that $|\xi_n| \le |\xi|$ and $\xi(x) = \lim_{n\to\infty} \xi_n(x)$ for every $x \in \mathcal{X}$. Since $\int_{\mathcal{X}} |\xi_n|\, dM_f^{\intercal} \le \int_{\mathcal{X}} |\xi|\, dM_f^{\intercal} < \infty$ for every $f \in \mathcal{H}$ and M is maximal, we get $\xi_n \in \mathcal{T}$, $n \ge 1$. Set $T_n := \int_{\mathcal{X}} \xi_n\, dM$, $n \ge 1$. Then, directly from the definition of $\int_{\mathcal{X}} \xi_n\, dM$, we get

$$|\langle (T_m - T_n)f, f\rangle| \le \int\limits_{\mathcal{X}} |\xi_m - \xi_n|\, dM_f^{\intercal}, \qquad f \in \mathcal{H},\ m, n \ge 1.$$

We have $\lim_{m\to\infty,\, n\to\infty} \int_{\mathcal{X}} |\xi_m - \xi_n|\, dM_f^{\intercal} = 0$ for every $f \in \mathcal{H}$ by the Lebesgue dominated convergence theorem, hence the above inequality implies that for every $f \in \mathcal{H}$, the sequence $\{\langle T_n f, f\rangle\}_{n=1}^{\infty}$ is convergent. It follows from the Banach–Steinhaus and the Riesz–Fischer theorems that there exists $T \in B(\mathcal{H})$ such that $\{T_n\}$ converges to T in the weak operator topology. Set

$$\int\limits_{\mathcal{X}} \xi\, dM := (\mathrm{WOT}) \lim_{n\to\infty} \int\limits_{\mathcal{X}} \xi_n\, dM.$$

Analysis similar to that presented above shows that the definition of $\int_{\mathcal{X}} \xi\, dM$ is independent of the choice of $\{\xi_n\}_{n=1}^{\infty}$ (consequently, both definitions of $\int_{\mathcal{X}} \xi\, dM$ coincide in case $\xi \in \mathcal{T}$). Next, by the Lebesgue dominated convergence theorem,

$$\int\limits_{\mathcal{X}} \xi\, dM_f^{\intercal} = \lim_{n\to\infty} \int\limits_{\mathcal{X}} \xi_n\, dM_f^{\intercal} = \lim_{n\to\infty} \Big\langle \int\limits_{\mathcal{X}} \xi_n\, dMf, f\Big\rangle$$
$$= \Big\langle \int\limits_{\mathcal{X}} \xi\, dMf, f\Big\rangle, \qquad f \in \mathcal{H},$$

which proves (i). Likewise, (A.2) implies

$$\Big|\Big\langle \int\limits_{\mathcal{X}} \xi\, dM f, g\Big\rangle\Big|^2 = \lim_{n\to\infty} \Big|\Big\langle \int\limits_{\mathcal{X}} \xi_n\, dM f, g\Big\rangle\Big|^2$$
$$\le \lim_{n\to\infty} \int\limits_{\mathcal{X}} |\xi_n|\, dM_f^{\intercal} \int\limits_{\mathcal{X}} |\xi_n|\, dM_g^{\intercal} = \int\limits_{\mathcal{X}} |\xi|\, dM_f^{\intercal} \int\limits_{\mathcal{X}} |\xi|\, dM_g^{\intercal}, \qquad f, g \in \mathcal{H},$$

which shows (vii).

