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The fixed points of holomorphic maps on a convex domain

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Abstract. We give a simple proof of the result that if D is a (not necessarily bounded) hyperbolic convex domain in $\mathbb{C}^n$ then the set V of fixed points of a holomorphic map $f : D \to D$ is a connected complex submanifold of D; if V is not empty, V is a holomorphic retract of D. Moreover, we extend these results to the case of convex domains in a locally convex Hausdorff vector space.

1. Introduction. In [15] J.-P. Vigué investigated the structure of the fixed point set of a holomorphic map from a bounded convex domain in $\mathbb{C}^n$ into itself. He proved the following. Let D be a bounded convex domain in $\mathbb{C}^n$. Then the set V of fixed points of a holomorphic map $f : D \to D$ is a connected complex submanifold of D and, if V is not empty, V is a holomorphic retract of D. His main tools were the results of Vesentini [13], [14] and Lempert [10], [11] about complex geodesics. However, his proof was rather long.

Our purpose in this article is to give a brief and simple proof of this theorem in the general case of (not necessarily bounded) hyperbolic convex domains in $\mathbb{C}^n$. Moreover, we shall investigate the fixed point sets of holomorphic maps from a convex domain in a locally convex Hausdorff vector space into itself.

We now recall some definitions and properties.

(i) We shall frequently make use of the Kobayashi pseudodistance d_M and the Carathéodory pseudodistance c_M on a complex manifold M (see Kobayashi [9]).

(ii) A complex manifold M is called *taut* [7] if whenever N is a complex manifold and $f_i : N \to M$ is a sequence of holomorphic maps, then either there exists a subsequence which converges uniformly on compact subsets to a holomorphic map $f : N \to M$ or a subsequence which is compactly divergent. In order for M to be taut, it suffices that this condition holds for

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$N = \Delta$, the unit disk in $\mathbb{C}^n$ [1]. Also, every complete hyperbolic complex space is taut, and a taut complex manifold is hyperbolic [7].

(iii) Let D be a domain in a locally convex Hausdorff topological vector space E. A holomorphic map $\varphi : \Delta \to D$ is called a *complex geodesic* [13] if $c_\Delta(\zeta_1, \zeta_2) = c_D(\varphi(\zeta_1), \varphi(\zeta_2))$ for all $\zeta_1, \zeta_2 \in \Delta$. Vesentini [13] proved that φ is a complex geodesic iff there exist two distinct points $\zeta_0, \zeta_1 \in \Delta$ such that $c_\Delta(\zeta_0, \zeta_1) = c_D(\varphi(\zeta_0), \varphi(\zeta_1))$.

The theorems of the present paper in the infinite-dimensional case were suggested by my friend Ngo Hoang Huy. I wish to thank him for his help.

2. The finite-dimensional case. In this section we always assume that D is a (not necessarily bounded) hyperbolic convex domain in $\mathbb{C}^n$ and $f : D \to D$ is a holomorphic map. Denote the fixed point set of f by $V = \mathrm{Fix}(f)$.

2.1. THEOREM. *If V is not empty then V is a holomorphic retract of D, i.e. there exists a holomorphic map $\varphi : D \to D$ such that $\varphi(D) \subset V$ and $\varphi|V = \mathrm{Id}$.*

Proof. The space $\mathrm{Hol}(D, \mathbb{C}^n)$ of all holomorphic maps $g : D \to \mathbb{C}^n$, endowed with the compact-open topology, is a locally convex Hausdorff vector space. Consider its subset $K = \{g \in \mathrm{Hol}(D, D) : g|V = \mathrm{Id}\}$ with the induced topology. Clearly, K is a nonempty convex subset of $\mathrm{Hol}(D, D)$. Since D is a hyperbolic convex domain, D is taut (see Barth [2]). Hence K is compact in $\mathrm{Hol}(D, \mathbb{C}^n)$.

