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Distortion function and quasisymmetric mappings

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Abstract. We study the relationship between the distortion function Φ_K and normalized quasisymmetric mappings. This is part of a new method for solving the boundary values problem for an arbitrary K-quasiconformal automorphism of a generalized disc on the extended complex plane.

Introduction. It is well known that a K-quasiconformal (K-qc) mapping F of a Jordan domain G onto a Jordan domain G' can be extended to a homeomorphism of their closures. It induces a homeomorphism f of the boundaries ∂G and $\partial G'$. In the case of $G = G' = H = \{z : \operatorname{Im} z > 0\}$ and a K-qc automorphism F of H that fixes the point at infinity, the induced homeomorphism f of $\mathbb{R}$ is a ϱ-quasisymmetric (ϱ-qs) function in the sense of the Beurling–Ahlfors condition

$$\text{(B-A)} \qquad \frac{1}{\varrho} \le \frac{f(x+t) - f(x)}{f(x) - f(x-t)} \le \varrho\,,$$

which holds for all $x \in \mathbb{R}$ and $t > 0$ with $\varrho = \lambda(K)$ (see [BA], [LV]). The class of all increasing homeomorphisms $f : \mathbb{R} \to \mathbb{R}$ satisfying (B-A) with a constant $\varrho \ge 1$ is called the *ϱ-qs class* on $\mathbb{R}$ and is denoted by $Q_{\mathbb{R}}(\varrho)$. By $Q^0_{\mathbb{R}}(\varrho)$ we will denote the subclass of $Q_{\mathbb{R}}(\varrho)$ consisting of all normalized ($f(0) = 0, f(1) = 1$) ϱ-qs functions on $\mathbb{R}$. A characterization of f in the case of K-qc automorphisms F of the unit disc $\Delta = \{z : |z| < 1\}$ with fixed point at zero was given by Krzyż [Kr1].

Neither of these characterizations comprises the general case of arbitrary K-qc automorphisms of H and Δ, respectively, and neither is "conformally" equivalent.

In order to build up a representation for the boundary values of an arbitrary K-qc automorphism of a generalized disc $D \subset \overline{\mathbb{C}}$, we need some new results on the relation between normalized ϱ-qs functions and the distortion function Φ_K.

The latter function gives a sharp upper bound in the quasiconformal version of the Schwarz Lemma [HP]: $|F(z)| \le \Phi_K(|z|)$ for each K-qc mapping of the disc Δ into itself with $F(0)=0$. Φ_K is defined by

$$\Phi_K(t)=\mu^{-1}\left(\frac{1}{K}\mu(t)\right) \tag{0.1}$$

where $\mu(t)$ stands for the conformal modulus of the unit disc slit along the real line from 0 to t, $0<t<1$, and is strictly decreasing with limits ∞ and 0 at 0 and 1, respectively. We may extend Φ_K to the closed interval [0,1] by setting $\Phi_K(0)=0$, $\Phi_K(1)=1$, for each $K>0$. Evidently $\Phi_K(t)\ge t$ for $K\ge 1$ and $\Phi_K(t)\le t$ for $0<K\le 1$, with equality in each case if and only if $K=1$. Clearly,

$$\begin{aligned}&\Phi_{K_1}\circ\Phi_{K_2}=\Phi_{K_1K_2}\,, \qquad \Phi_K^{-1}=\Phi_{1/K}\,,\\ &\Phi_2(t)=\frac{2\sqrt{t}}{1+t}\,, \qquad 0\le t\le 1\,.\end{aligned} \tag{0.2}$$

The explicit estimate

$$t^{1/K}\le\Phi_K(t)\le 4^{1-(1/K)}t^{1/K} \qquad 0\le t\le 1\,,\ K\ge 1\,, \tag{0.3}$$

was given by Wang [W] and Hübner [H].

A number of significant results concerning Φ_K were obtained by Anderson, Vamanamurphy and Vuorinen [AVV1], [AVV2] and others. One of them,

$$\Phi_K^2(t)+\Phi_{1/K}^2(\sqrt{1-t^2})=1\,, \qquad 0\le t\le 1\,,\ K>0\,, \tag{0.4}$$

is very useful in our present considerations.

1. New results on quasisymmetric functions. In this section we prove two auxiliary theorems on quasisymmetric functions. The first of them gives sharp Hölder type estimates for normalized ϱ-qs functions (those of Kelingos [Ke] are not sharp).

