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The existence of bounded solutions for differential equations in Hilbert spaces

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Abstract. The existence of bounded solutions for equations $x' = A(t)x + r(x,t)$ is proved, where the linear part is exponentially dichotomic and the nonlinear term r satisfies some weak conditions.

Introduction. We shall deal with the nonlinear differential equations of the form

$$x' = A(t)x + r(x,t) \tag{1}$$

where $A : \mathbb{R} \to L(H)$ is a continuous function taking values in the space $L(H)$ of bounded linear operators in a Hilbert space H and $r : H \times \mathbb{R} \to H$ is a nonlinear continuous mapping. We shall assume that the linear equation $x' = A(t)x$ is exponentially dichotomic, hence there exists the Main Green Function $G : \mathbb{R} \to L(H)$ such that

$$||G(t,s)|| \le Ne^{-\alpha|t-s|} \tag{2}$$

for all $t, s \in \mathbb{R}$, where N and α are some positive constants. The bounded solutions of (1) are the fixed points of the following operator $S : BC(\mathbb{R}, H) \to BC(\mathbb{R}, H)$:

$$Sx(t) := \int_{-\infty}^{\infty} G(t,s)r(x(s),s)\,ds \tag{3}$$

where $BC(\mathbb{R}, H)$ denotes the space of all bounded continuous functions $x : \mathbb{R} \to H$.

First, we shall consider the assumptions on r that yield the compactness of S. We need a criterion for compactness of sets in the space $BC(\mathbb{R}, E)$ (we replace the Hilbert space H by an arbitrary Banach space E since the result is of independent interest). Then we pass to other assumptions which imply that S is ultimately compact (in the sense of Sadovskiĭ [5]). We apply the

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appropriate degree theories in both cases: the Leray–Schauder theory and the Sadovskiĭ theory.

The second part of the paper is devoted to finding a priori bounds for fixed points of λS ($\lambda \in [0,1]$), under various assumptions on r (and, obviously, on A). Only for this purpose do we need a scalar product in the space.

1. Compact subsets of $BC(\mathbb{R}, E)$. Let E be a Banach space and let $BC(\mathbb{R}, E)$ be the space of all bounded continuous functions $x : \mathbb{R} \to E$ with the norm

$$||x||_\infty := \sup_{t\in\mathbb{R}} ||x(t)|| \,. \tag{4}$$

It is evident that $BC(\mathbb{R}, E)$ is a Banach space.

THEOREM 1. *For a set $D \subset BC(\mathbb{R}, E)$ to be relatively compact, it is necessary and sufficient that*:

1° $\{x(t) : x \in D\}$ *is relatively compact in E for any $t \in \mathbb{R}$*,

2° *for each $a > 0$, the family $D_a := \{x|[-a, a] : x \in D\}$ is equicontinuous*,

3° *D is stable at $\pm\infty$, i.e. for any $\varepsilon > 0$, there exist $T > 0$ and $\delta > 0$ such that if $||x(T) - y(T)|| \le \delta$ then $||x(t) - y(t)|| \le \varepsilon$ for $t \ge T$ and if $||x(-T) - y(-T)|| \le \delta$ then $||x(t) - y(t)|| \le \varepsilon$ for $t \le -T$ where x and y are arbitrary functions in D.*

Proof. *Sufficiency.* Suppose that conditions 1°–3° hold and take $\varepsilon > 0$. We shall construct a finite ε-net of D.

Let T and δ be positive constants chosen for ε by 3° and $\delta \le \varepsilon$. By 1°, we can find finite δ-nets: $x_1(T), \ldots, x_p(T)$ in $\{x(T) : x \in D\}$ and $y_1(-T), \ldots$ $\ldots, y_r(-T)$ in $\{x(-T) : x \in D\}$. Repeating the proof of the General Ascoli–Arzelà Theorem, we get a finite δ-net of the relatively compact set $D_T \subset C([-T, T], E)$ consisting of functions $z_1, \ldots, z_s$ which take one of the values $y_j(-T)$, $j = 1, \ldots, r$, at $-T$, and one of the values $x_i(T)$, $i = 1, \ldots, p$, at T. Then it is obvious that the set of all continuous functions $\mathbb{R} \to E$ which are obtained by "gluing" y_j, z_k and x_i together is a finite ε-net for D.

Necessity. If D is relatively compact, then 1° and 2° hold, since D_a, $a > 0$, are relatively compact in $C([-a, a], E)$ and by the General Ascoli–Arzelà Theorem. Suppose that D is not stable at $+\infty$, for instance, i.e. there exist $\varepsilon_0 > 0$ and sequences $(x_n), (y_n) \subset D$ and $(t_n) \subset \mathbb{R}$ such that

$$||x_n(n) - y_n(n)|| \le 1/n \,, \quad ||x_n(t_n) - y_n(t_n)|| > \varepsilon_0 \,, \quad t_n \ge n \,,$$

for any $n \in \mathbb{N}$.

Since D is relatively compact, it has a finite $\varepsilon_0/4$-net $z_1, \ldots, z_p$. We shall define $w_1, \ldots, w_p \in BC(\mathbb{R}, E)$ such that

$$||w_i(t) - z_i(t)|| \le \varepsilon_0/4\,, \qquad t \in \mathbb{R}\,, \tag{5}$$

for $i \le p$ and

$$||w_i(n) - w_j(n)|| \ge \varepsilon_0/8p \tag{6}$$

for $n \in \mathbb{N}$ and $i \ne j$. Let $w_1 = z_1$ and assume that $w_1, \ldots, w_k$ have already been defined. Fix $n \in \mathbb{N}$. If (6) is satisfied for $i \le k$ and w_j replaced by z_{k+1}, we put $h_{k+1,n} = z_{k+1}(n)$. If not, we choose $h_{k+1,n}$ arbitrarily from the set

$$\overline{B}(z_{k+1}(n), \varepsilon_0/4) \setminus \bigcup_{i=1}^{k} B(w_i(n), \varepsilon_0/8p)$$

where $B(u, a)$ (resp. $\overline{B}(u, a)$) stands for the open (resp. closed) ball with centre u and radius a. Then we define a multivalued mapping $\varPhi_{k+1} : \mathbb{R} \to 2^E$ by

$$\varPhi_{k+1}(t) := \begin{cases} \overline{B}(z_{k+1}(t), \varepsilon_0/4), & t \notin \mathbb{N}, \\ h_{k+1,n}, & t = n \in \mathbb{N}. \end{cases}$$

Rather simple calculations show that $\varPhi_{k+1}$ is lower semicontinuous. By the Michael Selection Theorem it has a continuous selection $w_{k+1} : \mathbb{R} \to E$. By induction, we obtain $w_i, \ldots, w_p$ having properties (5) and (6). Obviously, $\{w_1, \ldots, w_p\}$ is an $\varepsilon_0/2$-net for D.