The conditions (ii)–(vi) can be drawn from (i) via standard arguments. To prove (viii) take $\xi \in \mathcal{L}^1(M)$. Then, by (v), the operator $\int_{\mathcal{X}} |\xi|\, dM$ is positive and consequently

$$\|\xi\|_{1,M} = \sup\Big\{ \int\limits_{\mathcal{X}} |\xi|\, dM_f^{\intercal} : f \in \mathcal{H}, \|f\| = 1\Big\}$$
$$= \sup\Big\{\Big\langle \int\limits_{\mathcal{X}} |\xi|\, dM f, f\Big\rangle : f \in \mathcal{H}, \|f\| = 1\Big\} = \Big\| \int\limits_{\mathcal{X}} |\xi|\, dM\Big\|, \qquad \xi \in \mathcal{L}^1(M),$$

which shows (viii). The inequality (ix) can be derived from (vii) and (viii) as follows:

$$\Big\| \int\limits_{\mathcal{X}} \xi\, dM\Big\| = \sup\Big\{\Big|\Big\langle \int\limits_{\mathcal{X}} \xi\, dM f, g\Big\rangle\Big| : f, g \in H, \|f\| = 1 = \|g\|\Big\}$$
$$\le \sup\Big\{\Big(\int\limits_{\mathcal{X}} |\xi|\, dM_f^{\intercal}\Big)^{1/2}\Big(\int\limits_{\mathcal{X}} |\xi|\, dM_g^{\intercal}\Big)^{1/2} : f, g \in \mathcal{H}, \|f\| = 1 = \|g\|\Big\}$$
$$= \|\xi\|_{1,M} = \Big\| \int\limits_{\mathcal{X}} |\xi|\, dM\Big\|, \qquad \xi \in \mathcal{L}^1(M). \ ■$$

One of the consequences of Theorem A.1 is that a complex Borel function ξ on $\mathcal{X}$ is in $\mathcal{L}^p(M)$ if and only if $\|\xi\|_{p,M} < \infty$.

COROLLARY A.2. *Let $M : \mathfrak{R} \to B(\mathcal{H})$ be a maximal tight PO measure on a topological Hausdorff space $\mathcal{X}$ and let $p \ge 1$. Then*

$$\|\xi\|_{p,M} = \Big\| \int\limits_{\mathcal{X}} |\xi|^p\, dM\Big\|^{1/p} < \infty, \qquad \xi \in \mathcal{L}^p(M).$$

P r o o f. Indeed, if $\xi \in \mathcal{L}^p(M)$, then $|\xi|^p \in \mathcal{L}^1(M)$. Applying Theorem A.1(viii) to $|\xi|^p$ we get

$$\|\xi\|_{p,M} = (\|\, |\xi|^p\|_{1,M})^{1/p} = \Big\| \int\limits_{\mathcal{X}} |\xi|^p\, dM\Big\|^{1/p} < \infty. \ ■$$

The linear mapping $\mathcal{L}^1(M) \ni \xi \to \int_{\mathcal{X}} \xi\, dM \in B(\mathcal{H})$ can be regarded as a unique "linear extension" of the measure M, which is continuous in the following sense.

THEOREM A.3. *Let $M : \mathfrak{R} \to B(\mathcal{H})$ be a maximal tight PO measure on a topological Hausdorff space $\mathcal{X}$. Then the linear mapping $\Lambda : \mathcal{L}^1(M) \to B(\mathcal{H})$ defined by*

$$\Lambda(\xi) = \int_{\mathcal{X}} \xi \, dM \,, \qquad \xi \in \mathcal{L}^1(M) \,, \tag{i}$$

has the following properties:

(ii) $\Lambda(\chi_A) = M(A)$, $\quad A \in \mathfrak{R}$,

(iii) *if $\xi, \xi_n \in \mathcal{L}^1(M)$, $\xi_n \to \xi$ a.e. $[M]$ and $|\xi_n| \le \eta$ a.e. $[M]$ for some $\eta \in \mathcal{L}^1(M)$, then $\Lambda(\xi_n) \to \Lambda(\xi)$ in the strong operator topology.*

Conversely, if $\Lambda : \mathcal{L}^1(M) \to B(\mathcal{H})$ is a linear mapping satisfying (ii) *and* (iii), *then Λ is given by* (i).