Theorem 1. *Suppose that f is a normalized ϱ-qs function of $\mathbb{R}$. Then for each $m\in\mathbb{N}$*

$$\left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right)t^{\alpha_m}\le f(t)\le\left(1+\frac{1}{(\varrho+1)^m-1}\right)t^{\beta_m} \tag{1.1}$$

for $0\le t\le 1$ and $\varrho\ge 1$,

$$\begin{aligned}\left(\frac{2}{\varrho}-1\right)\left(1-\left(\frac{\varrho}{\varrho+1}\right)^m\right)(t_2-t_1)^{\alpha_m}&\le f(t_2)-f(t_1)\\ &\le(2\varrho-1)\left(1+\frac{1}{(\varrho+1)^m-1}\right)(t_2-t_1)^{\beta_m}\end{aligned} \tag{1.2}$$

for $0\le t_1\le t_2\le 1$ and $\varrho\ge 1$ (the left-hand bound in (1.2) *is essential for*

$1 \le \varrho \le 2$), *and*

$$\left(1 + \frac{1}{(\varrho+1)^m - 1}\right) t^{\beta_m} \le f(t) \le \left(1 - \left(\frac{\varrho}{\varrho+1}\right)^m\right)^{-1} t^{\alpha_m} \tag{1.3}$$

for $t \ge 1$ *and* $\varrho \ge 1$, *where*

$$\begin{aligned} \alpha_m &= \log_{1-2^{-m}} \left(1 - \left(\frac{\varrho}{\varrho+1}\right)^m\right), \\ \beta_m &= \log_{1-2^{-m}} \left(1 - \left(\frac{1}{\varrho+1}\right)^m\right). \end{aligned} \tag{1.4}$$

Proof. Let $m \in \mathbb{N}$ and $c_m = 1 - 2^{-m}$. By induction on m one can prove the inequalities

$$\begin{aligned} &\left(\frac{\varrho}{\varrho+1}\right)^m f(a) + \left(1 - \left(\frac{\varrho}{\varrho+1}\right)^m\right) f(b) \\ &\quad \le f((1-c_m)a + c_m b) \le \left(\frac{1}{\varrho+1}\right)^m f(a) + \left(1 - \left(\frac{1}{\varrho+1}\right)^m\right) f(b) \end{aligned}$$

for $a, b \in [0,1]$; the case $m = 1$, i.e.

$$\frac{\varrho}{\varrho+1} f(a) + \frac{1}{\varrho+1} f(b) \le f\left(\frac{a+b}{2}\right) \le \frac{1}{\varrho+1} f(a) + \frac{\varrho}{\varrho+1} f(b),$$

is equivalent to the (B-A) condition. Induction with respect to n gives

$$c_m^{n\alpha_m} = \left(1 - \left(\frac{\varrho}{\varrho+1}\right)^m\right)^n \le f(c_m^n) \le \left(1 - \left(\frac{1}{\varrho+1}\right)^m\right)^n = c_m^{n\beta_m}$$

for $n = 0, 1, 2, \ldots$

Since f is strictly increasing, for every $t \in [c_m^n, c_m^{n-1}]$, $m, n = 1, 2, \ldots$, we have

$$\begin{aligned} f(t) &\le f(c_m^{n-1}) \le (c_m^{n-1})^{\beta_m} \le (c_m^{-1} t)^{\beta_m} = c_m^{-\beta_m} t^{\beta_m}, \\ f(t) &\ge f(c_m^n) \ge (c_m^n)^{\alpha_m} \ge (c_m t)^{\alpha_m} = c_m^{\alpha_m} t^{\alpha_m}. \end{aligned}$$

This yields (1.1) because $[0,1] = \{0\} \cup \bigcup_{n=1}^{\infty} [c_m^n, c_m^{n-1}]$ for each $m \in \mathbb{N}$.

For every $t_1 \in [0,1]$ the function

$$g_{t_1}(t) = \frac{f(t+t_1) - f(t_1)}{f(1+t_1) - f(t_1)} \tag{1.5}$$

belongs to $Q_{\mathbb{R}}^0(\varrho)$ provided that $f \in Q_{\mathbb{R}}(\varrho)$. Hence, by (1.1) with $t = t_2 - t_1$,

$$\begin{aligned} f(t_2) - f(t_1) &\le (f(1+t_1) - f(t_1)) \left(1 + \frac{1}{(\varrho+1)^m - 1}\right) (t_2 - t_1)^{\beta_m}, \\ f(t_2) - f(t_1) &\ge (f(1+t_1) - f(t_1)) \left(1 + \left(\frac{\varrho}{\varrho+1}\right)^m\right) (t_2 - t_1)^{\alpha_m} \end{aligned}$$

for any $m \in \mathbb{N}$. By (1.5) and the definition of quasisymmetry we see that

$$\frac{1}{\varrho} g_1(t_1) - f(t_1) + 1 \le f(1+t_1) - f(t_1) \le \varrho g_1(t_1) - f(t_1) + 1\,.$$