Let us return to the sequences (x_n) and (y_n). We can choose a subsequence (x_{n_m}) contained in an $\varepsilon_0/2$-neighbourhood of one element w_j. By (6) and $x_n(n) - y_n(n) \to 0$, y_{n_m} is in the same ball for sufficiently large m. Hence,

$$||x_{n_m}(t) - y_{n_m}(t)|| \le 2 \cdot \varepsilon_0/2 = \varepsilon_0$$

for each $t \in \mathbb{R}$, which is impossible for $t = t_{n_m}$. ∎

Remark. It is surprising that the set

$$\bigcup_{x \in D} \bigcup_{t \in \mathbb{R}} x(t) \tag{7}$$

need not be relatively compact in E if D is so in $BC(\mathbb{R}, E)$. Counterexamples are only possible for infinite-dimensional E but are very simple: one-point sets. However, if $x(\mathbb{R})$ is relatively compact for any $x \in D$ and D has the same property, then the set (7) is also relatively compact.

2. Compactness of the operator S. Now, we consider the nonlinear integral Hammerstein operator $S : BC(\mathbb{R}, E) \to BC(\mathbb{R}, E)$ given by (3) where the Green function $G : \mathbb{R}^2 \to L(E)$ satisfies (2) and $r : E \times \mathbb{R} \to E$ is continuous.

THEOREM 2. *If, in addition, r has the following properties*:

(8) $r(\cdot,t) : E \to E$ *is completely continuous for any* $t \in \mathbb{R}$,

(9) *there exists a bounded continuous function* $b : \mathbb{R} \to E$ *such that for any* M *and* $\varepsilon > 0$, *there is* $T > 0$ *such that* $||r(x,t) - b(t)|| \le \varepsilon$ *where* $||x|| \le M$ *and* $|t| \ge T$,

then S is completely continuous.

Proof. We shall show that the image of $\{x \in BC(\mathbb{R},E) : ||x||_\infty \le M\}$ under S is relatively compact.

First, we prove condition 1° of Theorem 1. Fix $t \in \mathbb{R}$ and $\varepsilon > 0$. We shall find a finite ε-net for $A := \{Sx(t) : ||x||_\infty \le M\}$. Choose $T > 0$ such that

$$||r(x,s) - b(s)|| \le \frac{\alpha}{4N}\varepsilon \tag{10}$$

for $||x|| \le M$ and $|s| > T$. Set

$$x_0(t) = \int\limits_{|s|>T} G(t,s)b(s)\,ds\,.$$

Then $x_0 \in BC(\mathbb{R},E)$. By (2) and (10)

$$\Big\|\int\limits_{|s|>T} G(t,s)(r(x(s),s) - b(s))\,ds\Big\| \le \frac{\alpha}{4N}\varepsilon \int\limits_{\mathbb{R}} Ne^{-\alpha|t-s|}\,ds = \frac{\varepsilon}{2}\,.$$

Consider $B := \{\int_{|s|\le T} G(t,s)r(x(s),s)\,ds : ||x||_\infty \le M\}$. It is easy to see that the set $\{r(x,s) : ||x|| \le M,\ |s| \le T\}$ is relatively compact ((8) and the continuity of r). Since $G(t,\cdot)$ is continuous for $s < t$ and $s > t$ and it has finite limits as $s \to t^\pm$, the set

$$Z := \{G(t,s)r(x,s) : ||x|| \le M\,,\ |s| \le T\}$$

is also relatively compact. But the integrals in B belong to the convex hull $2T\,\overline{\mathrm{conv}}\,Z$, so there exists a finite $\varepsilon/2$-net of B: $x_1,\ldots,x_p$. Now, we easily see that $x_0 + x_1,\ldots,x_0 + x_p$ constitute an ε-net of A:

$$\begin{aligned}||Sx(t) - (x_0 + x_j)|| &\le \Big\|\int\limits_{|s|\le T} G(t,s)r(x(s),s)\,ds - x_j\Big\| \\ &\quad + \Big\|\int\limits_{|s|>T} G(t,s)(r(x(s),s) - b(s)))\,ds\Big\| \le 2\frac{\varepsilon}{2} = \varepsilon\,.\end{aligned}$$

Now, we prove condition 2°. Fix $a > 0$, $\varepsilon > 0$, and take T_1 from (9) such that

$$||r(x,s) - b(s)|| \le ||b||_\infty + 1, \quad ||x|| \le M,\ |s| > T_1\,.$$

Next, we choose $T \geq T_1$ such that

$$\int_{|s|>T} e^{-\alpha|t-s|} ds \leq \frac{\varepsilon}{3 \cdot 2N(||b||_\infty + 1)}$$

for all $t \in [-a, a]$. Let

$$M_1 = \sup_{|s|\leq T,\ ||x||\leq M} ||r(x,s)|| \,. \tag{11}$$

Since G is uniformly continuous on $|t| \leq a$, $|s| \leq T$, $s > t$ and on $|t| \leq a$, $|s| \leq T$, $s < t$, there exists $\delta > 0$ such that $|t' - t| \leq \delta$ implies $||G(t', s) - G(t, s)|| \leq \varepsilon/12M_1T$ for any $|s| \leq T$ and either $t, t' \in [-a, \min(s, a))$ or $t, t' \in (\max(s, -a), a]$. One can assume that $\delta \leq \varepsilon/12NM_1$. Finally, if $||x||_\infty \leq M$ and $t, t' \in [-a, a]$, $|t' - t| \leq \delta$, then

$$\begin{aligned}
||Sx(t') - Sx(t)|| \leq & \int_{|s|\leq T,\, s\notin[t,t']} ||G(t', s) - G(t, s)|| \cdot ||r(x(s), s)||\, ds \\
& + 2 \int_{|s|\leq T,\, s\in[t,t']} N||r(x(s), s)||\, ds \\
& + 2 \int_{|s|>T} \sup_{|t|\leq a} ||G(t, s)||\, ||r(x(s), s) - b(s)||\, ds \\
& + 2 \int_{|s|>T} \sup_{|t|\leq a} ||G(t, s)||\, ||b||_\infty\, ds \\
\leq 2T \frac{\varepsilon}{12M_1T} M_1 & + 2\delta N M_1 + (4||b||_\infty + 2)N \frac{\varepsilon}{6N(||b||_\infty + 1)} < \varepsilon \,.
\end{aligned}$$

