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The Oka–Weil theorem in topological vector spaces

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Abstract. It is shown that a sequentially complete topological vector space X with a compact Schauder basis has WSPAP (see Definition 2) if and only if X has a pseudo-homogeneous norm bounded on every compact subset of X.

The problem of approximation of holomorphic functions by polynomials in Banach spaces has been investigated by P. L. Noverraz [6], R. Aron and M. Schottenloher [1]. In 1973 C. Matyszczyk [5] generalized the results of these authors to the Fréchet space case. He showed that a Fréchet space with BAP has SPAP if and only if it has a continuous norm. In this note, we study the approximation of holomorphic functions by polynomials in topological vector spaces. In order to obtain the main results (Theorem 3 and 4) some notions for a topological vector space X should be introduced.

Definition 1. We say that a sequence of operators $A_n : X \to Y$ ($n = 1, 2, \ldots$) converges *almost uniformly* on an open set Q in X to an operator $A : X \to Y$ if $A_n(x) \to A(x)$ uniformly on every compact subset K of Q.

Definition 2. We say that X has the *bounded approximation property*, shortly BAP (resp. *compact approximation property*, shortly CAP) if there exists a sequence of finite-dimensional operators pointwise (resp. almost uniformly) convergent to the identity operator on X.

Moreover, we say that X has a *compact Schauder basis* if X has a Schauder basis $\{e_j\}$ such that $\{S_n(x) = \sum_{j=1}^{n} e_j^*(x)e_j\}$ converges almost uniformly to the identity operator on X.

Note that if X is either a complete metric vector space or a barrelled locally convex space with BAP, then X has CAP.

Definition 3. X is said to have the *strong polynomial approximation property*, shortly SPAP, if for every open polynomially convex subset Q of

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X and for every holomorphic function f on Q, there exists a sequence of polynomials almost uniformly convergent to f on Q.

In the case where the above property holds for open polynomially convex subsets Q of X of the form $Q = \bigcup_{n=1}^{\infty} \operatorname{Int} F_n$, where the F_n are closed subsets of X contained in Q, X is said to have WSPAP.

PROPOSITION 4. *Let X be a topological vector space with a compact Schauder basis. If X has WSPAP, then X has a norm bounded on every compact subset of X.*

Proof. Let $\{e_j\}$ be a compact Schauder basis in X.

a) We first show that there exists a sequence $\{\lambda_j\} \in \mathbb{C}^{\infty}$ such that $\lambda_{j_k} e_{j_k} \not\to 0$ for any subsequence $\{\lambda_{j_k}\}$ of $\{\lambda_j\}$. Assume that D is an open polynomially convex set in $\mathbb{C}$ consisting of infinitely many connected components, $D = \bigcup_{j=1}^{\infty} D_j$, with $0 \in D$. Put

$$G = \bigcup_{j=2}^{\infty} D_j e_1 + M,$$

where $D_j e_1 = \{\lambda e_1 : \lambda \in D_j\}$ and $M = \overline{\operatorname{span}\{e_j\}}_{j \geq 2}$. On G, consider the holomorphic function f given by

$$f(z) = e_j^*(z) \quad \text{for } z \in D_j e_1 + M.$$

By hypothesis, there is a sequence of polynomials $\{P_n\}$, almost uniformly convergent to f on G. For each $j \in \mathbb{N}$, consider the restriction $P_n | D_j e_1 + \mathbb{C} e_j$. Since on every compact subset of $D_j e_1 + \mathbb{C} e_j$, this sequence converges uniformly to $e_j^*(z) = z_j$, where $z = z_1 e_1 + z_j e_j$, there exists n_j such that P_{n_j} depends on z_j. Thus there exists $z_1^j \in \mathbb{C}$ such that $|z_1^j| < 1/j$ and $P_{n_j}(z_1^j, z_j)$ depends on z_j. Therefore, there exists $\lambda_j \in \mathbb{C}$ such that $|P_{n_j}(z_1^j, \lambda_j)| > j$. We claim that $\{\lambda_j\}_{j=2}^{\infty}$ is the desired sequence. Indeed, assume that there exists a subsequence $\{\lambda_{j_k}\}$ of $\{\lambda_j\}$ such that $\lambda_{j_k} e_{j_k} \to 0$. Consider the compact set in G given by