Since

$$|g(t) - t| \le \frac{\varrho - 1}{\varrho + 1}$$

for all $g \in Q^0_{\mathbb{R}}(\varrho)$, $\varrho \ge 1$ and $0 \le t \le 1$ (see [Kr2]), we have

$$t_1 - \frac{\varrho-1}{\varrho+1} \le g_1(t_1) \le t_1 + \frac{\varrho-1}{\varrho+1}$$

for $t_1 \in [0,1]$ and $\varrho \ge 1$. Consequently,

$$\begin{aligned} f(1+t_1) - f(t_1) &\le \varrho\left(x_1 + \frac{\varrho-1}{\varrho+1}\right) - x_1 + \frac{\varrho-1}{\varrho+1} + 1 \\ &= (\varrho-1)x_1 + \varrho \le 2\varrho - 1 \end{aligned}$$

and

$$\begin{aligned} f(1+t_1) - f(t_1) &\ge \frac{1}{\varrho}\left(x_1 - \frac{\varrho-1}{\varrho+1}\right) - x_1 - \frac{\varrho-1}{\varrho+1} + 1 \\ &= \left(\frac{1}{\varrho} - 1\right)x_1 - \frac{\varrho-1}{\varrho} + 1 \ge \frac{2}{\varrho} - 1\,. \end{aligned}$$

Hence

$$\frac{2}{\varrho} - 1 \le f(1+t_1) - f(t_1) \le 2\varrho - 1\,.$$

The left-hand estimate is essential for $1 \le \varrho \le 2$ but asymptotically sharp.

The inequality (1.3) can be derived in much the same way as (1.1). For $m = 1$ the inequalities (1.1) and (1.3) reduce to those of Kelingos while (1.2) is better.

Now we prove

Lemma. *Let $f : [a,b] \to \mathbb{R}$ be strictly increasing and concave. Then*

$$\frac{f(t+s_t) - f(t)}{f(t) - f(t-s_t)} \le \frac{f(t+s) - f(t)}{f(t) - f(t-s)} = \mathcal{F}(t,s) \le 1 \tag{1.6}$$

for all $t \in (a,b)$ and $0 < s \le s_t = \min\{b-t, t-a\}$.

Proof. Let $t \in (a,b)$ and $0 < s < s_t$, and set $d = s_t - s$. By the concavity of f we have

$$\begin{aligned} f(t-s) &\ge \frac{d}{s_t} f(t-s_t) + \frac{s}{s_t} f(t)\,, \\ f(t+s) &\ge \frac{s}{s_t} f(t) + \frac{d}{s_t} f(t+s_t)\,. \end{aligned}$$

Therefore

$$f(t)-f(t-s)\le \frac{d}{s_t}(f(t)-f(t-s_t))\,,$$
$$f(t+s)-f(t)\ge \frac{d}{s_t}(f(t+s_t)-f(t))\,.$$

Since f is strictly increasing,

$$\frac{f(t+s)-f(t)}{f(t)-f(t-s)}\ge \frac{f(t+s_t)-f(t)}{f(t)-f(t-s_t)}\,.$$

Using once again the concavity of f gives $f(t)\ge \frac{1}{2}f(t-s)+\frac{1}{2}f(t+s)$, and so $f(t+s)-f(t)\le f(t)-f(t-s)$, which completes the proof.

This lemma has a very practical application. It means that the qs order ϱ of a given concave and increasing homeomorphism f on $[a,b]$ is attained on the upper frame of the domain of $\mathcal{F}$.