It remains to prove condition 3°. Take $\varepsilon > 0$ and $T_1 > 0$ such that

$$||r(x, s) - b(s)|| \leq \frac{\varepsilon\alpha}{8N}$$

for $|s| > T_1$ and $||x|| \leq M$. Let M_1 be given by (11) where T is replaced by T_1. We can find sufficiently large T such that, for $|t| \geq T$ and $|s| \leq T_1$,

$$||G(t, s)|| \leq N e^{-\alpha(T-T_1)} < \frac{\varepsilon}{8M_1T_1} \,.$$

For $|t| \geq T$ and $||x||_\infty$, $||y||_\infty \leq M$, we have

$$\begin{aligned}
||Sx(t) - Sy(t)|| \leq & \; 2 \int_{|s|>T_1} ||G(t, s)|| \sup_{||x||\leq M} ||r(x, s) - b(s)||\, ds \\
& + 2 \int_{|s|\leq T_1} ||G(t, s)|| M_1\, ds \\
& \leq \frac{\varepsilon\alpha}{4N} \int_{\mathbb{R}} N e^{-\alpha|t-s|}\, ds + 2M_1 \frac{\varepsilon}{8M_1T_1} 2T_1 = \varepsilon \,.
\end{aligned}$$

We have verified a stronger condition than 3°, namely,

$$\lim_{t\to\pm\infty} \sup_{||x||_\infty,||y||_\infty\le M} ||Sx(t)-Sy(t)||=0\,.$$

This ends the proof.

Remark. In fact, we have only needed the following properties of G: the estimate (2), measurability with respect to the second variable, piecewise continuity with respect to the first variable.

3. Condensing property of S. One can use many degree theories to get the solvability of our integral equation. In the case of infinite-dimensional space E, it is usually effective to apply some measures of noncompactness.

Let ψ be a real nonnegative function defined on the family of all bounded subsets of a Banach space E having the following properties:

(a) $\psi(\overline{\operatorname{conv}}\,X)=\psi(X)$,
(b) $X\subset Y\Rightarrow \psi(X)\le\psi(Y)$,
(c) $\psi(X\cup Y)=\max(\psi(X),\psi(Y))$,
(d) $\psi(X+Y)\le\psi(X)+\psi(Y)$,
(e) for any $\varepsilon>0$, there exists $\delta>0$ such that

$$\operatorname{diam} X\le\delta \quad\text{implies}\quad \psi(X)\le\varepsilon\,,$$

(f) $\psi(X)=0$ if and only if X is relatively compact,
(g) $\psi(A(X))\le||A||\psi(X)$ for each bounded linear operator $A\in L(E)$.

We shall say that ψ is a *measure of noncompactness* in E.

Remarks. Sadovskiĭ's ([5]) measures of noncompactness have only property (a) but all functions of this type used in practice are: monotonic (b), semiadditive (c), algebraically subadditive (d), continuous with respect to the Hausdorff distance at one-point sets (e) and regular (f). For example, the *Kuratowski measure of noncompactness* ([3])

$$\alpha(X):=\inf\{d>0: X=X_1\cup\ldots\cup X_n \text{ for some } n\in\mathbb{N},\\ \operatorname{diam} X_j\le d\,,\ j=1,\ldots,n\}\,,$$

and the *Hausdorff measure of noncompactness*

$$\beta(X):=\inf\{\varepsilon>0: X \text{ admits a finite } \varepsilon\text{-net}\}$$

have all properties (a)–(g). In the case of Hilbert space with a fixed complete orthonormal system $\{e_n:n\in\mathbb{N}\}$ one can also consider the function

$$\gamma(X):=\limsup_{n\to\infty}\sup_{x\in X}\Big(\sum_{k=n}^{\infty}|(x,e_k)|^2\Big)^{1/2}.$$

It satisfies (a)–(f), and (g) only for operators A commuting with all projectors R_n onto $\overline{\mathrm{Lin}}\{e_k : k \geq n\}$, the closed linear subspace spanned by $e_k,\ k \geq n$.

Every measure of noncompactness ψ in E induces a function Ψ defined on the family of bounded subsets of $BC(\mathbb{R}, E)$ by

$$\Psi(D) := \psi\{x(t) : x \in D,\ t \in \mathbb{R}\}\,.$$

It is easy to see that Ψ has properties (a)–(e) (see [5]; in fact we need only (a)). Moreover, if $\Psi(D) = 0$, the functions in D are equicontinuous on any compact interval and D is stable at $\pm\infty$, then D is a relatively compact subset of $BC(\mathbb{R}, E)$.

Let Ω be an open bounded subset of a Banach space E and let $S : \overline{\Omega} \to E$ be a continuous map. Define the following transfinite sequence:

$$\begin{aligned}
&\Omega_1 := \overline{\mathrm{conv}}\, S(\Omega),\\
&\Omega_\alpha := \overline{\mathrm{conv}}\, S(\Omega \cap \Omega_{\alpha-1}) \quad \text{if } \alpha \text{ is a successor ordinal},\\
&\Omega_\alpha := \bigcap_{\beta<\alpha} \Omega_\beta \quad \text{if } \alpha \text{ is a limit ordinal}\,.
\end{aligned}$$

It is well known ([5]) that there is α such that $\Omega_\alpha = \Omega_{\alpha+1} = \ldots$ The set Ω_α is then called the *ultimate domain* of S and denoted by Ω_∞. When Ω_∞ is compact, the map S is said to be *ultimately compact.* If the ultimately compact mapping S has no fixed points on the boundary of Ω, then one can define a topological degree of $I - S$ (where I is the identity operator) on the set Ω at the point 0. One restricts S to Ω_∞, then extends $S|\Omega_\infty$ to a compact map $S_* : \overline{\Omega} \to \Omega_\infty$ fixed point free on the boundary, and finally one defines

$$\deg(I - S, \Omega, 0) := \deg_{\mathrm{LS}}(I - S_*, \Omega, 0)$$

where $\deg_{\mathrm{LS}}$ stands for the Leray–Schauder degree.

Consider a function ψ in E satisfying only (a), (b) and (f) and a ψ-*condensing* operator $S : \overline{\Omega} \to E$, i.e.

$$\psi(S(X)) < \psi(X) \quad \text{if } X \text{ is not relatively compact in } \overline{\Omega}\,.$$

It is easy to see that S is ultimately compact—the crucial point is property (f). But we want to use this construction in the space $BC(\mathbb{R}, E)$ with the function Ψ without property (f). In this case, we have to require that S is not only Ψ-condensing on $\overline{\Omega}$ but $S(\overline{\Omega})$ is stable at $\pm\infty$ and is composed of functions which are equicontinuous on each bounded interval. Then the ultimate domain is compact and the degree theory can be applied.