Proof. Assume that Λ is given by (i). Then (ii) follows from Theorem A.1. To prove (iii), take $\xi, \xi_n, \eta \in \mathcal{L}^1(M)$ such that $\xi_n \to \xi$ a.e. $[M]$ and $|\xi_n| \le \eta$ a.e. $[M]$. Without loss of generality we can assume that $\xi_n \ge 0$ and $\xi = 0$. Set $T_n := \int_{\mathcal{X}} \xi_n \, dM$, $n \ge 1$. It follows from Theorem A.1 that $T_n \ge 0$ and $\langle T_n f, f\rangle = \int_{\mathcal{X}} \xi_n \, dM_f^{\intercal}$ for all $n \ge 1$ and $f \in \mathcal{H}$. Thus, by the Lebesgue dominated convergence theorem, $\langle T_n f, f\rangle \to 0$ for every $f \in \mathcal{H}$. In virtue of the Banach–Steinhaus theorem, there exists $\alpha > 0$ such that $\|T_n\| \le \alpha$ for each $n \ge 1$. Consequently,

$$\begin{aligned} \|T_n f\|^2 &= \|T_n^{1/2} T_n^{1/2} f\|^2 \le \|T_n^{1/2}\|^2 \|T_n^{1/2} f\|^2 \\ &= \|T_n\| \langle T_n f, f\rangle \le \alpha \langle T_n f, f\rangle \,, \qquad f \in \mathcal{H} \,, \end{aligned}$$

which implies that $T_n \to 0$ in the strong operator topology.

Assume now that $\Lambda : \mathcal{L}^1(M) \to B(\mathcal{H})$ is a linear mapping having the properties (ii) and (iii). Let $\mathcal{T}$ be as in the proof of Theorem A.1. Then (ii) implies that $\Lambda(\xi) = \int_{\mathcal{X}} \xi \, dM$ for every $\xi \in \mathcal{T}$. If $\xi \in \mathcal{L}^1(M)$, then there exists a sequence $\{\xi_n\}_{n=1}^{\infty} \subset \mathcal{T}$ such that $\xi_n \to \xi$ (pointwise) and $|\xi_n| \le |\xi|$ (see the proof of Theorem A.1). Since the linear mappings Λ and $\mathcal{L}^1(M) \ni \xi \to \int_{\mathcal{X}} \xi \, dM \in B(\mathcal{H})$ have the property (iii), we get $\Lambda(\xi) = (\mathrm{SOT}) \lim_{n\to\infty} \Lambda(\xi_n) = (\mathrm{SOT}) \lim_{n\to\infty} \int_{\mathcal{X}} \xi_n \, dM = \int_{\mathcal{X}} \xi \, dM$. ∎

Define an equivalence relation $\approx$ in the set of all complex Borel functions on $\mathcal{X}$ as follows:

$$\xi \approx \eta \iff \xi = \eta \text{ a.e.} \, [M] \,.$$

In case M is maximal and $p \ge 1$, the restriction of $\approx$ to $\mathcal{L}^p(M)$ can be described as follows:

$$\xi \approx \eta \iff \|\xi - \eta\|_{p,M} = 0 \,, \qquad \xi, \eta \in \mathcal{L}^p(M) \,.$$

Denote by $L^p(M)$ the quotient space $\mathcal{L}^p(M)/{\approx}$ and by $\|\cdot\|_{p,M}$ the quotient norm.

