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On topological invariants of vector bundles

by Zbigniew Szafraniec (Gdańsk)

Abstract. Let $E \to W$ be an oriented vector bundle, and let $\mathrm{X}(E)$ denote the Euler number of E. The paper shows how to calculate $\mathrm{X}(E)$ in terms of equations which describe E and W.

Introduction. Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$, $n - k > 0$, be a C^1-map such that $W = F^{-1}(0)$ is compact and $\operatorname{rank}[DF(x)] \equiv k$ at every $x \in W$. From the implicit function theorem W is an $(n-k)$-dimensional C^1-manifold.

Let $G_1, \ldots, G_s : \mathbb{R}^n \to \mathbb{R}^m$, where $m = s+n-k$, be a family of C^1-vector functions, and assume that the vectors $G_1(x), \ldots, G_s(x)$ are linearly independent for every $x \in W$. Define

$$\begin{aligned} E &= \{(x,y) \in W \times \mathbb{R}^m \mid y \perp G_i(x),\ i = 1, \ldots, s\} \\ &= \Big\{(x,y) \in W \times \mathbb{R}^m \;\Big|\; \sum y_j G_i^j(x) = 0,\ i = 1, \ldots, s\Big\}. \end{aligned}$$

Clearly E is an $(n-k)$-dimensional vector bundle over W. In particular, if $s = k$ and $G_i = \operatorname{grad} F_i$ then E becomes TW. Later we shall describe how to orient W and E.

Let $\mathrm{X}(E)$ be the Euler number of the bundle E (see [1], Chapter 5.2). The problem is how to calculate $\mathrm{X}(E)$ in terms of F and $G_1, \ldots, G_s$.

Let $S_R = \{(x,\lambda) \in \mathbb{R}^n \times \mathbb{R}^s \mid \|x\|^2 + \|\lambda\|^2 = R^2\}$, and let $H : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m \times \mathbb{R}^k$ be the map given by

$$H(x,\lambda) = \Big(\sum_{i=1}^{s} \lambda_i G_i(x)\,, F(x)\Big),$$

where $\lambda = (\lambda_1, \ldots, \lambda_s)$.

Take $R > 0$ such that $W \subset \{x \in \mathbb{R}^n \mid \|x\| < R\}$. It is easy to see that $H|S_R : S_R \to \mathbb{R}^m \times \mathbb{R}^k - \{0\}$. Since $n + s = m + k$, the topological degree

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$\deg(H|S_R)$ of the map $H|S_R$ is well defined. We shall prove (see Theorem 4) that

$$\mathrm{X}(E) = (-1)^{n(s+k)+k} \deg(H|S_R)\,.$$

As a corollary we get a formula (see Theorem 5) which expresses the Euler characteristic $\chi(W)$ in terms of F. A very similar formula has been proved in [2]. The advantage of the present work is that it is usually easy to find the appropriate value of R. The same is not necessarily true in [2].

2. Preliminaries. We assume that every space $\mathbb{R}^n$, $n > 0$, has the canonical orientation corresponding to its canonical ordered basis.

Let $F = (F_1, \ldots, F_k) : \mathbb{R}^n \to \mathbb{R}^k$ be a C^1-map as above. For each $x \in W$ there is a natural inclusion $T_x W \subset \mathbb{R}^n$. Vectors $\xi_{k+1}, \ldots, \xi_n \in T_x W$ are said to be positively oriented if $\operatorname{grad} F_1(x), \ldots, \operatorname{grad} F_k(x)$, $\xi_{k+1}, \ldots, \xi_n$ form a positively oriented basis in $\mathbb{R}^n$. From now on we assume W to be equipped with this orientation.

Let $G_1, \ldots, G_s : \mathbb{R}^n \to \mathbb{R}^m$, $m = s + n - k$, and the vector bundle E over W be as in the introduction. If $E(x)$ is the fibre of E over $x \in W$ then there is a natural inclusion $E(x) \subset \mathbb{R}^m$. Vectors $v_{s+1}, \ldots, v_m \in E(x)$ are said to be positively oriented if $G_1(x), \ldots, G_s(x), v_{s+1}, \ldots, v_m$ form a positively oriented basis in $\mathbb{R}^m$. So E is an $(n-k)$-dimensional oriented vector bundle.