Consider the continuous operator

$$T : \mathrm{Hol}(D, D) \to \mathrm{Hol}(D, D), \qquad g \mapsto f \circ g\,.$$

It is easy to see that $T(K) \subset K$. By the Schauder fixed point theorem (see Edwards [4]), there exists $\varphi \in K$ such that $f \circ \varphi = \varphi$, i.e. $\varphi(D) \subset V$. Since $\varphi|V = \mathrm{Id}$, V is a holomorphic retract of D. ∎

By a result of Rossi (see Fischer [5, p. 102]), we deduce the following

2.2. COROLLARY. *The fixed point set V of f is a complex submanifold of D.*

2.3. PROPOSITION. *For any two distinct fixed points x and y of f, there exists a complex geodesic φ which passes through x, y and satisfies $\varphi(\Delta) \subset V = \mathrm{Fix}(f)$.*

Proof. $\mathrm{Hol}(\Delta, \mathbb{C}^n)$, endowed with the compact-open topology, is a locally convex Hausdorff vector space. Assume that $x, y \in V$ and $x \neq y$. Choose $\eta \in \Delta$ such that $c_\Delta(0, \eta) = c_D(x, y)$. Consider the subset $\Gamma = \{g \in \mathrm{Hol}(\Delta, D) : g(0) = x,\ g(\eta) = y\}$ of $\mathrm{Hol}(\Delta, \mathbb{C}^n)$ with the induced topology.

By the results of Lempert [10], [11] and Royden–Wong [12], we have $c_D(x,y) = d_D(x,y) = \delta_D(x,y) = \inf\{c_\Delta(0,\zeta) : \exists\varphi : \Delta \to D$ holomorphic with $\varphi(0) = x,\ \varphi(\zeta) = y\}$. Thus there exists a sequence $\{\varphi_n\} \subset \mathrm{Hol}(\Delta, D)$ and a sequence $\{\zeta_n\} \subset \Delta$ such that $\varphi_n(0) = x$, $\varphi_n(\zeta_n) = y$ and $\lim_{n\to\infty} c_\Delta(0,\zeta_n) = c_D(x,y) < \infty$. We can assume that $\{\zeta_n\}$ converges to a point $\zeta_0 \in \Delta$. Since D is taut [2], we may assume that $\{\varphi_n\}$ converges in $\mathrm{Hol}(\Delta, D)$ to a map $\varphi_0 \in \mathrm{Hol}(\Delta, D)$. Clearly $\varphi_0(0) = x$, $\varphi_0(\zeta_0) = y$ and $c_\Delta(0,\zeta_0) = c_D(x,y)$.

Take an automorphism T of Δ such that $T(0) = 0$, $T(\eta) = \zeta_0$. Then $\varphi_0 \circ T \in \Gamma$. Thus Γ is a nonempty convex subset of $\mathrm{Hol}(\Delta, D)$. On the other hand, since D is taut, Γ is compact in $\mathrm{Hol}(\Delta, \mathbb{C}^n)$.

Consider the continuous operator

$$T : \mathrm{Hol}(\Delta, D) \to \mathrm{Hol}(\Delta, D), \qquad g \mapsto f \circ g\,.$$

It is easy to see that $T(\Gamma) \subset \Gamma$. By the Schauder fixed point theorem, there is $\varphi \in \Gamma$ such that $f \circ \varphi = \varphi$, i.e. $\varphi(\Delta) \subset V$. ∎

Corollary 2.2 and Proposition 2.3 yield the following

2.4. THEOREM. *The fixed point set V of f is a connected complex submanifold of D.*

2.5. PROPOSITION. *Assume that V is a one-dimensional connected complex submanifold of D. Then the following are equivalent*:

(i) *V is the fixed point set of some holomorphic map $f : D \to D$.*
(ii) *V is the image of some complex geodesic $\varphi : \Delta \to D$.*
(iii) *V is a holomorphic retract of D.*

Proof. (i)⇒(ii). Assume that $V = \mathrm{Fix}(f)$, where $f : D \to D$ is a holomorphic map. Take two distinct $x, y \in V$. By Proposition 2.3, there exists a complex geodesic which passes through x, y and satisfies $\varphi(\Delta) \subset V$. Then $\varphi(\Delta) = V$, because $\varphi(\Delta)$ is open and closed in V.