THEOREM A.4. *Let $M : \mathfrak{R} \to B(\mathcal{H})$ be a maximal tight PO measure on a topological Hausdorff space $\mathcal{X}$ and let $p \geq 1$. Then $(L^p(M), \|\cdot\|_{p,M})$ is a Banach space.*

Proof. Let $\{\xi_k\}_{n=1}^{\infty} \subset \mathcal{L}^p(M)$ with $\sum_{k=1}^{\infty} \|\xi_k\|_{p,M} < \infty$. Denote by A the Borel set $\{x \in \mathcal{X} : \sum_{k=1}^{\infty} |\xi_k(x)| < \infty\}$ and put $\mathbb{S} := \{f \in \mathcal{H} : \|f\| = 1\}$. Then, by Fatou's lemma and the Minkowski inequality,

$$\Big(\int_{\mathcal{X}} \Big(\sum_{k=1}^{\infty} |\xi_k| \Big)^p dM_f^{\intercal} \Big)^{1/p} \leq \sum_{k=1}^{\infty} \Big(\int_{\mathcal{X}} |\xi_k|^p \, dM_f^{\intercal} \Big)^{1/p}$$
$$\leq \sum_{k=1}^{\infty} \|\xi_k\|_{p,M} < \infty, \quad f \in \mathbb{S},$$

which implies that $M_f^{\intercal}(\mathcal{X} \setminus A) = 0$ for every $f \in \mathcal{H}$. Since M is maximal, $\mathcal{X} \setminus A \in \mathfrak{R}$. Define a Borel function ξ on $\mathcal{X}$ by $\xi := \chi_A \sum_{k=1}^{\infty} \xi_k$. Then, again by Fatou's lemma and the Minkowski inequality,

$$\Big(\int_{\mathcal{X}} \Big|\xi - \sum_{k=1}^{n} \xi_k \Big|^p dM_f^{\intercal} \Big)^{1/p} = \Big(\int_{A} \Big| \sum_{k=n+1}^{\infty} \xi_k \Big|^p dM_f^{\intercal} \Big)^{1/p}$$
$$\leq \sum_{k=n+1}^{\infty} \Big(\int_{\mathcal{X}} |\xi_k|^p \, dM_f^{\intercal} \Big)^{1/p} \leq \sum_{k=n+1}^{\infty} \|\xi_k\|_{p,M}, \quad f \in \mathbb{S},$$

which implies that $\lim_{n\to\infty} \|\xi - \sum_{k=1}^{n} \xi_k\|_{p,M} \leq \lim_{n\to\infty} \sum_{k=n+1}^{\infty} \|\xi_k\|_{p,M} = 0$. We have proved that any absolutely convergent series in $L^p(M)$ is convergent in $L^p(M)$, which completes the proof. ∎

Let M be a maximal tight PO measure on $\mathcal{X}$. Let $\mathcal{L}^{\infty}(M)$ stand for the linear space of all complex Borel functions on $\mathcal{X}$ which are $M_f^{\intercal}$-essentially bounded for every $f \in \mathcal{H}$. Set

$$\|\xi\|_{\infty,M} := \sup\{\|\xi\|_{L^{\infty}(M_f^{\intercal})} : f \in \mathcal{H}, \|f\| = 1\}, \quad \xi \in \mathcal{L}^{\infty}(M).$$

Using the classical Hölder inequality, one can show that if $p, q \in [1, \infty]$, $p^{-1} + q^{-1} = 1$, $\xi \in \mathcal{L}^p(M)$ and $\eta \in \mathcal{L}^q(M)$, then $\xi\eta \in \mathcal{L}^1(M)$ and $\|\xi\eta\|_{1,M} \leq \|\xi\|_{p,M}\|\eta\|_{q,M}$. It turns out that $\|\xi\|_{\infty,M} < \infty$ for every $\xi \in \mathcal{L}^{\infty}(M)$. This is a consequence of the following proposition.