Another immediate application of the lemma yields

THEOREM 2. *Suppose that $f:D\to\mathbb{R}$ is strictly increasing and concave. Then f is ϱ-qs on D in each of the following cases*:

(i) *D=(a,b) and*

$$\min\left\{\inf_{t\in(a,(a+b)/2]}\frac{f(2t-a)-f(t)}{f(t)-f(a)},\inf_{t\in[(a+b)/2,b)}\frac{f(b)-f(t)}{f(t)-f(2t-b)}\right\}=\frac{1}{\varrho}>0\,. \tag{1.7}$$

(ii) *$D=(b,\infty)$ and*

$$\inf_{t\in(b,\infty)}\frac{f(2t-b)-f(t)}{f(t)-f(b)}=\frac{1}{\varrho}>0\,. \tag{1.8}$$

(iii) *$D=(\infty,a)$ and*

$$\inf_{t\in(-\infty,a)}\frac{f(a)-f(t)}{f(t)-f(2t-a)}=\frac{1}{\varrho}>0\,. \tag{1.9}$$

(iv) *$D=\mathbb{R}$ and*

$$\inf_{t\in\mathbb{R}}\lim_{x\to\infty}\frac{f(t+x)-f(t)}{f(t)-f(t-x)}=\frac{1}{\varrho}>0\,.$$

2. Main results

THEOREM 3. *For each $K\ge 1$, there exists $\varrho\ge 0$ such that the function Φ_K is ϱ-qs on $[0,1]$ with*

$$\varrho\le\varrho_0=\max\{2^{5K-3},2^{2-3/K}(1-\Phi_K(1/2))^{-1}\}\,. \tag{2.1}$$

P r o o f. By the definition, Φ_K is concave for each $K > 1$. Let $t \in (0, 1/2]$. Then, by the lemma and by (0.3) we have

$$\frac{\Phi_K(2t) - \Phi_K(t)}{\Phi_K(t)} = \frac{\Phi_K(2t) - \Phi_K(2t\frac{1}{2})}{\Phi_K(t)} \ge \frac{\Phi_K(2t)}{\Phi_K(t)}(1 - \Phi_K(1/2))$$
$$\ge \frac{(2t)^{1/K}}{4^{1-(1/K)}t^{1/K}}(1 - \Phi_K(1/2)) = \frac{8^{1/K}}{4}(1 - \Phi_K(1/2))\,.$$

For $t \in [1/2, 1)$, using (0.4) and (0.3) for $0 < K \le 1$ we have

$$\frac{\Phi_K(1) - \Phi_K(t)}{\Phi_K(t) - \Phi_K(2t-1)} \ge \frac{1 - \Phi_K(t)}{1 - \Phi_K(2t-1)} = \frac{1 - \Phi_K^2(t)}{1 - \Phi_K^2(2t-1)} \cdot \frac{1 + \Phi_K(2t-1)}{1 + \Phi_K(t)}$$
$$\ge \frac{\Phi_{1/K}^2(\sqrt{1-t^2})}{\Phi_{1/K}^2(\sqrt{1-(2t-1)^2})} \cdot \frac{1}{2} \ge \frac{(4^{1-K}(\sqrt{1-t^2})^K)^2}{(\sqrt{1-(2t-1)^2})^{2K}} \cdot \frac{1}{2}$$
$$= \frac{16^{1-K}}{2}\left(\frac{1-t^2}{4t-4t^2}\right)^K = 8 \cdot 4^{-3K}\left(1 + \frac{1}{t}\right)^K \ge 8 \cdot 2^{-6K}2^K = 8 \cdot 2^{-5K}\,,$$

which completes the proof.

Now, using Theorem 1 we prove a very useful theorem (see [Z]).

THEOREM 4 (subordination principle). *Suppose that f is a ϱ-qs function of $[0,1]$ onto itself. Then for each $\varrho \ge 1$ there is a constant $K = K(\varrho)$ such that*

$$\Phi_{1/K}^2(\sqrt{t}) \le f(t) \le \Phi_K^2(\sqrt{t}) \quad \textit{for } 0 \le t \le 1\,, \tag{2.2}$$

where

(2.3)

$$K \le \nu(\varrho) = \begin{cases} \dfrac{e^{2\sqrt{\varrho-1}}}{1 - 2^{-m}e^{1/m}}\,,\ m = \mathrm{Ent}\{1/\sqrt{\varrho-1}\}\,, & 1 \le \varrho \le 5/4, \\ 3.41\log_2(1+\varrho)\,, & 5/4 < \varrho \le 6, \\ (\log 2)\left(1 - \dfrac{1}{\log_2(\frac{2}{\varrho}\log_2(1+\varrho))}\right)(1+\varrho) & \varrho > 6, \end{cases}$$

with $\nu(\varrho) \cong (\log 2)(1+\varrho)$ as $\varrho \to \infty$.