Let

$$E' = \{(x, y) \in W \times \mathbb{R}^m \mid y \in \operatorname{span}(G_1(x), \ldots, G_s(x))\}\,.$$

Then E' is a trivial vector bundle over W such that $E \oplus E'$ is trivial.

Let $p : W \to E$ be a C^1-section of E such that $p(\overline{x}) = 0$, for some $\overline{x} \in W$. There are C^1-sections $v_{s+1}, \ldots, v_m : U \to E$ defined in some open neighbourhood U of $\overline{x}$ in W such that $v_{s+1}(\overline{x}), \ldots, v_m(\overline{x})$ are linearly independent and positively oriented in $E(\overline{x})$. The sections $v_{s+1}, \ldots, v_m$ define a trivialization of E over U, and thus there are unique C^1-functions $t_{s+1}, \ldots, t_m : U \to \mathbb{R}$ such that $p = \sum_{i=s+1}^m t_i v_i$ over U. Let $(x_{k+1}, \ldots, x_n)$ be a positively oriented coordinate system in some neighbourhood of $\overline{x}$ in W.

Definition. $\operatorname{ind}(p, \overline{x}) = \operatorname{sign} \dfrac{\partial(t_{s+1}, \ldots, t_m)}{\partial(x_{k+1}, \ldots, x_n)}(\overline{x})$.

One can prove that the definition of $\operatorname{ind}(p, \overline{x})$ does not depend on the choice of $v_{s+1}, \ldots, v_m$ and $(x_{k+1}, \ldots, x_n)$. Note that if the section p is transversal to the zero-section at $\overline{x}$ then $\operatorname{ind}(p, \overline{x})$ is the index of p at x (see [1], Chapter 5.2).

Let $P = (P^1, \ldots, P^m) : \mathbb{R}^n \to \mathbb{R}^m$ be a C^1-vector function. There are sections $p : W \to E$, $p' : W \to E'$ such that $P|W = p + p'$. Let $\widetilde{H} : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m \times \mathbb{R}^k$ be given by

$$\widetilde{H}(x, \lambda) = \Big(P(x) + \sum_{d=1}^{s} \lambda_d G_d(x), F(x)\Big).$$

LEMMA 1. *A point $\overline{x} \in \mathbb{R}^n$ is in $p^{-1}(0) \subset W$ if and only if there is a unique $\overline{\lambda} \in \mathbb{R}^s$ such that $\widetilde{H}(\overline{x}, \overline{\lambda}) = 0$.*

Proof. ($\Rightarrow$) If $p(\overline{x}) = 0$ then $P(\overline{x}) = p'(\overline{x}) \in E'(\overline{x})$, where $E'(\overline{x})$ is the fibre of E' over $\overline{x}$. The vectors $G_1(\overline{x}), \ldots, G_s(\overline{x})$ form a basis in $E'(\overline{x})$, and thus there is a unique $\overline{\lambda} = (\overline{\lambda}_1, \ldots, \overline{\lambda}_s) \in \mathbb{R}^s$ such that

$$P(\overline{x}) + \sum_{d=1}^{s} \overline{\lambda}_d G_d(\overline{x}) = 0.$$

Since $\overline{x} \in W = F^{-1}(0)$, we get $\widetilde{H}(\overline{x}, \overline{\lambda}) = 0$.

($\Leftarrow$) Clearly $\overline{x} \in W$, $P(\overline{x}) = p(\overline{x}) + p'(\overline{x}) \in \operatorname{span}(G_1(\overline{x}), \ldots, G_s(\overline{x}))$, and so $p(\overline{x}) = 0$. ∎

From now on we assume that $\overline{x} \in p^{-1}(0)$. Let $\overline{\lambda} \in \mathbb{R}^s$ be as in Lemma 1. Since $n + s = m + k$, the derivative matrix $D\widetilde{H}(\overline{x}, \overline{\lambda})$ is a square matrix.

LEMMA 2. $\operatorname{ind}(p, \overline{x}) = (-1)^{n(s+k)+k} \operatorname{sign} \det[D\widetilde{H}(\overline{x}, \overline{\lambda})]$.