THEOREM 3. *Let $G : \mathbb{R}^2 \to L(E)$ be continuous for $s \neq t$, have finite limits as $s \to t^\pm$ and satisfy the estimate* (2). *Let $r : E \times \mathbb{R} \to E$ be* *continuous, bounded on bounded sets, and let it satisfy*

$$\psi(r(D,s)) \le k(s)\psi(D) \tag{12}$$

for any bounded set $D \subset E$ and $s \in \mathbb{R}$ where ψ is a measure of noncompactness in E having all properties (a)–(g) *and $k : \mathbb{R} \to [0,\infty)$ is a locally integrable function such that*

$$k(t) < \frac{\alpha}{2N}\,, \qquad t \in \mathbb{R}\,. \tag{13}$$

Suppose that r satisfies (9) *and the limit function $b : \mathbb{R} \to E$ appearing in* (9) *has the property that*

$$\Big\{ \int\limits_{-\infty}^{\infty} G(t,s)b(s)\,ds : t \in \mathbb{R}\Big\} \tag{14}$$

is relatively compact in E. Then the integral operator $S : BC(\mathbb{R},E) \to BC(\mathbb{R},E)$ given by (3) *is ultimately compact on each bounded subset of $BC(\mathbb{R},E)$.*

Proof. The second and the third parts of the proof of Theorem 2 are based only on (9) and the properties of G, so the image of $\{x \in BC(\mathbb{R},E) : \|x\|_\infty \le M\}$ under S is stable at $\pm\infty$ and its elements are equicontinuous on bounded intervals. Hence, we should verify that S is Ψ-condensing.

Fix a bounded set $X \subset BC(\mathbb{R},E)$ and fix $x \in X$. Then, for any $T > 0$,

$$\begin{aligned} Sx(t) = &\int\limits_{|s|\le T} G(t,s)r(x(s),s)\,ds + \int\limits_{|s|>T} G(t,s)(r(x(s),s) - b(s)))\,ds \\ &+ \int\limits_{\mathbb{R}} G(t,s)b(s)\,ds - \int\limits_{|s|\le T} G(t,s)b(s)\,ds\,. \end{aligned}$$

The third summand takes values in the relatively compact set (14). By (2) and (g), the same is true for the fourth summand. Applying properties (d) and (f) of ψ, we get

$$\begin{aligned} \psi\{Sx(t) : x \in X,\ t \in \mathbb{R}\} &\le \psi\Big\{ \int\limits_{|s|\le T} G(t,s)r(x(s),s)\,ds : x \in X,\ t \in \mathbb{R}\Big\} \\ &\quad + \psi\Big\{ \int\limits_{|s|>T} G(t,s)(r(x(s),s) - b(s))\,ds : x \in X\,,\ t \in \mathbb{R}\Big\} \\ &=: a_1 + a_2 \end{aligned} \tag{15}$$

for every $T > 0$.

Take $\varepsilon > 0$. There exists $T > 0$ such that $a_2 \le \varepsilon/2$ by (9), (2) and (e).

We shall find a bound for a_1. First, choose $T_1 > 0$ such that

$$\psi\Big\{ \int\limits_{|s|\le T} G(t,s)r(x(s),s)\,ds : x\in X\,,\ |t|>T_1\Big\} \le \frac{\varepsilon}{4} \tag{16}$$

(this is possible due to (2)). Now, divide $[-T_1, T_1]$ into small intervals $-T_1 = t_0 < t_1 < \ldots < t_n = T_1$ in such a way that

$$||G(t,s) - G(t_i,s)|| \le \frac{\varepsilon}{8MT} \tag{17}$$

for $|s| \le T$ and $t \in [t_{i-1}, t_i]$, $i = 1,\ldots,n$, where M is an upper bound for r on X. Hence, by (c) and (d),

$$\begin{aligned}
&\psi\Big\{ \int\limits_{|s|\le T} G(t,s)r(x(s),s)\,ds : |t|\le T_1,\ x\in X\Big\} \\
&\quad\le \max_{1\le i\le n} \psi\Big\{ \int\limits_{|s|\le T} G(t_i,s)r(x(s),s)\,ds : x\in X\Big\} \\
&\qquad + \max_{1\le i\le n} \psi\Big\{ \int\limits_{|s|\le T} (G(t,s)-G(t_i,s))r(x(s),s)\,ds : \\
&\qquad\qquad x\in X,\ t\in[t_{i-1},t_i]\Big\}\,.
\end{aligned} \tag{18}$$

Applying (17) and (g), we see that the second term is not greater than $\varepsilon/4$. In order to estimate the first one, we divide $[-T, T]$ arbitrarily: $-T = s_0 < s_1 < \ldots < s_m = T$, and fix $i = 1,\ldots,n$. Since

$$\int\limits_a^b f(t)\,dt \in (b-a)\,\overline{\mathrm{conv}}\{f(t) : t\in[a,b]\}\,, \tag{19}$$

we can make the following calculations:

$$\begin{aligned}
&\psi\Big\{ \int\limits_{|s|\le T} G(t_i,s)r(x(s),s)\,ds : x\in X\Big\} \\
&\quad\le \sum_{j=1}^m \psi\Big\{ \int\limits_{s_{j-1}}^{s_j} G(t_i,s)r(x(s),s)\,ds : x\in X\Big\} \\
&\quad\le \sum_{j=1}^m (s_j - s_{j-1})\psi[\overline{\mathrm{conv}}\{G(t_i,s)r(x(s),s) : \\
&\qquad\qquad x\in X,\ s\in[s_{j-1},s_j]\}] \\
&\quad\le \sum_{j=1}^m (s_j - s_{j-1}) N e^{-\alpha|t_i-\xi_j|} \sup_{s_{j-1}\le s\le s_j} k(s)\varPsi(X)
\end{aligned}$$

where we have applied, in turn, (d), (18), (a), (2), (g) and (12); here $\xi_j = s_{j-1}$ or s_j, depending on which makes $|t-\xi_j|$ smaller. Summing up (15), (16), (18) and the last inequality, we obtain

$$\Psi(S(X)) \le \Psi(X)N \max_{1\le i\le n} \int\limits_{|s|\le T} e^{-\alpha|t_i-s|}k(s)\,ds + \varepsilon$$

(recall that k is locally integrable). We can let $T\to\infty$ and $\varepsilon\to 0$, therefore it suffices to show that

$$\int\limits_{-\infty}^{\infty} e^{-\alpha|t-s|}k(s)\,ds < \frac{1}{N} \tag{20}$$

for any $t\in\mathbb{R}$. Consider the functions

$$h_1(t) := e^{-\alpha t}\int\limits_{-\infty}^{t} e^{\alpha s}k(s)\,ds\,, \qquad h_2(t) := e^{\alpha t}\int\limits_{t}^{\infty} e^{-\alpha s}k(s)\,ds\,.$$

If h_1 takes its maximum value at t_1, then $h_1'(t_1) = 0$, hence $h_1(t_1) = k(t_1)/\alpha$. If h_1 tends to this maximum as $t\to\pm\infty$, then this limit equals $\lim_{t\to\pm\infty} k(t)/\alpha$. The same argument works for h_2, so condition (20) is satisfied and this ends the proof. ■

Remarks. The theorem is also true if the measure of noncompactness ψ has property (g) only for $G(t,s)$, $t,s\in\mathbb{R}$, or if (g) is replaced by a weaker condition:

$$\psi(A(X)) \le c||A||\psi(X)$$

where $c>0$ is a constant.