(ii)⇒(iii). Assume that $\varphi : \Delta \to D$ is a complex geodesic and $V = \varphi(\Delta)$. Take two distinct points $z_1, z_2 \in \Delta$. We have $c_D(\varphi(z_1), \varphi(z_2)) = \sup\{c_\Delta(0, g(\varphi(z_2))) : g \in \mathrm{Hol}(D, \Delta)$ with $g(\varphi(z_1)) = 0\}$. By the normality of $\mathrm{Hol}(D, \Delta)$, there exists $g \in \mathrm{Hol}(D, \Delta)$ such that

$$c_D(\varphi(z_1), \varphi(z_2)) = c_\Delta(g(\varphi(z_1)), g(\varphi(z_2)))\,.$$

Hence $c_\Delta(z_1, z_2) = c_\Delta(g \circ \varphi(z_1), g \circ \varphi(z_2))$. Thus $g \circ \varphi$ is an automorphism of Δ having two distinct fixed points z_1, z_2. By the Schwarz lemma, $g \circ \varphi = \mathrm{Id}$. Therefore $\varphi \circ g : D \to \varphi(\Delta)$ is a retraction on $\varphi(\Delta) = V$.

(iii)⇒(i). The proof follows immediately from the definition of a holomorphic retract of D. ∎

From Proposition 2.5 we have the following

2.6. COROLLARY. *Let f be a holomorphic map of a hyperbolic convex domain D in $\mathbb{C}^2$ into itself having a fixed point in D. Then one of the following cases necessarily occurs*:

(i) *f has a unique fixed point.*

(ii) *The fixed point set of f is the image of a complex geodesic $\varphi : \Delta \to D$.*

(iii) *f is the identity map.*

3. The infinite-dimensional case. Assume that D is a domain in a locally convex Hausdorff vector space E.

The Kobayashi pseudodistance d_D on D is defined as in [6]. If d_D is a distance and if the topology defined by d_D is equivalent to the relative topology of D in E, the domain D is said to be *hyperbolic* (see [6]).

In this section we always assume that D is a convex domain in a locally convex Hausdorff vector space E such that $\overline{D}$ is contained in a hyperbolic domain D' of E and $f : D \to D$ is a holomorphic map such that the image $f(D)$ of f is contained in some compact convex subset K of E.

3.1. THEOREM. *If the fixed point set V of f is not empty then V is a holomorphic retract of D.*

Proof. The space $\mathrm{Hol}(D,E)$, endowed with the compact-open topology, is a locally convex Hausdorff vector space. Consider its subset $N = \{g \in \mathrm{Hol}(D,D) : g|V = \mathrm{Id} \text{ and } g(D) \subset K\}$ with the induced topology. Then N is a nonempty convex subset of $\mathrm{Hol}(D,E)$.

Now we prove that N is compact in $\mathrm{Hol}(D,E)$. Suppose that a sequence $\{g_n\} \subset N$ converges in $\mathrm{Hol}(D,E)$ to a map $g \in \mathrm{Hol}(D,E)$. Clearly $g|V = \mathrm{Id}$ and $g(D) \subset K$. We must prove that $g(D) \subset D$. Indeed, we have $D = \bigcap_{\gamma \in \partial D} \{x^*_\gamma < a_\gamma\}$, where x^*_γ are (real) linear functionals on E. Therefore $x^*_\gamma \circ g$ is plurisubharmonic on D, $x^*_\gamma \circ g(z) \le a_\gamma$ for all $z \in D$ and $x^*_\gamma \circ g(z) < a_\gamma$ for all $z \in V$. By the maximum principle, $x^*_\gamma \circ g(z) < a_\gamma$ for all $z \in D$, i.e. $g(D) \subset D$. Thus N is a closed subset in $\mathrm{Hol}(D,E)$.

Now we prove that $\mathrm{Hol}(D,D)$ is an even family [8]. Indeed, let $x \in D$, $y \in E$ be any points and let U be a neighbourhood of y in E. Without loss of generality we can assume that $y \in \overline{D} \subset D'$.

Take $r > 0$ such that $B_r = \{q \in D' : d_{D'}(y,q) < r\} \subset U$. Since D is hyperbolic, $V = \{p \in D : d_D(x,p) < r/2\}$ is an open neighbourhood of x in D. Analogously, the ball $W = B_{r/2} = \{q \in D' : d_{D'}(y,q) < r/2\}$ is an open neighbourhood of y in E. It is easy to see that $\widetilde{f}(V) \subset U$ whenever $\widetilde{f}(x) \in W$ (for all $\widetilde{f} \in \mathrm{Hol}(D,D)$).

By Arzelà–Ascoli's theorem (see [8, Theorems 7.6 and 7.21]), N is compact in $\mathrm{Hol}(D, E)$.