$$K = \{z_1^{j_k} e_1 + \lambda_{j_k} e_{j_k}, 0\}.$$

Since there exists $l \geq 2$ such that $0 \in D_l e_1 + M$, for k sufficiently large we have

$$z_1^{j_k} e_1 + \lambda_{j_k} e_{j_k} \in D_l e_1 + M.$$

Hence

$$f(z_1^{j_k} e_1 + \lambda_{j_k} e_{j_k}) = e_l^*(z_1^{j_k} e_1 + \lambda_{j_k} e_{j_k}) = 0$$

for $j_k > l$ and

$$\|P_{n_{j_k}} - f\|_K \geq |P_{n_{j_k}}(z_1^{j_k} e_1 + \lambda_{j_k} e_{j_k}) - f(z_1^{j_k} e_1 + \lambda_{j_k} e_{j_k})| \geq j_k \quad \text{for } j_k > 1.$$

Thus $\|P_{n_{j_k}} - f\|_K \nrightarrow 0$. This contradicts the almost uniform convergence of $\{P_n\}$ to f.

b) Since $\{e_j\}$ is a Schauder basis of X, we have

$$0 = \lim_{j\to\infty} e_j^*(x)e_j = \lim_{j\to\infty} (e_j^*(x)/\lambda_j)\lambda_j e_j$$

for each $x \in X$. Put

$$\rho(x) = \sup_{j\in\mathbb{N}} |e_j^*(x)/\lambda_j|, \qquad \text{where } \lambda_1 = 1.$$

Then ρ is a norm on X. Since $\{e_j\}$ is a compact Schauder basis, ρ is bounded on every compact subset of X. The proposition is proved.

DEFINITION 5. A function $\rho : X \to \mathbb{R}$ is said to be a *pseudo-homogeneous seminorm* of degree $p > 0$ if it satisfies the following conditions:

1) $\rho(x) \ge 0$, $\forall x \in X$,
2) $\rho(\lambda x) = |\lambda|^p \rho(x)$, $\forall x \in X$ and $\forall \lambda \in \mathbb{C}$,
3) $\rho(x + y) \le \rho(x) + \rho(y)$, $\forall x, y \in X$.

In the case where $\rho(x) = 0$ if and only if $x = 0$, this pseudo-homogeneous seminorm is said to be a *pseudo-homogeneous norm*.

PROPOSITION 6. *Let X be a sequentially complete topological vector space with CAP. If X has a pseudo-homogeneous norm bounded on every compact subset of X, then X has WSPAP.*

Proof. Let $\{A_j\}$ be a sequence of finite-dimensional operators almost uniformly converging to the identity operator on X and let Q be an open polynomially convex set in X such that

$$Q = \bigcup_{n=1}^{\infty} \operatorname{Int} F_n = \bigcup_{n=1}^{\infty} F_n,$$

where the F_n are closed sets in X and $F_n \subseteq F_{n+1}$, $\forall n \ge 1$. Put

$$Q_j = \{x \in Q : \|x\| < j\} \quad \text{and} \quad K_j = \overline{F_j \cap Q_j \cap A_j(X)},$$

where $\|\cdot\|$ is a pseudo-homogeneous norm bounded on every compact subset of X. Then

$$K_j \subseteq F_j \cap A_j(X) \subset Q \cap A_j(X), \qquad \forall j \ge 1.$$

Since the topology of $A_j(X)$ is defined by $\|\cdot\|\,|A_j(X)$, K_j is compact in $Q \cap A_j(X)$. Thus by polynomial convexity of $Q \cap A_j(X)$, according to the Oka–Weil theorem there exists a polynomial P_j on $A_j(X)$ such that