Proof. We can find a coordinate system $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ such that

$$\frac{\partial F_i}{\partial x_j}(\overline{x}) = 0, \tag{1}$$

for every $1 \le i \le k$, $j \ge k + 1$. Let

$$A = \left[\frac{\partial F_i}{\partial x_j}(\overline{x})\right]_{1 \le i, j \le k}.$$

From (1) and from the fact that $\operatorname{rank}[DF(\overline{x})] = k$ we deduce that $\det[A] \ne 0$.

For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ we write $x = (x', x'')$, where $x' = (x_1, \ldots, x_k) \in \mathbb{R}^k$, $x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k}$. From the implicit function theorem there is a germ of a C^1-function $\psi = (\psi_1, \ldots, \psi_k) : (\mathbb{R}^{n-k}, \overline{x}'') \to (\mathbb{R}^k, \overline{x}')$ such that

$$F_i(\psi(x''), x'') \equiv 0, \quad 1 \le i \le k. \tag{2}$$

Since $\operatorname{graph} \psi = W$ in some neighbourhood of $\overline{x}$, we can treat $(x_{k+1}, \ldots, x_n)$ as a coordinate system in some neighbourhood of $\overline{x}$ in W. From (1)

(3) the coordinate system $(x_{k+1}, \ldots, x_n)$ is positively oriented if and only if $\det[A] > 0$.

From (1), (2), for every $1 \le i \le k$, $j \ge k+1$ we have

$$\frac{\partial}{\partial x_j}[F_i(\psi(x''), x'')](\overline{x}'') = \frac{\partial F_i}{\partial x_1}(\overline{x})\frac{\partial \psi_1}{\partial x_j}(\overline{x}'') + \ldots + \frac{\partial F_i}{\partial x_k}(\overline{x})\frac{\partial \psi_k}{\partial x_j}(\overline{x}'') = 0\,.$$

The matrix A is non-singular, and therefore

$$\frac{\partial \psi_i}{\partial x_j}(\overline{x}'') = 0\,, \quad \text{for } 1 \le i \le k\,,\ j \ge k+1\,. \tag{4}$$

There are C^1-vector maps $V_{s+1}, \ldots, V_m : \mathbb{R}^n \to \mathbb{R}^m$ defined in some neighbourhood of $\overline{x}$ such that $V_{s+1}(\overline{x}), \ldots, V_m(\overline{x})$ form a positively oriented basis in $E(\overline{x})$. Write $G_d = (G_d^1, \ldots, G_d^m)$, $1 \le d \le s$, and $V_d = (V_d^1, \ldots, V_d^m)$, $s+1 \le d \le m$. Since $s < m$, after an orientation preserving change of coordinates in $\mathbb{R}^m$ we may assume that

$$\begin{aligned} G_d^i(\overline{x}) &= \delta_{di}\,, \quad \text{for } 1 \le d \le s\,,\\ V_d^i(\overline{x}) &= \delta_{di}\,, \quad \text{for } s+1 \le d \le m\,, \end{aligned} \tag{5}$$

where δ_{di} is the Kronecker delta.

There are C^1-functions $T_1, \ldots, T_m : \mathbb{R}^n \to \mathbb{R}$ defined in a neighbourhood of $\overline{x}$ such that

$$P = \sum_{d=1}^{s} T_d G_d + \sum_{d=s+1}^{m} T_d V_d\,.$$

Since $p(\overline{x}) = 0$, we have $(T_1(\overline{x}), \ldots, T_s(\overline{x})) = -\overline{\lambda}$ and $T_{s+1}(\overline{x}) = \ldots = T_m(\overline{x}) = 0$. Let $\theta : (\mathbb{R}^{n-k}, \overline{x}'') \to (W, \overline{x})$ be given by $\theta(x'') = (\psi(x''), x'')$, and let $p^i = P^i \circ \theta$, $t_i = T_i \circ \theta$, $g_d^i = G_d^i \circ \theta$ and $v_d^i = V_d^i \circ \theta$. Then

$$(t_1(\overline{x}''), \ldots, t_s(\overline{x}'')) = -\overline{\lambda}\,, \quad t_{s+1}(\overline{x}'') = \ldots = t_m(\overline{x}'') = 0\,. \tag{6}$$