PROPOSITION A.5. *Let $M : \mathfrak{R} \to B(\mathcal{H})$ be a maximal tight PO measure on a topological Hausdorff space $\mathcal{X}$. Then a Borel function $\xi : \mathcal{X} \to \mathbb{C}$ is in $\mathcal{L}^{\infty}(M)$ if and only if $\|\xi\|_{\infty,M} < \infty$. Moreover,*

$$\|\xi\|_{\infty,M} = \min\{\alpha \geq 0 : M(\{x \in \mathcal{X} : |\xi(x)| > \alpha\}) = 0\}, \quad \xi \in \mathcal{L}^{\infty}(M).$$

Proof. First we show that

$$M_{f+g}^{\intercal}(A)^{1/2} \leq M_f^{\intercal}(A)^{1/2} + M_g^{\intercal}(A)^{1/2}, \quad f, g \in \mathcal{H}, \ A \in \mathfrak{B}(X). \tag{A.3}$$

Indeed, since for every $C \in \mathfrak{C}(X)$, $\langle M(C)(\cdot), (-)\rangle$ is a semi-inner product on $\mathcal{H}$, (A.3) holds for all $f, g \in \mathcal{H}$ and $C \in \mathfrak{C}(\mathcal{X})$. However, $M^{\intercal}_{f+g}$, $M^{\intercal}_{f}$, $M^{\intercal}_{g}$ are $\mathfrak{C}(\mathcal{X})$-inner regular, so the inequality extends to the whole class $\mathfrak{B}(\mathcal{X})$.

Take $\xi \in \mathcal{L}^{\infty}(M)$. Setting $A(\alpha) := \{x \in \mathcal{X} : |\xi(x)| > \alpha\}$ for $\alpha \in \mathbb{R}$, we define $\varrho : \mathcal{H} \to \mathbb{R}_{+}$ by

$$\varrho(f) := \|\xi\|_{L^{\infty}(M^{\intercal}_{f})} = \inf\{\alpha \geq 0 : M^{\intercal}_{f}(A(\alpha)) = 0\}, \qquad f \in \mathcal{H}.$$

Then

$$\text{(A.4)} \qquad \varrho(f+g) \leq \max\{\varrho(f), \varrho(g)\}, \qquad f, g \in \mathcal{H},$$

for if $\alpha > \max\{\varrho(f), \varrho(g)\}$, then $M^{\intercal}_{f}(A(\alpha)) = 0 = M^{\intercal}_{g}(A(\alpha))$ and consequently, by (A.3), $M^{\intercal}_{f+g}(A(\alpha)) = 0$, which implies that $\varrho(f+g) \leq \alpha$.

The next property of ϱ is easily seen to be true:

$$\text{(A.5)} \qquad \varrho(\alpha f) = \varrho(f), \qquad \alpha \in \mathbb{C} \setminus \{0\},\ f \in \mathcal{H}.$$

Now we show that ϱ is lower semicontinuous, i.e. $\mathcal{H}(\varepsilon) := \{g \in \mathcal{H} : \varrho(g) \leq \varepsilon\}$ is closed for any $\varepsilon \geq 0$. For if $\{f_n\} \subset \mathcal{H}(\varepsilon)$ converges to an $f \in \mathcal{H}$ and $\alpha > \varepsilon$, then $M^{\intercal}_{f_n}(A(\alpha)) = 0$ for all n. Consequently, $M^{\intercal}_{f_n}(C) = 0$ for every $C \in \mathfrak{C}(\mathcal{X})$ such that $C \subset A(\alpha)$ and for all n. This in turn implies that

$$M^{\intercal}_{f}(C) = \langle M(C)f, f\rangle = \lim_{n\to\infty} \langle M(C)f_n, f_n\rangle = 0, \qquad C \in \mathfrak{C}(\mathcal{X}),\ C \subset A(\alpha).$$

Since $M^{\intercal}_{f}$ is $\mathfrak{C}(\mathcal{X})$-inner regular, $M^{\intercal}_{f}(A(\alpha)) = 0$. Thus $\varrho(f) \leq \alpha$ for every $\alpha > \varepsilon$ or equivalently $f \in \mathcal{H}(\varepsilon)$, which shows that $\mathcal{H}(\varepsilon)$ is closed.