The compactness of the set (14), in the case when G is the Main Green Function for the equation $x' = A(t)x$, means that the unique bounded solution of the nonhomogeneous equation

$$x' = A(t)x + b(t)$$

has a relatively compact range. This happens, for example, when A and b are almost periodic functions ([4]).

It is surprising that condition (12) has been assumed first in the work of Goebel and Rzymowski ([2]) to get a local solution of the equation $x' = r(x,t)$ with $x(t_0) = x_0$ in a Banach space.

Now, we are in a position to consider the nonlinear equation (1) assuming that the linear equation $x' = A(t)x$ is exponentially dichotomic (see [1]) and r satisfies the conditions of Section 2 or Section 3. Instead of (1), we deal with the homotopical family of equations

$$x' = A(t)x + \xi r(x,t)\,, \qquad \xi\in[0,1]\,. \tag{21}$$

We shall add some assumptions on r (and A) which guarantee that all bounded solutions of (21) are contained in a certain ball. Via the appropriate degree theory, this will imply the existence of bounded solutions for (1). From now on the Banach space E is replaced by a Hilbert space H.

4. The case of A constant. Let $A \in L(H)$ and suppose $\operatorname{Sp} A$ does not meet the imaginary axis. Then we can define two projectors P_+ and P_- by

$$P_\pm := -\frac{1}{2\pi i} \int_{\Gamma_\pm} (A - \lambda I)^{-1}\, d\lambda$$

where Γ_+ is a positively oriented Jordan closed curve encircling $\operatorname{Sp} A \cap \{\lambda : \operatorname{Re} \lambda > 0\}$, and Γ_- is a similar curve for $\operatorname{Sp} A \cap \{\lambda : \operatorname{Re} \lambda < 0\}$. We have

$$P_+ + P_- = I\,,$$

i.e. the operators project onto complementary subspaces (not necessarily orthogonal).

The so-called *Main Green Function*

$$G_A(t) := \begin{cases} \exp At \circ P_- & \text{for } t > 0, \\ -\exp At \circ P_+ & \text{for } t < 0, \end{cases}$$

has the property $\|G_A(t)\| \le N \exp(-\alpha|t|)$ for any $t \in \mathbb{R}$, and putting $G(t,s) := G_A(t-s)$ in (3) leads to bounded solutions of the equation

$$x' = Ax + r(x,t)\,.$$

It is known ([5]) that $H_\pm = P_\pm(H)$ are A-invariant subspaces. Since $\operatorname{Sp}(A|H_-)$ is contained in the left open half-plane, then, by the General Lyapunov Theorem, there exists a strictly positive operator $W_- \in L(H_-)$ such that

$$\operatorname{Re}(W_- A x_-, x_-) \le -\lambda_- \|x_-\|^2 \tag{22}$$

for each $x_- \in H_-$ where λ_- is a positive constant. Similarly, there exists a strictly positive operator $W_+ \in L(H_+)$ such that

$$\operatorname{Re}(W_+ A x_+, x_+) \ge \lambda_+ \|x_+\|^2 \tag{23}$$

for $x_+ \in H_+$ where $\lambda_+ > 0$.

Suppose that

$$\limsup_{\|x\| \to \infty} \|P_\pm r(x,t)\| / \|P_\pm x\| = L_\pm(t) < \infty \tag{24}$$

uniformly with respect to $t \in \mathbb{R}$ and that

$$\sup_t L_+(t) < \lambda_+ / \|W_+\|\,, \qquad \sup_t L_-(t) < \lambda_- / \|W_-\|\,. \tag{25}$$

THEOREM 4. *Under the above assumptions, the equation $x' = Ax + r(x,t)$ has a bounded solution.*

Proof. It is obvious that if we replace r by ξr, $\xi \in [0,1]$, then (25) also holds. Hence, it suffices to find an a priori bound for bounded solutions of our equation ($\xi = 1$).

Let $\phi : \mathbb{R} \to H$ be a bounded solution and $\phi_\pm := P_\pm \phi$. Take $\varepsilon > 0$ such that

$$\varepsilon < \min(\lambda_+/||W_+|| - \sup_t L_+(t),\ \lambda_-/||W_-|| - \sup_t L_-(t))\,. \tag{26}$$

By (24), there exists $M > 0$ such that

$$||P_\pm r(x,t)|| \le (L(t) + \varepsilon)||P_\pm x|| \tag{27}$$

for $||x|| \ge M$ and $t \in \mathbb{R}$. Since $W_\pm$ is strictly positive, the norm

$$||x_\pm||_\pm := \sqrt{(W_\pm x_\pm, x_\pm)}$$

is equivalent to the original one on $H_\pm$, i.e.

$$a_\pm ||x_\pm|| \le ||x_\pm||_\pm \le b_\pm ||x_\pm|| \tag{28}$$

for $x_+ \in H_+$ and $x_- \in H_-$. Define two differentiable functions $g_\pm : \mathbb{R} \to \mathbb{R}$ by

$$g_+(t) := ||\phi_+(t)||_+^2\,, \qquad g_-(t) := ||\phi_-(t)||_-^2\,.$$

There exists $t_1 \in \mathbb{R}$ such that

$$\begin{gathered} g_+(t) \le g_+(t_1) + 1\,, \qquad t \in \mathbb{R}\,, \\ |g_+'(t_1)| \le \delta\,, \qquad g_+(t_1) > \tfrac{1}{2} \sup_t g_+(t)\,, \end{gathered} \tag{29}$$

where δ is a sufficiently small constant. In fact, this is evident if the lowest upper bound of g_+ belongs to the range $g_+(\mathbb{R})$ and if this bound is the limit of a sequence $g_+(t_n)$ where $t_n \to \pm\infty$. Otherwise, g_+ is monotonic for $t \ge t_0$ (or $t \le t_0$) and $\lim_{t\to\infty} g_+'(t) = 0$. In the last case, t_1 can also be found. Similarly, there exists $t_2 \in \mathbb{R}$ such that

$$\begin{gathered} g_-(t) \le g_-(t_2) + 1\,, \qquad t \in \mathbb{R}\,, \\ |g_-'(t_2)| \le \delta\,, \qquad g_-(t_2) > \tfrac{1}{2} \sup_t g_-(t)\,, \end{gathered} \tag{29'}$$