P r o o f. By Theorem 1, since $1 - f(1-t)$ is ϱ-qs and f is a ϱ-qs mapping of $[0,1]$ onto itself, for every $m \in \mathbb{N}$ we have

$$f(t) \le \min\{c_m^{-\beta_m}t^{\beta_m}, 1 - c_m^{\alpha_m}(1-t)^{\alpha_m}\}\,, \quad t \in [0,1]\,.$$

Let $\lambda \in (0, c_m)$ and

$$K_{\lambda,m} = \max\left\{\frac{1}{\beta_m}\frac{\log_{1/c_m}\lambda}{\log_{1/c_m}\lambda + 1}, \alpha_m\frac{\log_{1/c_m}(1-\lambda) - 1}{\log_{1/c_m}(1-\lambda)}\right\}\,.$$

Then

$$c_m^{-\beta_m} t^{\beta_m} \le t^{1/K_{\lambda,m}} \qquad \text{for } 0 \le t \le \lambda\,,$$
$$(1-t)^{K_{\lambda,m}} \le c_m^{\alpha_m}(1-t)^{\alpha_m} \qquad \text{for } \lambda \le t \le 1\,.$$

Now, by the Wang and Hübner inequalities (0.3) and (1.1)

$$f(t) \le \Phi^2_{K_{\lambda,m}}(\sqrt{t}) \quad \text{for } 0 \le t \le \lambda\,,$$

and by (0.2) and (0.4)

$$f(t) \le 1 - c_m^{\alpha_m}(1-t)^{\alpha_m} \le 1-(1-t)^{K_{\lambda,m}} \le 1 - \Phi^2_{1/K_{\lambda,m}}(\sqrt{1-t^2})$$
$$= \Phi^2_{K_{\lambda,m}}(\sqrt{t}) \quad \text{for } \lambda \le t \le 1\,.$$

Then

$$f(t) \le \Phi^2_K(\sqrt{t}) \quad \text{for } 0 \le t \le 1\,,$$

where

(2.4) $$K = \min_{m=1,2,\dots} \min_{0<\lambda<c_m} K_{\lambda,m} \le \min_{m=1,2,\dots} K_{\lambda_m,m}$$

and λ_m is the solution of

$$\frac{\log_{1/c_m}\lambda_m}{1+\log_{1/c_m}\lambda_m} = \alpha_m\beta_m \frac{\log_{1/c_m}(1-\lambda_m)-1}{\log_{1/c_m}(1-\lambda_m)}\,,$$

Consider first the case when $1 \le \varrho \le 5/4$. We have the following estimates:

$$\alpha_m = \frac{\log(1-(\frac{\varrho}{\varrho+1})^m)}{\log(1-2^{-m})} \le \left(\frac{2\varrho}{1+\varrho}\right)^m \frac{1}{1-(\frac{\varrho}{\varrho+1})^m} \le \varrho^m \frac{1}{1-(\frac{\varrho}{\varrho+1})^m}$$
$$\le \left(1+\frac{1}{m^2}\right)^m \frac{1}{1-2^{-m}e^{1/m}} \le \frac{e^{1/m}}{1-2^{-m}e^{1/m}} \quad \text{for } 1 \le \varrho \le 1+1/m^2\,.$$

Similarly, we obtain the estimate

$$\beta_m \ge (1-2^{-m})e^{-1/(2m)} \quad \text{for } 1 \le \varrho \le 1+1/m^2\,.$$

Suppose that $m \ge 2$ is the smallest possible number for which the above inequalities (2.4) are satisfied with $\lambda = 1/2$. Then

$$K \le K_{1/2,m} \le \max\left\{\frac{1}{\beta_m}\cdot\frac{1}{1-\log_2(1-2^{-m})}, \alpha_m(1+\log_2(1-2^{-m}))\right\}$$
$$\le \max\left\{\frac{e^{1/(2m)}}{(1-2^{-m})(1-\log_2(1-2^{-m}))}, \frac{e^{1/m}}{1-2^{-m}e^{1/m}}(1+\log_2(1-2^{-m}))\right\}$$
$$\le \max\left\{\frac{e^{1/(2m)}}{(1-2^{-m})(1-\log_2(1-2^{-m}))}, \frac{e^{1/m}(1-\log_2^2(1-2^{-m}))}{(1-2^{-m}e^{1/m})(1-\log_2(1-2^{-m}))}\right\}$$
$$\le \max\left\{\frac{e^{1/(2m)}}{(1-2^{-m})(1-\log_2(1-2^{-m}))}, \frac{e^{1/m}}{(1-2^{-m}e^{1/m})(1-\log_2(1-2^{-m}))}\right\}$$