Consider the continuous operator

$$T : \mathrm{Hol}(D, D) \to \mathrm{Hol}(D, D), \quad g \mapsto f \circ g\,.$$

Obviously $T(N) \subset N$. By the Schauder fixed point theorem, there is $\varphi \in N$ such that $f \circ \varphi = \varphi$. As in Theorem 2.1, we have $\varphi(D) \subset V$ and $\varphi|V = \mathrm{Id}$. Thus V is a holomorphic retract of D. ■

3.2. THEOREM. *For any two distinct fixed points x and y of f, there exists a complex geodesic $\varphi : \Delta \to D$ which passes through x, y and satisfies $\varphi(\Delta) \subset \mathrm{Fix}(f)$.*

P r o o f. Consider the space $\mathrm{Hol}(\Delta, E)$ with the compact-open topology.

By our assumption, D is a hyperbolic convex domain and hence $c_D(x, y) = d_D(x, y) = \delta_D(x, y) = \inf\{c_\Delta(0, \zeta) : \exists\varphi : \Delta \to D$ holomorphic with $\varphi(0) = x,\ \varphi(\zeta) = y\}$ (see [3]). Thus there exist a sequence $\{\varphi_n\} \subset \mathrm{Hol}(\Delta, D)$ and a sequence $\{\zeta_n\} \subset \Delta$ such that $\varphi_n(0) = x,\ \varphi_n(\zeta_n) = y$ and $\lim_{n\to\infty} c_\Delta(0, \zeta_n) = c_D(x, y) < \infty$. We can assume that $\{\zeta_n\}$ converges to a point $\zeta_0 \in \Delta$ and $|\zeta_i| \le r < 1$ for all $i \ge 0$. Put $\psi_n = f \circ \varphi_n$ for all $n \ge 1$.

Consider the subset $A = \{\theta \in \mathrm{Hol}(\Delta, D) : \theta(0) = x,\ \theta(\zeta) = y$ for some $|\zeta| \le r$ and $\theta(\Delta) \subset K\}$ of $\mathrm{Hol}(\Delta, E)$ with the induced topology. Reasoning as in Theorem 3.1, we find that A is closed in $\mathrm{Hol}(\Delta, E)$ and $\mathrm{Hol}(\Delta, D)$ is an even family. By Arzelà–Ascoli's theorem, A is compact.

Since $\{\psi_n\} \subset A$, we can assume that $\{\psi_n\}$ converges in $\mathrm{Hol}(\Delta, D)$ to a map $\psi_0 \in \mathrm{Hol}(\Delta, D)$. We have $\psi_0(0) = x,\ \psi_0(\zeta_0) = y$ and $c_\Delta(0, \zeta_0) = c_D(x, y)$, i.e. ψ_0 is a complex geodesic passing through x and y.

Consider the subset $N = \{\varphi \in \mathrm{Hol}(\Delta, D) : \varphi(0) = x,\ \varphi(\zeta_0) = y$ and $\varphi(\Delta) \subset K\}$ of $\mathrm{Hol}(\Delta, E)$ with the induced topology. Just as in Theorem 3.1, N is closed in $\mathrm{Hol}(\Delta, E)$ and hence it is a nonempty compact convex subset of $\mathrm{Hol}(\Delta, E)$.

Consider the continuous operator

$$T : \mathrm{Hol}(\Delta, D) \to \mathrm{Hol}(\Delta, D), \quad g \mapsto f \circ g\,.$$

Again as in Theorem 3.1, there is $\varphi \in N$ such that $f \circ \varphi = \varphi$, i.e. $\varphi(\Delta) \subset \mathrm{Fix}(f)$. ■

Theorems 3.1 and 3.2 yield the following

3.3. COROLLARY. *Let D be a bounded convex domain in a Banach complex space E. Assume that $f : D \to D$ is a holomorphic map whose image $f(D)$ is contained in some compact convex subset K of E. Then*

(i) $\mathrm{Fix}(f)$ *is a holomorphic retract of D if* $\mathrm{Fix}(f) \ne \emptyset$.

(ii) *For any two distinct fixed points x, y of f, there exists a complex geodesic $\varphi : \Delta \to D$ passing through x, y and satisfying $\varphi(\Delta) \subset \mathrm{Fix}(f)$.*

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