$$\|P_j - f\|_{K_j} < 1/j.$$

We shall prove that $\{P_j\}$ converges almost uniformly to f on Q. Let K be a compact subset of Q. Take n_0 such that $K \subset \operatorname{Int} F_{n_0}$. Then there exists a neighbourhood V of zero in X such that

$$K + V \subseteq K + \overline{V} \subseteq \operatorname{Int} F_{n_0} . \tag{1}$$

Since $A_j(x) \to x$ uniformly on K, we get

$$A_j(K) \subseteq K + V \quad \text{ for } j \geq j_0 . \tag{2}$$

From (1) and (2) we have

$$A_j(K) \subseteq F_{n_0} \subseteq F_j , \quad \forall j \geq j_1 = \max\{j_0, n_0\} . \tag{3}$$

On the other hand, since $\bigcup_{j \geq j_1} A_j(K)$ is relatively compact and $\| \cdot \|$ is bounded on every relatively compact subset of X, it follows that $\bigcup_{j \geq j_1} A_j(K) \subset Q_{j_2}$ for some $j_2 \geq j_1$. Hence

$$A_j(K) \subset Q_j , \quad \forall j \geq j_2 . \tag{4}$$

From (3) and (4) we get

$$A_j(K) \subset Q_j \cap F_j \cap A_j(X) \subset K_j , \quad \forall j \geq j_2 .$$

Hence

$$\begin{aligned} \|P_j A_j - f\|_K &\leq \|P_j A_j - f A_j\|_K + \|f A_j - f\|_K \\ &= \|P_j - f\|_{A_j(K)} + \|f A_j - f\|_K \leq \|P_j - f\|_{K_j} + \|f A_j - f\|_K \\ &< 1/j + \|f A_j - f\|_K \quad \text{ for } j \geq j_2 . \end{aligned}$$

Thus by the continuity of f and since $\{A_j\}$ converges almost uniformly to the identity operator we infer that $\|P_j A_j - f\|_K \to 0$ as $j \to \infty$. The proposition is proved.

From Propositions 4 and 5 we get the following

THEOREM 7. *Let X be a sequentially compete vector space with a compact Schauder basis. Then X has WSPAP if and only if X has a pseudo-homogeneous norm bounded on every compact subset of X.*

We now consider SPAP for the class of pseudo-homogeneous topological vector spaces.

DEFINITION 8. A topological vector space X is said to be *pseudo-homogeneous* if its topology can be defined by a family of pseudo-homogeneous seminorms.

In the case where the family of pseudo-homogeneous seminorms can be chosen countable and X is complete, X is said to be a *pseudo-homogeneous Fréchet space*.

Denote by $P(X)$ the family of all pseudo-homogeneous continuous seminorms on X. For each $p \in P(X)$, put

$$U_p = \{x \in X : p(x) \leq 1\} .$$

It is easy to see that

$$p(x) = \inf\{\lambda^{\rho_p} > 0 : x/\lambda \in U_p\}\,,$$

where ρ_p is the homogeneous degree of p.

We note that if $U_p \subseteq U_q$, then $\operatorname{Ker} p \subseteq \operatorname{Ker} q$, and if $p(x_\alpha) \to 0$, then $q(x_\alpha) \to 0$. Thus we can define a continuous linear map $\omega(p,q) : \widehat{X/\operatorname{Ker} p} \to \widehat{X/\operatorname{Ker} q}$. Obviously $\widehat{X} = \varprojlim \{\widehat{X/\operatorname{Ker} p} : p \in P(X)\}$.