$$\le \frac{e^{1/m}}{(1-2^{-m})(1-\log_2(1-2^{-m}))} \le \frac{e^{1/m}}{1-2^{-m}e^{1/m}}$$

where $m < \operatorname{Ent}\{1/\sqrt{\varrho-1}\}$. Since

$$\frac{1}{m} < \frac{\sqrt{\varrho-1}}{1-\sqrt{\varrho-1}} \le 2\sqrt{\varrho-1}$$

we obtain

$$K \le \nu(\varrho) = \frac{e^{2\sqrt{\varrho-1}}}{1-2^{-\operatorname{Ent}\{1/\sqrt{\varrho-1}\}}e^{\operatorname{Ent}\{\sqrt{\varrho-1}\}}}\,.$$

It is easy to see that $\nu(\varrho) \to 1$ as $\varrho \to 1$.

Consider now the case $1 \le \varrho \le 6$. By setting $m = 1$ and $\lambda = 1/4$ we have

$$\begin{aligned}
K &\le \min_{0<\lambda<c_1} K_{\lambda,1} \le K_{1/4,1}\\
&= \max\left\{\frac{1}{\beta_1}\cdot\frac{\log_{1/c_1}(1/4)}{\log_{1/c_1}(1/4)+1}, \alpha_1\frac{\log_{1/c_1}(3/4)-1}{\log_{1/c_1}(3/4)}\right\}\\
&= \max\left\{\frac{2}{\log_2(1+(1/\varrho))}, \log_2(1+\varrho)\frac{\log_2 3-3}{\log_2 3-2}\right\}\\
&\le \frac{\log_2(3/8)}{\log_2(3/4)}\log_2(1+\varrho) < 3.41\log_2(1+\varrho) = \nu(\varrho) \quad \text{for } 5/4 < \varrho \le 6\,.
\end{aligned}$$

To obtain the last case we set $m = 1$, $\alpha_1 = \alpha$, $\beta_1 = \beta$, and $\varrho > 6$. Then we have

$$\begin{aligned}
\alpha\beta\log 2 &= \log_2(1+\varrho)\cdot\log_2\left(1+\frac{1}{\varrho}\right)\cdot\log 2 < \frac{1}{\varrho}\log_2(1+\varrho) < \frac{1}{2}\\
&< \frac{\log^3 2}{2(1-\log 2)}\,.
\end{aligned}$$

Hence

$$2^{(1/(\alpha\beta))+1} \ge 2\left(\frac{\log 2}{\alpha\beta} + \frac{\log^2 2}{2(\alpha\beta)^2}\right) \ge \frac{1}{\alpha\beta\log 2}\,,$$

and so $\alpha\beta < 1/(r-1)$ with $r = -\log(\alpha\beta\log 2)$. By setting $\lambda = 2^{-r}$ we arrive at

$$\begin{aligned}
K \le K_{\lambda,1} &= \max\left\{\frac{1}{\beta}\cdot\frac{r}{r-1}, \alpha\left(1-\frac{1}{\log_2(1-2^{-r})}\right)\right\}\\
&\le \max\left\{\frac{1}{\beta}, \frac{r}{r-1}, \alpha(1+(\log 2)2^r)\right\} \le \max\left\{\frac{1}{\beta}\cdot\frac{r}{r-1}, \alpha+\frac{1}{\beta}\right\}\\
&\le \frac{1}{\beta}\cdot\frac{r}{r-1}\,.
\end{aligned}$$

Then

$$
\begin{aligned}
K &\le \frac{1}{\log_2(1+1/\varrho)}\left(1-\frac{1}{\log_2(\alpha\beta\log 4)}\right)\\
&\le (\log 2)(\varrho+1)\left(1-\frac{1}{\log_2(\alpha\beta\log 4)}\right)\\
&\le (\log 2)\left(1-\frac{1}{\log_2(\frac{2}{\varrho}\log_2(1+\varrho))}\right)(\varrho+1)=\nu(\varrho)\qquad \text{for } \varrho>6\,.
\end{aligned}
$$

Asymptotically $\nu(\varrho)\cong(\log 2)(\varrho+1)$ as $\varrho\to\infty$. To obtain the left-hand side inequality of (2.2) we notice that $g(t)=1-f(1-t)$ is a ϱ-qs function if so is f. Substituting $1-t=x$ we have $f(x)\ge 1-\Phi_K^2(\sqrt{1-x})=\Phi_{1/K}^2(\sqrt{x})$.

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