If $||\phi(t_1)|| \le M$ and $||\phi(t_2)|| \le M$ then $||\phi_+(t_1)|| \le M$ and $||\phi_-(t_2)|| \le M$, hence $||\phi_+(t)||_+ \le b_+M + 1$, $||\phi_-(t)||_- \le b_-M + 1$ for each $t \in \mathbb{R}$ by (28) and (29), (29′). Applying once more (28), we obtain

$$||\phi(t)|| \le a_+^{-1}(b_+M + 1) + a_-^{-1}(b_-M + 1)\,. \tag{30}$$

If, however, either $||\phi(t_1)|| > M$ or $||\phi(t_2)|| > M$, then we get a contradiction. In the first case,

$$||P_+ r(\phi(t_1), t_1)|| \le (L(t_1) + \varepsilon)||\phi_+(t_1)||$$

from (27) and we have

$$\begin{aligned} g_+'(t_1) &= 2\,\mathrm{Re}(W_+A\phi_+(t_1),\phi_+(t_1)) + 2\,\mathrm{Re}(W_+P_+r(\phi(t_1),t_1),\phi_+(t_1)) \\ &\ge 2[\lambda_+ - ||W_+||(L_+(t_1)+\varepsilon)]||\phi_+(t_1)||^2 \ge 2c_+b_+^{-1}g_+(t_1) > c_+b_+^{-1}\sup_t g_+(t) \end{aligned}$$

where $c_+ = \lambda_+ - ||W_+||(\sup L_+(t) + \varepsilon) > 0$. It suffices to take $\delta = c_+b_+^{-1}\sup g_+(t)$ in (29). The inequality $||\phi(t_2)|| > M$ leads to a contradiction in the same way.

Therefore, we have the a priori estimate (30). ∎

5. A nonautonomous linear part. Let A depend on $t \in \mathbb{R}$ and suppose the linear equation $x' = A(t)x$ is exponentially dichotomic (in the sense of Daletskiĭ and Kreĭn [1]), i.e. $H = H_+ \oplus H_-$ and solutions starting at $t = 0$ from H_+ satisfy

$$||x(t)|| \le N_+e^{-\alpha_+(s-t)}||x(s)||\,, \qquad t \le s\,, \tag{31}$$

where the constants N_+ and $\alpha_+ > 0$ are independent of the solution, and similarly,

$$||x(t)|| \le N_-e^{-\alpha_-(t-s)}||x(s)||\,, \qquad s \le t\,, \tag{32}$$

for solutions starting from H_- (N_- and α_- are positive again). The difference between the above case and the case of A constant consists in the fact that, now, the subspaces H_+ and H_- rotate as t varies. More precisely, if P_+ and P_- are complementary projectors onto H_+ and H_-, then $P_\pm U(t) \ne U(t)P_\pm$ where U is the Cauchy operator of our equation. Let us introduce the notations:

$$P_+(t) = U(t)P_+U^{-1}(t)\,, \qquad P_-(t) = U(t)P_-U^{-1}(t)$$

for $t \in \mathbb{R}$. Obviously, $P_+(t)$ and $P_-(t)$ are complementary projectors and solutions passing at time t_0 through $P_+(t_0)H$ (resp. $P_-(t_0)H$) satisfy (31) (resp. (32)). We assume additionally that

$$\sup_t ||P_\pm(t)|| < \infty\,, \tag{33}$$

which has a geometric meaning that the angle between $P_+(t)H$ and $P_-(t)H$ cannot be arbitrarily small (see [1]).

Denote by $U(t,s) = U(t)U^{-1}(s)$ the evolution operator. It is easy to see that $U(t,s)$ transforms isomorphically $P_\pm(s)H$ onto $P_\pm(t)H$ and (31)–(33) imply

$$\begin{aligned} ||U(t,s)|P_-(s)H|| &\le N_1^-e^{-\alpha_-(t-s)}\,, \qquad s \le t\,, \\ ||U(t,s)|P_+(s)H|| &\le N_1^+e^{-\alpha_+(s-t)}\,, \qquad t \le s\,, \end{aligned} \tag{34}$$

Suppose that similar estimates hold for the inverse operators:

$$(35)\qquad \begin{aligned} ||U(s,t)|P_-(t)H|| &\le \frac{1}{N_2^-}e^{+\beta_-(t-s)}\,, \qquad s\le t\,,\\ ||U(s,t)|P_+(t)H|| &\le \frac{1}{N_2^+}e^{+\beta_+(s-t)}\,, \qquad t\le s\,, \end{aligned}$$

where β_-, β_+ are positive. Conditions (34) and (35) are equivalent to

$$(36)\qquad N_2^- e^{-\beta_-(t-s)}||x|| \le ||U(t)P_-U^{-1}(s)x|| \le N_1^- e^{-\alpha_-(t-s)}||x||$$

for $s\le t$ and $x\in P_-(s)H$, and

$$(37)\qquad N_2^+ e^{-\beta_+(s-t)}||x|| \le ||U(t)P_+U^{-1}(s)x|| \le N_1^+ e^{-\alpha_+(s-t)}||x||$$

for $t\le s$ and $x\in P_+(s)H$.

Recall that the Main Green Function is $G(t,s)=U(t)P_-U^{-1}(s)$ for $s<t$ and $G(t,s)=-U(t)P_+U^{-1}(s)$ for $t<s$. This implies the estimate (2) with $N=\max(N_1^+,N_1^-)$ and $\alpha=\min(\alpha_+,\alpha_-)$. In order to prove the existence of bounded solutions for (1), we should find an a priori bound for them.

Let

$$W_+(t) := \int\limits_{-\infty}^{t} U^{*-1}(t)P_+^*U^*(s)U(s)P_+U^{-1}(t)\,ds$$

for $t\in\mathbb{R}$ (the convergence of this integral is a consequence of (37)). By the second half of (37),

$$(W_+(t)x,x) = \int\limits_{-\infty}^{t} ||U(s)P_+U^{-1}(t)x||^2\,ds \ge \frac{1}{2\beta_+}(N_2^+)^2||x||^2$$

for $x\in P_+(t)H$, and similarly,

$$(W_+(t)x,x) \le \frac{1}{2\alpha_+}(N_1^+)^2||x||^2\,,$$

so the norms $||\ ||_{+t}$ defined on $P_+(t)H$ by the formula

$$||x||_{+t} := \sqrt{(W_+(t)x,x)}$$

are equivalent to the norm $||\ ||$ uniformly with respect to $t\in\mathbb{R}$. Analogously, the operator

$$W_-(t) := \int\limits_{t}^{\infty} U^{*-1}(t)P_-^*U^*(s)U(s)P_-U^{-1}(t)\,ds$$

enables us to define on $P_-(t)H$ the norm

$$||x||_{-t} := \sqrt{(W_-(t)x,x)}$$

equivalent to $||\ ||$ uniformly.