THEOREM 9. *Let X be a pseudo-homogeneous Fréchet space and let τ be a pseudo-homogeneous continuous topology on X such that every τ-compact set is compact in X. Then the following properties are equivalent*:

(i) *every subspace of X with BAP has SPAP*,
(ii) *there exists a pseudo-homogeneous continuous norm on X*,
(iii) *X does not contain a subspace isomorphic to $\mathbb{C}^\infty$*,
(iv) *(X,τ) does not contain a subspace isomorphic to $\mathbb{C}^\infty$*,
(v) *every subspace of (X,τ) with BAP has WSPAP*,
(vi) *(X,τ) has a pseudo-homogeneous norm bounded on every compact subset of X.*

To prove the theorem, we first prove the following

PROPOSITION 10. *Let X be a pseudo-homogeneous Fréchet space. Then the following properties are equivalent*:

(i) *every subspace of X with BAP has SPAP*,
(ii) *there exists a pseudo-homogeneous continuous norm on X*,
(iii) *X does not contain a subspace isomorphic to $\mathbb{C}^\infty$.*

Proof. (i)$\Rightarrow$(iii) is an immediate consequence of Proposition 4.

(iii)$\Rightarrow$(ii). Let $\{p_n\}$ be an increasing sequence of pseudo-homogeneous seminorms defining the topology of X. If X does not have a pseudo-homogeneous continuous norm, then $\dim \operatorname{Ker} p_n = \infty$ $\forall n \geq 1$. Since $\operatorname{Ker} p_{n+1} \subseteq \operatorname{Ker} p_n$, $\forall n \geq 1$, we can choose $e_1 \in \operatorname{Ker} p_1$ with $p_2(e_1) \neq 0$. Since $\dim \operatorname{Ker} p_2 = \infty$ and $\operatorname{Ker} p_3 \subseteq \operatorname{Ker} p_2$, we find $e_2 \in \operatorname{Ker} p_2$ such that $\{e_1, e_2\}$ are linearly independent and $p_3(e_2) \neq 0$. Continuing this process, we get a linearly independent sequence $\{e_n\}$ such that $e_n \in \operatorname{Ker} p_n$, $\forall n \geq 1$ and $p_n(e_m) = 0$ for $m > n$. Put $X_0 = \overline{\operatorname{span}\{e_n\}}$. Then $\dim X_0/\operatorname{Ker} p_n < \infty$, $\forall n \geq 1$. Thus $X_0 = \varprojlim X_0/\operatorname{Ker} p_n \cong \mathbb{C}^\infty$. This contradicts (iii).

(ii)$\Rightarrow$(i) is an immediate consequence of Proposition 6.

Proof of Theorem 9. We shall prove that (i)$\Rightarrow$(vi)$\Rightarrow$(v)$\Rightarrow$(iv)$\Rightarrow$(iii)$\Rightarrow$(ii)$\Rightarrow$(i).

(i)$\Rightarrow$(vi). By Proposition 10, we have (i)$\Rightarrow$(ii), and (ii)$\Rightarrow$(vi) is trivial. Hence (i)$\Rightarrow$(vi).

(vi)⇒(v) is an immediate consequence of Proposition 6, and (v)⇒(iv) follows from Proposition 4. We now prove that (iv)⇒(iii). Let X contain a subspace X_0 isomorphic to $\mathbb{C}^\infty$. Since $\dim X_0/\operatorname{Ker} p < \infty$ for every $p \in P(X,\tau)$ it follows that (X_0,τ) is a locally convex space. By a result of Martineau [4], we have $(X_0,\tau) \cong X_0 = \mathbb{C}^\infty$.

Finally, the implications (iii)⇒(ii)⇒(i) follow from Proposition 10.

Corollary 11 (Theorem 2.12 of [5]). *If X is a Fréchet space with BAP, then the following properties are equivalent*:

(i) *X has SPAP*,
(ii) *there is a continuous norm on X*,
(iii) *X contains no subspace isomorphic to $\mathbb{C}^\infty$.*

Examples 12. 1. The following example shows that there is a locally convex space with WSPAP which does not have a continuous norm.