Since $\mathcal{H} = \bigcup_{n=1}^{\infty} \mathcal{H}(n)$ and $\mathcal{H}(n)$, $n \geq 1$, are closed, we can apply the Baire theorem to get $n_0 \geq 1$ such that $\operatorname{int}(\mathcal{H}(n_0)) \neq \emptyset$. Using (A.4) and (A.5) one can show that $\mathcal{H}(n_0) = \mathcal{H}$, which implies that $\|\xi\|_{\infty,M} \leq n_0 < \infty$.

Since M is maximal, the family $\mathfrak{R}$ has the following property:

$$\text{(A.6)} \qquad \text{if } A_n \in \mathfrak{R} \text{ and } M(A_n) = 0 \text{ for } n \geq 1, \quad \text{then } \bigcup_n A_n \in \mathfrak{R} \text{ and } M\Big(\bigcup_n A_n\Big) = 0.$$

Using (A.6), one can easily verify the other part of the conclusion. ∎

The definition of $L^p(M)$ can be extended to the case of $p = \infty$ by setting $L^{\infty}(M) := \mathcal{L}^{\infty}(M)/\approx$. In case M is maximal, Proposition A.5 leads to

$$\xi \approx \eta \ \Leftrightarrow\ \|\xi - \eta\|_{\infty,M} = 0, \qquad \xi, \eta \in \mathcal{L}^{\infty}(M),$$

which implies that the quotient norm in $L^{\infty}(M)$ is well defined. Denote it by $\|\cdot\|_{\infty,M}$. Using Proposition A.5 and the property (A.6) one can show that $(L^{\infty}(M), \|\cdot\|_{\infty,M})$ is a Banach space. In other words, Theorem A.4 holds for $p = \infty$.

It is worthwhile to notice that the definition of $L^p(M)$, $1 \leq p \leq \infty$, carries over without substantial changes to the case of a semispectral measure

M defined on an arbitrary σ-algebra $\mathfrak{A}$ of subsets of a set $\mathcal{X}$. A careful verification of the proofs shows that all the results of the appendix remain true for such an M. Even in this case $L^p(M)$ can be essentially larger than $L^\infty(M)$ for all $p\in[1,\infty)$. However, if M is a spectral measure, all the spaces $L^p(M)$, $p\geq 1$, coincide with $L^\infty(M)$.

PROPOSITION A.6. *Let $E:\mathfrak{A}\to B(\mathcal{H})$ be a spectral measure and let $p\in[1,\infty)$. Then $L^p(E)=L^\infty(E)$ and $\|\cdot\|_{p,E}=\|\cdot\|_{\infty,E}$.*

Proof. Let $\mathbb{S}:=\{f\in\mathcal{H}:\|f\|=1\}$. For $f\in\mathbb{S}$, set $\mu_f(\cdot):=\langle E(\cdot)f,f\rangle=\|E(\cdot)f\|^2$ and $\mathcal{H}_f:=\bigvee\{E(A)f:A\in\mathfrak{A}\}$. Then $\mathcal{H}_f$ reduces E to a spectral measure E_f in $\mathcal{H}_f$. Moreover, there exists a (unique) unitary operator $U_f\in B(\mathcal{H}_f,L^2(\mu_f))$ such that $U_f(E(A)f)=\chi_A$ for all $A\in\mathfrak{A}$. Thus the measure $F_f:=U_fE_fU_f^{-1}$ acts in $L^2(\mu_f)$ according to the formula

$$F_f(A)\xi=\chi_A\xi\,,\qquad A\in\mathfrak{A}\,,\ \xi\in L^2(\mu_f)\,.$$

This implies that for every $h\in\mathcal{H}_f$,

$$\begin{aligned}\mu_h(A)=\langle F_f(A)U_f(h),U_f(h)\rangle&=\langle\chi_A\cdot U_f(h),U_f(h)\rangle\\&=\int\limits_A|U_f(h)|^2\,d\mu_f\,,\qquad A\in\mathfrak{A}\,,\end{aligned}$$