Consider a bounded solution $\phi : \mathbb{R} \to H$ of equation (1) and its projections $\phi_\pm(t) = P_\pm(t)\phi(t)$, $t \in \mathbb{R}$. We have

$$||\phi_+(t)||^2_{+t} = \int_{-\infty}^{t} ||U(s)P_+U^{-1}(t)\phi(t)||^2\, ds$$

and

$$\begin{aligned}(38)\qquad \frac{d}{dt}||\phi_+(t))||^2_{+t} &= ||\phi_+(t)||^2 \\ &\quad - 2\,\mathrm{Re}\int_{-\infty}^{t} (U(s)P_+U^{-1}(t)A(t)\phi(t), U(s)P_+U^{-1}(t)\phi(t))\, ds \\ &\quad + 2\,\mathrm{Re}\int_{-\infty}^{t} (U(s)P_+U^{-1}(t)\phi'(t), U(s)P_+U^{-1}(t)\phi(t))\, ds \\ &= ||\phi_+(t)||^2 + 2\,\mathrm{Re}(W_+(t)r(\phi(t),t),\phi(t)) \\ &= ||\phi_+(t)||^2 + 2\,\mathrm{Re}(W_+(t)P_+(t)r(\phi(t),t),\phi_+(t))\,,\end{aligned}$$

$$(39)\qquad \frac{d}{dt}||\phi_-(t)||^2_{-t} = -||\phi(t)||^2 + 2\,\mathrm{Re}(W_-(t)P_-(t)r(\phi(t),t),\phi_-(t))\,.$$

THEOREM 5. *Suppose that the estimates* (36) *and* (37) *hold and* $r : H \times \mathbb{R} \to H$ *satisfies the assumptions of Section* 2 *or Section* 3. *Let* $L_\pm : \mathbb{R} \to \mathbb{R}_+$ *be defined by* (24), *where we replace* $P_\pm$ *by* $P_\pm(t)$, *and*

$$(40)\qquad \max\{\sup_t L_+(t)||W_+(t)||\,,\ \sup_t L_-(t)||W_-(t)||\} < 1/2\,.$$

Then equation (1) *has a bounded solution.*

Proof. Take a positive ε less than

$$[1 - 2\sup L_\pm(t)||W_\pm(t)||]/2\sup||W_\pm(t)||\,.$$

By (24), there exists $M > 0$ such that, for $||x|| > M$ and $t \in \mathbb{R}$,

$$||P_\pm(t)r(x,t)|| \le (L_\pm(t) + \varepsilon)||P_\pm x||\,.$$

Now, take two bounded functions

$$g_+(t) := ||\phi_+(t)||^2_{+t}\,, \qquad g_-(t) := ||\phi_-(t)||^2_{-t}$$

and two numbers $t_1, t_2 \in \mathbb{R}$ satisfying (29) and (29′). If $||\phi(t_1)|| \le M$ and $||\phi(t_2)|| \le M$, then the first inequalities of (29) and (29′) and the uniform equivalence of the relevant norms lead to an a priori bound for $||\phi(t)||$. The inequality $||\phi(t_1)|| > M$ is impossible, since it implies (see (38))

$$\begin{aligned} g_+'(t_1) &\ge ||\phi_+(t)||^2 - 2||W_+(t)||(L_+(t)+\varepsilon)||\phi_+(t)||^2 \\ &\ge c_+||\phi_+(t)||^2 \ge (2c_+\alpha_+/(N_1^+)^2)g_+(t_1) \\ &> (c_+\alpha_+/(N_1^+)^2)\sup g_+(t) \end{aligned}$$

and it suffices to take $\delta = (c_+\alpha_+/(N_1^+)^2)\sup g_+$ in (29). The second inequality $||\phi(t_2)|| > M$ contradicts (39) after similar calculations. ∎

Remark. Condition (40) holds, for instance, if

$$L(N_1^+)^2/\alpha_+ < 1\,, \qquad L(N_1^-)^2/\alpha_- < 1\,,$$

where $L = \sup L(t)$.

One can choose another way of finding bounded solutions. It is known ([1], p. 230) that if $x' = A(t)x$ is exponentially dichotomic then there exists an operator function $Q : \mathbb{R} \to L(H)$ with invertible values such that $||Q(t)|| \le q$, $||Q^{-1}(t)|| \le q$ for $t \in \mathbb{R}$, and it defines a change of variables $y = Q(t)x$ which transforms the linear equation to the form $y' = B(t)y$ where all $B(t)$ commute with P_+ and P_-. The last condition means that the equation $y' = B(t)y$ splits into two independent equations:

$$y_+' = B_+(t)y_+\,, \qquad y_-' = B_-(t)y_-$$

where $y_\pm \in P_\pm(H)$. Moreover, solutions of the "plus equation" satisfy the estimate (31) and those of the "minus equation" the estimate (32). The studied nonlinear equation (1) is equivalent (after changing variables) to the following system:

$$\begin{aligned} y_+' &= B_+(t)y_+ + P_+Q(t)r(Q^{-1}(t)y,t)\,, \\ y_-' &= B_-(t)y_- + P_-Q(t)r(Q^{-1}(t)y,t)\,. \end{aligned} \tag{44}$$

THEOREM 6. *Let the upper limits*

$$\limsup_{||x||\to\infty} \frac{||P_\pm Q(t)r(x,t)||}{||P_\pm Q(t)x||} = L_\pm(t)$$

be finite and suppose that

$$\begin{aligned} &\inf_t\{\lambda_{\min}(\operatorname{Re} B_+(t)) - L_+(t)\} > 0\,, \\ &\sup_t\{\lambda_{\max}(\operatorname{Re} B_-(t)) + L_-(t)\} < 0\,. \end{aligned} \tag{42}$$

Here, $\operatorname{Re} C = \frac{1}{2}(C + C^*)$ *is a Hermitian operator and* $[\lambda_{\min}(D), \lambda_{\max}(D)]$ *is the smallest interval containing the spectrum of a Hermitian operator* D,

$$\begin{aligned} \lambda_{\min}(D) &= \inf\{(Dx,x) : ||x|| \le 1\}\,, \\ \lambda_{\max}(D) &= \sup\{(Dx,x) : ||x|| \le 1\}\,. \end{aligned}$$