Let $Z : \mathbb{R}^n \to \mathbb{R}$ be a C^1-function and let $z : \mathbb{R}^{n-k} \to \mathbb{R}$ be given by $z = Z \circ \theta$. From (4) we have

$$\frac{\partial z}{\partial x_j}(\overline{x}'') = \sum_{i=1}^{k} \frac{\partial Z}{\partial x_i}(\overline{x})\frac{\partial \psi_i}{\partial x_j}(\overline{x}'') + \frac{\partial Z}{\partial x_j}(\overline{x}) = \frac{\partial Z}{\partial x_j}(\overline{x})\,, \tag{7}$$

for $k+1 \le j \le n$.

Take $i \in \{s+1, \ldots, m\}$, $j \in \{k+1, \ldots, n\}$. Then

$$p^i = \sum_{d=1}^{s} t_d g_d^i + \sum_{d=s+1}^{m} t_d v_d^i\,,$$

and therefore, from (5) and (6),

$$\frac{\partial p^i}{\partial x_j}(\overline{x}'') = -\sum_{d=1}^{s} \overline{\lambda}_d \frac{\partial g_d^i}{\partial x_j}(\overline{x}'') + \frac{\partial t_i}{\partial x_j}(\overline{x}'')\,,$$

and so, from (7), we have

$$\frac{\partial t_i}{\partial x_j}(\overline{x}'') = \frac{\partial P^i}{\partial x_j}(\overline{x}) + \sum_{d=1}^{s} \overline{\lambda}_d \frac{\partial G_d^i}{\partial x_j}(\overline{x}) \,.$$

Let m_{ij} be the above expression, and let $M = [m_{ij}]_{s+1\le i\le m, k+1\le j\le m}$. From (1) and (5) it is easy to see that the derivative matrix $D\widetilde{H}(\overline{x}, \overline{\lambda})$ has the form

$$\begin{bmatrix} ? & ? & I \\ ? & M & 0 \\ A & 0 & 0 \end{bmatrix},$$

where I is the $s \times s$ identity matrix, so $\det[D\widetilde{H}(\overline{x}, \overline{\lambda})] = (-1)^{n(s+k)+k} \times \det[M]\det[A]$. By (3),

$$\operatorname{ind}(p, \overline{x}) = (-1)^{n(s+k)+k} \operatorname{sign}\det[D\widetilde{H}(\overline{x}, \overline{\lambda})] \,. \ \blacksquare$$

3. Main theorem. Let $H : \mathbb{R}^n \times \mathbb{R}^s \to \mathbb{R}^m \times \mathbb{R}^k$ be given by

$$H(x, \lambda) = \Big(\sum_{i=1}^{s} \lambda_i G_i(x), F(x) \Big) \,.$$

LEMMA 3. $H^{-1}(0) = W \times \{0\}$.

Proof. If $(x, \lambda) \in H^{-1}(0)$ then $F(x) = 0$, i.e. $x \in W$. By our assumption, the vectors $G_1(x), \ldots, G_s(x)$ are linearly independent, and so $\lambda = 0$. $\blacksquare$

Let $B_R = \{(x, \lambda) \mid \|x\|^2 + \|\lambda\|^2 < R^2\}$ and $S_R = \partial B_R$. Since W is compact, by the above lemma there is $R > 0$ such that $H^{-1}(0) \subset B_R$. Hence $H|S_R : S_R \to \mathbb{R}^m \times \mathbb{R}^k - \{0\}$. Let $\deg(H|S_R)$ be the topological degree of $H|S_R$.

THEOREM 4. $\mathrm{X}(E) = (-1)^{n(s+k)+k} \deg(H|S_R)$.

Proof. Let $D_R = \{x \in \mathbb{R}^n \mid \|x\| < R\}$. For each C^1-map $P : \mathbb{R}^n \to \mathbb{R}^m$ there are sections $p : W \to E$, $p' : W \to E'$ such that $P|W = p + p'$. For each $\varepsilon > 0$ we can choose P so that

(1) $$\sup_{x \in D_R} \|P(x)\| < \varepsilon \,,$$

(2) if $p(x) = 0$ then $\operatorname{ind}(p, x) \neq 0$, i.e. p is transversal to the zero-section.