Denote by $(C[0,1],\tau)$ the space of all continuous functions on [0,1] equipped with the topology τ defined by uniform convergence on all convergent sequences of [0,1] and all seminorms defined by $\{e_j^*\}$, where $\{e_j\}$ is the Schauder basis in $C[0,1]$. Then $(C[0,1],\tau)$ has the following properties:

a) $(C[0,1],\tau)$ *is sequentially complete with a compact Schauder basis* $\{e_j\}$. This property follows from the fact that every convergent sequence in $(C[0,1],\tau)$ is convergent in $C[0,1]$.

b) *Every τ-compact subset is compact in $C[0,1]$.*

c) $(C[0,1],\tau)$ *does not have a continuous norm.* Indeed, let p be a continuous norm on $(C[0,1],\tau)$. Then there exists a sequence $\{t_k\}$ convergent in [0,1] and $n \in N$ such that for some constant $C > 0$ we have

$$p(f) \le C \max\{\sup_k |f(t_k)|, \max_{1\le j\le n} |e_j^*(f)|\}$$

for every $f \in C[0,1]$. Obviously this is impossible.

d) $(C[0,1],\tau)$ *does not contain a subspace isomorphic to* $\mathbb{C}^\infty$. Indeed, suppose E is such a subspace. Consider the identity map $(E,\|\cdot\||E) \to (E,\tau|E)$, where $\|f\| = \sup\{|f(t)| : t \in [0,1]\}$. Since E is closed in $C[0,1]$ and $(E,\|\cdot\||E)$ is a Banach space, by the open mapping theorem we get $(E,\|\cdot\||E) \cong (E,\tau|E) \cong \mathbb{C}^\infty$. This is impossible.

From a), b) and from Theorem 9 it follows that $(C[0,1],\tau)$ has WSPAP. On the other hand, by c), $(C[0,1],\tau)$ does not have a continuous norm.

2. Now we consider a class of spaces in which every closed ball is polynomially convex.

a) *Let X be a topological vector space with the Grothendieck approximation property and let ρ be a continuous translation invariant metric on X.*

If $\rho(x,0)$ is plurisubharmonic on X, then for every $x \in X$ and $r > 0$, the closed ball

$$S(x,r) = \{y \in X : \rho(x,y) \le r\}$$

is polynomially convex.

Indeed, let $z \notin S(x,r)$. Then there is k such that $A_k(z) \notin S(x,r)$, where $\{A_j\}$ is the sequence of Grothendieck's approximation. Since $S(x,r) \cap A_k(X)$ is polynomially convex, there exists a polynomial P on $A_k(X)$ such that

$$|P(A_k(z))| > 1 \quad \text{and} \quad \|P\|_{S(x,r)\cap A_k(X)} \le 1\,.$$

Put $\widetilde{P} = PA_k$. Then $\widetilde{P}$ is a polynomial on X such that

$$|\widetilde{P}(z)| > 1 \quad \text{and} \quad \|\widetilde{P}\|_{S(x,r)} \le 1$$

b) *Consider the space $L^p = L^p(X,\mu)$, $0 < p < 1$, with the metric*

$$\rho(x,y) = \int_X |x(t) - y(t)|^p \, d\mu \qquad \textit{for } x, y \in L^p\,.$$

Then $\rho(x,0)$ is plurisubharmonic on L^p.