which yields

$$d\mu_h=|U_f(h)|^2\,d\mu_f\,,\qquad h\in\mathcal{H}_f\,. \tag{A.7}$$

We now return to our proof. Without loss of generality we can assume that $E\neq 0$. Take $\xi\in\mathcal{L}^p(E)$ and set $A(\alpha):=\{x\in\mathcal{X}:|\xi(x)|>\alpha\}$ for $\alpha\in\mathbb{R}$. We show that $E(A(\alpha))=0$ for every $\alpha>\|\xi\|_{p,E}$. Suppose, contrary to our claim, that $E(A(\alpha))\neq 0$ for some $\alpha>\|\xi\|_{p,E}$. Then there exists $f\in\mathbb{S}$ such that $E(A(\alpha))f\neq 0$. This implies that $\mu_f(A(\alpha))>0$. Set $\varphi:=\mu_f(A(\alpha))^{-1/2}\chi_{A(\alpha)}$ and $h:=U_f^{-1}(\varphi)$. Then $h\in\mathcal{H}_f\cap\mathbb{S}$ and, by (A.7),

$$\alpha\leq\Big(\int\limits_{\mathcal{X}}|\xi|^p|\varphi|^2\,d\mu_f\Big)^{1/p}=\Big(\int\limits_{\mathcal{X}}|\xi|^p\,d\mu_h\Big)^{1/p}\leq\|\xi\|_{p,E}\,,$$

which contradicts $\alpha>\|\xi\|_{p,E}$. Thus $\xi\in\mathcal{L}^\infty(E)$ and $\|\xi\|_{\infty,E}\leq\|\xi\|_{p,E}$.

Assume now that $\xi\in\mathcal{L}^\infty(E)$. Take $\alpha>\|\xi\|_{\infty,E}$. Then, by Proposition A.5, we have $E(A(\alpha))=0$, which implies

$$\sup_{f\in\mathbb{S}}\Big(\int\limits_{\mathcal{X}}|\xi|^p\,d\mu_f\Big)^{1/p}\leq\alpha(\sup_{f\in\mathbb{S}}\langle E(\mathcal{X})f,f\rangle)^{1/p}=\alpha\|E(\mathcal{X})\|^{1/p}=\alpha\,.$$

The last equality follows from the fact that $E(\mathcal{X})$ is a nonzero orthogonal projection on $\mathcal{H}$. Thus $\xi\in\mathcal{L}^p(E)$ and $\|\xi\|_{p,E}\leq\|\xi\|_{\infty,E}$. ■

We end the appendix with an example of a semispectral measure M acting in an infinite-dimensional complex Hilbert space for which all the spaces $L^p(M)$, $p \in [1, \infty)$, are essentially larger than $L^\infty(M)$.

Example A.7. Let $(\mathcal{X}, \mathfrak{A}, \mu)$ be a probability space and let $\{\varrho_n\}_{n=1}^\infty$ be a sequence of measurable functions on $\mathcal{X}$ such that $0 \le \varrho_n(x) \le \varrho_{n+1}(x) \le 1$ and $\lim_{k\to\infty} \varrho_k(x) = 1$ for all $x \in \mathcal{X}$ and $n \ge 1$. Define a semispectral measure $M : \mathfrak{A} \to B(l^2)$ by

$$M(A)\{\alpha_n\} := \Big\{\alpha_n \int\limits_A \varrho_n \, d\mu\Big\}, \qquad A \in \mathfrak{A}, \ \{\alpha_n\} \in l^2 .$$

Here l^2 stands for the Hilbert space of all square summable complex sequences. Using the Lebesgue monotone convergence theorem and Proposition A.5, one can show that $L^p(M) = L^p(\mu)$ and $\|\cdot\|_{p,M} = \|\cdot\|_{L^p(\mu)}$ for all $p \in [1, \infty]$. Thus taking an appropriate measure μ and a sequence $\{\varrho_n\}_{n=1}^\infty$ leads to the desired example.

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