Let $\widetilde{H} = \widetilde{H}(x, \lambda) = (P(x) + \sum_{i=1}^{s} \lambda_i G_i(x), F(x))$. From (1) and Lemma 3 we can show (using Cramer's rule) that $\widetilde{H}^{-1}(0)$ lies close to $W \times \{0\}$ and thus, for small ε, $\widetilde{H}^{-1}(0) \subset B_R$. The manifold W is compact and so, from (2) and Lemma 1, $\widetilde{H}^{-1}(0)$ is finite, say $\widetilde{H}^{-1}(0) = \{(x^1, \lambda^1), \ldots, (x^m, \lambda^m)\}$.

Then $p^{-1}(0) = \{x^1, \ldots, x^m\}$ and according to the definition of $\mathrm{X}(E)$ (see [1], Chapter 5.2) and Lemma 2

$$\mathrm{X}(E) = \sum_{j=1}^{m} \operatorname{ind}(p, x^j) = (-1)^{n(s+k)+k} \sum_{j=1}^{m} \operatorname{sign}\det[D\widetilde{H}(x^j, \lambda^j)]\,.$$

Clearly the last sum equals $\deg(\widetilde{H}|S_R)$, and since $H|S_R$ and $\widetilde{H}|S_R$ are homotopic for ε small enough, we conclude that

$$\mathrm{X}(E) = (-1)^{n(s+k)+k} \deg(H|S_R)\,. \ \blacksquare$$

Clearly $TW = \{(x,y) \in W \times \mathbb{R}^n \mid y \perp \operatorname{grad} F_i(x),\ i = 1, \ldots, k\}$. It is well known that $\mathrm{X}(TW) = \chi(W)$, where $\chi(W)$ is the Euler characteristic of W. Let $H : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n \times \mathbb{R}^k$ be given by

$$H(x, \lambda) = \Big(\sum_{i=1}^{k} \lambda_i \operatorname{grad} F_i(x), F(x)\Big)\,.$$

As above, there is $R > 0$ such that $H^{-1}(0) \subset B_R$ and so we have a continuous map $H|S_R : S_R \to \mathbb{R}^n \times \mathbb{R}^k - \{0\}$. As a consequence of Theorem 4 we have

Theorem 5. $\chi(W) = (-1)^k \deg(H|S_R)$. $\blacksquare$

A very similar version of the above theorem has been proved in [2].

Example 1. Let $W = S^2 = \{x \in \mathbb{R}^3 \mid x_1^2 + x_2^2 + x_3^2 - 1 = 0\}$, let $G = G(x) = (3 + x_1x_2 - x_3^2, x_1x_2 - x_2, x_1 - x_2x_3)$, and let $E_1 = \{(x,y) \in S^2 \times \mathbb{R}^3 \mid y \perp G(x)\}$. Then

$$\begin{aligned} H &= H(x, \lambda) \\ &= (3\lambda + x_1x_2\lambda - x_3^2\lambda, x_1x_2\lambda - x_2\lambda, x_1\lambda - x_2x_3\lambda, x_1^2 + x_2^2 + x_3^2 - 1) \end{aligned}$$

and $R = 2$. Thanks to a computer program written by Marek Izydorek and Sławomir Rybicki from the Mathematical Department of the Technical University of Gdańsk we have been able to calculate that $\deg(H|S_2) = 0$, so $\mathrm{X}(E) = 0$.

Example 2. Let $G = G(x) = (3x_1 + x_1x_2^2, 3x_2 + x_2x_3, 3x_3)$, and let $E_2 = \{(x,y) \in S^2 \times \mathbb{R}^3 \mid y \perp G(x)\}$. Then

$$H = H(x, \lambda) = (3x_1\lambda + x_1x_2^2\lambda, 3x_2\lambda + x_2x_3\lambda, 3x_3\lambda, x_1^2 + x_2^2 + x_3^2 - 1)$$

and $R = 2$. As above we have calculated that $\deg(H|S_2) = -2$, so $\mathrm{X}(E) = 2$.

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INSTITUTE OF MATHEMATICS
UNIVERSITY OF GDAŃSK
WITA STWOSZA 57
80-952 GDAŃSK, POLAND

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