Then equation (1) *has a bounded solution.*

Proof. It suffices to find an a priori estimate for bounded solutions of (41). Let $\phi = (\phi_+, \phi_-)$ be such a solution and let ε be a positive number less than the infimum in (42) and less than the opposite of the supremum in (42). Take $M > 0$ such that

$$||P_\pm Q(t) r(x,t)|| \le (L_\pm(t) + \varepsilon)||P_\pm Q(t)x|| \tag{43}$$

for $||x|| > M$ and $t \in \mathbb{R}$, and consider two functions

$$g_\pm(t) = ||\phi_\pm(t)||^2, \quad t \in \mathbb{R}.$$

They are bounded, so one can find $t_1, t_2 \in \mathbb{R}$ satisfying (29) and (29′). If $||\phi_+(t_1)|| \le Mq$ and $||\phi_-(t_2)|| \le Mq$, then

$$||\phi(t)|| \le 2\sqrt{M^2q^2 + 1}, \quad t \in \mathbb{R},$$

which is an a priori bound we have looked for. In fact, the inequality $||\phi_+(t_1)|| > Mq$ leads to

$$\begin{aligned} g'_+(t_1) &= 2(\operatorname{Re} B_+(t_1)\phi_+(t_1), \phi_+(t_1)) \\ &\quad + 2\operatorname{Re}(P_+Q(t_1)R(Q^{-1}(t_1)\phi(t_1), t_1), \phi_+(t_1)) \\ &\ge 2\lambda_{\min}(\operatorname{Re} B_+(t_1))||\phi_+(t_1)||^2 \\ &\quad - 2(L_+(t_1) + \varepsilon)||P_+Q(t_1)Q^{-1}(t_1)\phi(t_1)||\,||\phi_+(t_1)|| \\ &= 2[\lambda_{\min}(\operatorname{Re} B_+(t_1)) - L_+(t_1) - \varepsilon]||\phi_+(t_1)||^2 \\ &> 2M^2q^2[\lambda_{\min}(\operatorname{Re} B_+(t_1)) - L_+(t_1) - \varepsilon], \end{aligned}$$

where we can use (43) since

$$||Q^{-1}(t_1)\phi(t_1)|| \ge \frac{1}{||Q(t_1)||}||\phi(t_1)|| > M.$$

It suffices to take $\delta = 2M^2q^2 \inf_t\{\lambda_{\min}(\operatorname{Re} B_+(t)) - L_+(t) - \varepsilon\}$ in (29) in order to get a contradiction. Similar arguments show that $||\phi_-(t_2)||$ cannot be greater than Mq. ■

6. Another type of assumptions. We do not change the conditions on r of Sections 2 and 3, but assumptions on $A(t)$, $t \in \mathbb{R}$, will be stronger than above (or partially stronger) and the growth assumptions on r will be replaced by angular ones.

THEOREM 7. *Suppose that one of the following conditions holds*:

(i) $\sup_t \lambda_{\max}(\operatorname{Re} A(t)) = -\lambda_- < 0$ *and*

$$\limsup_{||x||\to\infty} \operatorname{Re}(r(x,t), x)/||x|| \le 0$$

uniformly with respect to $t \in \mathbb{R}$,

(ii) $\inf_t \lambda_{\min}(\operatorname{Re} A(t)) = \lambda_+ > 0$ *and*

$$\liminf_{||x||\to\infty} \operatorname{Re}(r(x,t),x)/||x|| \ge 0$$

uniformly in $t \in \mathbb{R}$,

(iii) $\lambda_{\max}(\operatorname{Re} A(t)) < 0$ *and*

$$\sup_t [\limsup_{||x||\to\infty} \operatorname{Re}(r(x,t),x)/||x||^2 - \lambda_{\max}(\operatorname{Re} A(t))] < 0\,,$$

(iv) $\lambda_{\min}(\operatorname{Re} A(t)) > 0$ *and*

$$\inf_t [\liminf_{||x||\to\infty} \operatorname{Re}(r(x,t),x)/||x||^2 - \lambda_{\min}(\operatorname{Re} A(t))] > 0\,.$$

Moreover, *in* (iii) *assume that for*

$$\Lambda(t) := -\int_0^t \lambda_{\max}(\operatorname{Re} A(s))\,ds$$

the integral $\int_{-\infty}^t e^{\Lambda(s)}\,ds$ *is convergent and*

$$\sup_t e^{-\Lambda(t)} \int_{-\infty}^t e^{\Lambda(s)}\,ds = N_0 < \infty\,. \tag{44}$$

Similarly in (iv), *for* $\Lambda(t) := \int_0^t \lambda_{\min}(\operatorname{Re} A(s))\,ds$, *assume the convergence of* $\int_t^\infty e^{-\Lambda(s)}\,ds$ *and the boundedness of* $e^{\Lambda(t)} \int_t^\infty e^{-\Lambda(s)}\,ds$.

Then equation (1) *has a bounded solution. The last two assumptions on* Λ *are satisfied*, *for example*, *if*

$$-\lambda_{\max}(\operatorname{Re} A(t)),\ \lambda_{\min}(\operatorname{Re} A(t)) \le c|t|^{-\gamma}$$

for each $t \in \mathbb{R}$ *where* c *and* γ *are positive constants*, $\gamma < 1$.

Proof. (i) The Ważewski–Wintner inequalities ([6]) imply that all solutions of $x' = A(t)x$ satisfy

$$||x(t)|| \le Ne^{-\lambda_-(t-s)}||x(s)||\,, \qquad s \le t\,,$$

i.e. we have exponential dichotomy with $P_+ = 0$, $P_- = I$. Hence, we only need an a priori bound. Let $\phi : \mathbb{R} \to H$ be a bounded solution of (1) and $\varepsilon \in (0, \lambda_-)$. There exists $M > 0$ such that

$$\operatorname{Re}(r(x,t),x) \le \varepsilon ||x||^2$$

for $||x|| > M$ and $t \in \mathbb{R}$. We put

$$g_-(t) := ||\phi(t)||^2$$

and $\delta = 2(\lambda_- - \varepsilon)M^2$ in (29′). Let t_2 be defined there. If $||\phi(t_2)|| \le M$, we can estimate $||\phi(t)||$ on $\mathbb{R}$. If $||\phi(t_2)|| > M$, we have

$$\frac{d}{dt}||\phi(t_2)||^2 \le -2(\lambda_- - \varepsilon)||\phi(t_2)||^2 < -2(\lambda_- - \varepsilon)M^2\,,$$

which contradicts (29′).

The proof of (ii) is quite similar.

(iii) Now, the evolution operator is the Green Function but condition (2) is not satisfied, in general. However, by the Ważewski–Wintner inequalities, we get

$$||U(t,s)|| = ||G(t,s)|| \le \exp(\Lambda(s) - \Lambda(t))$$

for $s \le t$. It is easy to see that the above estimate suffices for the proof of Theorems 2 and 3 ($G(t,s) = 0$ for $s > t$ and one should replace (13) by $k(t) < N_0,\ t \in \mathbb{R}$). The proof of a priori bounds is a simple repetition of the above arguments.

The proof of (iv) is similar. ■

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