Indeed, for every complex line in L^p

$$L(\xi) = x + \xi y, \quad \xi \in \mathbb{C}, \text{ where } (x,y) \in L^p \times L^p \setminus \{0\}$$

put

$$\varphi(\xi) = \int_X |x + \xi y|^p \, d\mu\,.$$

We first prove that if x, y are simple functions, then $\varphi(\xi)$ is subharmonic on $\mathbb{C}$. Let $x = \sum_{i=1}^n a_i \chi_{A_i}$ and $y = \sum_{j=1}^m b_j \chi_{B_j}$ where χ_{A_i} and χ_{B_j} are the characteristic functions of A_i and B_j respectively. Then we have

$$\begin{aligned}
\varphi(\xi) &= \int_X \Big|\sum_{i=1}^n a_i \chi_{A_i} + \xi \sum_{j=1}^m b_j \chi_{B_j}\Big|^p \, d\mu \\
&= \sum_{i,j} \int_{A_i \cap B_j} |a_i \chi_{A_i} + \xi b_j \chi_{B_j}|^p \, d\mu = \sum_{i,j} |a_i + \xi b_j|^p \mu(A_i \cap B_j) \\
&= \sum_{i,j} |a_i + \xi b_j|^p \alpha_{ij} = \sum_{i,j} [(a_i + \xi b_j)(\overline{a}_i + \overline{\xi b_j})]^{p/2} \alpha_{ij}\,,
\end{aligned}$$

where $\alpha_{ij} = \mu(A_i \cap B_j)$. Hence

$$\partial\varphi/\partial\xi = \sum_{i,j} (p/2)\alpha_{ij}[(a_i + \xi b_j)(\overline{a}_i + \overline{\xi}\overline{b}_j)]^{p/2-1} b_j(\overline{a}_i + \overline{\xi}\overline{b}_j)\,.$$

$$\begin{aligned}
\partial^2\varphi/\partial\overline{\xi}\partial\xi = \sum_{i,j} &(p/2)(p/2-1)\alpha_{ij}[(a_i + \xi b_j)(\overline{a}_i + \overline{\xi}\overline{b}_j)]^{p/2-2} b_j \overline{b}_j (a_i + \xi b_j) \\
&\times (\overline{a}_i + \overline{\xi}\overline{b}_j) + (p/2)\alpha_{ij} b_j \overline{b}_j [(a_i + \xi b_j)(\overline{a}_i + \overline{\xi}\overline{b}_j)]^{p/2-1}
\end{aligned}$$

$$= (p/4)\sum_{i,j}\alpha_{ij}|b_j|^2[(a_i+\xi b_j)(\overline{a}_i+\overline{\xi b}_j)]^{p/2-1}\ge 0$$

for $\xi\in\mathbb{C}\setminus\bigcup_{i,j}\{\xi : a_i+\xi b_j=0\}$. From this and from the continuity of φ on $\mathbb{C}$ it follows that φ is subharmonic on $\mathbb{C}$.

Let now $(x,y)\in L^p\times L^p\setminus\{0\}$. Then there exists two sequences of simple functions $\{x_n\}$ and $\{y_n\}$ such that $\int_X|x-x_n|^p\,d\mu\to 0$ and $\int_X|y_n-y|^p\,d\mu\to 0$. Put

$$\varphi(\xi)=\int\limits_X|x-\xi y|^p\,d\mu\,,\qquad \varphi_n(\xi)=\int\limits_X|x_n-\xi y_n|^p\,d\mu\,.$$

Then

$$|\varphi(\xi)-\varphi_n(\xi)|\le\int\limits_X|x-x_n|^p\,d\mu+|\xi|^p\int\limits_X|y-y_n|^p\,d\mu\to 0$$

uniformly on every compact subset of $\mathbb{C}$. Thus $\varphi(\xi)$ is subharmonic on $\mathbb{C}$.

From the first example it follows that if $L^p(X,\mu)$ has the Grothendieck approximation, then every closed ball in $L^p(X,\mu)$ is polynomially convex.

In the case where $L^p(X,\mu)$ does not have the Grothendieck approximation, no closed ball in $S(x,r)$ can be polynomially convex. For example, consider the space $L^p[0,1]$, $0<p<1$. It is known that $(L^p[0,1])'=\{0\}$. This implies that every polynomial on $L^p[0,1]$ is constant. Thus no closed ball in $L^p[0,1]$ is polynomially convex.

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