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Some theorems on double integrals over rectangles

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1. Notation. Suppose $a \leqslant x \leqslant b$ and $c \leqslant y \leqslant d$. We shall denote by $[a, x; c, y]$ and $\{a, x; c, y\}$, or briefly by $[x; y]$ and $\{x; y\}$ a and c being fixed, the rectangles

$$\{(s, t)\colon a \leqslant s \leqslant x,\ c \leqslant t \leqslant y\} \quad \text{and} \quad \{(s, t)\colon a < s \leqslant x,\ c < t \leqslant y\},$$

respectively. By $\rangle b; d\langle$ we shall denote the poligonal line, formed by two sides of the greatest rectangle $[a, b; c, d] = [b; d]$, joining the points (b, c) and (a, d) and passing through (a, c).

In theorem 1 we consider functions $f(s, t)$, $g(s, t)$, $\varphi(s, t)$ bounded and Riemann-integrable in the rectangle $[a, b; c, d] = [b; d]$. The function $\varphi(s, t)$ is non-negative and non-increasing in $[b; d]$ in each variable, separately, and such that for any pair t', t'', where $c \leqslant t' < t'' < d$, the difference $\varphi(s, t') - \varphi(s, t'')$ is non-increasing with respect to s in the interval $\langle a, b)$. Moreover, we assume that $\varphi(s, t)$ is not identical to zero in $\{b; d\}$. The integrals are taken in the sense of Riemann. If $a > 0$ and $c > 0$ then for example, $\varphi(s, t) = (s+t)^{-\alpha}$ and $\varphi(s, t) = (st)^{-\alpha}$ $(\alpha > 0)$ satisfy the above conditions in any rectangle $[b; d]$.

In theorems 2 and 3 the function $f(s, t)$ is Lebesgue-integrable over $[0, 1; 0, 1] = [1; 1]$, the function $\varphi(s, t)$ satisfies the same conditions as before, in this square, and the integrals are taken in the Lebesgue sense.

2. An analogue of a Biernacki theorem. At first we shall give the fundamental

LEMMA 1. *Let* u_{ij} $(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n)$ *be arbitrary numbers and let* v_{ij} *and* ω_{ij} *be such that*

$$\sum_{\mu=1}^{i} \sum_{\nu=1}^{j} v_{\mu\nu} > 0 \ (\textit{or} < 0) \qquad (1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n),$$

$$0 \leqslant \omega_{i+1,n} \leqslant \omega_{in}, \qquad 0 \leqslant \omega_{m,j+1} \leqslant \omega_{mj} \qquad (1 \leqslant i \leqslant m-1,\ 1 \leqslant j \leqslant n-1),$$

$$\omega_{i+1,j} - \omega_{i+1,j+1} \leqslant \omega_{ij} - \omega_{i,j+1} \qquad (1 \leqslant i \leqslant m-1,\ 1 \leqslant j \leqslant n-1).$$

Write

$$M_1 = \min_{(i,j)} \frac{\sum\limits_{\mu=1}^{i}\sum\limits_{\nu=1}^{j} u_{\mu\nu}}{\sum\limits_{\mu=1}^{i}\sum\limits_{\nu=1}^{j} v_{\mu\nu}}, \qquad M_2 = \max_{(i,j)} \frac{\sum\limits_{\mu=1}^{i}\sum\limits_{\nu=1}^{j} u_{\mu\nu}}{\sum\limits_{\mu=1}^{i}\sum\limits_{\nu=1}^{j} v_{\mu\nu}}$$

$$(1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n).$$

If, moreover, $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} v_{ij}\omega_{ij} \neq 0$, *then*

$$M_1 \leqslant \frac{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} u_{ij}\omega_{ij}}{\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} v_{ij}\omega_{ij}} \leqslant M_2. \tag{1}$$

Proof. We replace the numerator of the quotient (1) by the right-hand side of the identity ([3], p. 16)

$$\sum_{i=1}^{m}\sum_{j=1}^{n} u_{ij}\omega_{ij} = \sum_{i=1}^{m-1}\sum_{j=1}^{n-1} U_{ij}(\omega_{ij} - \omega_{i,j+1} - \omega_{i+1,j} + \omega_{i+1,j+1}) + \tag{2}$$

$$+ \sum_{i=1}^{m-1} U_{in}(\omega_{in} - \omega_{i+1,n}) + \sum_{j=1}^{n-1} U_{mj}(\omega_{mj} - \omega_{m,j+1}) + U_{mn}\omega_{mn},$$

where

$$U_{ij} = \sum_{\mu=1}^{i}\sum_{\nu=1}^{j} u_{\mu\nu};$$

similarly we write the denominator of this quotient.

To obtain the inequality (1) it is sufficient to use the following: if $B_k > 0$ (or < 0), $c_k \geqslant 0$ $(k = 1, 2, \ldots, r)$, c_k are not all equal to zero, and if

$$S_1 \leqslant \frac{A_k}{B_k} \leqslant S_2 \qquad (k = 1, 2, \ldots, r),$$

S_1 and S_2 being constants, then

$$S_1 \leqslant \frac{\sum\limits_{k=1}^{r} A_k c_k}{\sum\limits_{k=1}^{r} B_k c_k} \leqslant S_2.$$

Now, we shall present the following (compare [1], pp. 123-126; [2], pp. 79-81)

THEOREM 1. *Let*

$$H(x, y) = \begin{cases} \dfrac{\iint\limits_{[x;y]} f(s,t)\,dsdt}{\iint\limits_{[x;y]} g(s,t)\,dsdt} & \text{for} \quad (x, y) \in \{b; d\}, \\[2ex] \dfrac{\int_a^x f(s,c)\,ds}{\int_a^x g(s,c)\,ds} & \text{for} \quad (x, y) \in \rangle b; d\langle,\ x > a, \\[2ex] \dfrac{\int_c^y f(a,t)\,dt}{\int_c^y g(a,t)\,dt} & \text{for} \quad (x, y) \in \rangle b; d\langle,\ y > c, \\[2ex] \dfrac{f(a,c)}{g(a,c)} & \text{for} \quad (x, y) = (a, c). \end{cases}$$

1° *If* $\iint\limits_{[x;y]} g(s,t)\,dsdt \neq 0$ *for every point* $(x, y) \in \{b; d\}$, *then* $\iint\limits_{[b;d]} g(s,t)\varphi(s,t)\,dsdt \neq 0$ *and*

$$\inf_{(x,y)} H(x, y) \leqslant \frac{\iint\limits_{[b;d]} f(s,t)\varphi(s,t)\,dsdt}{\iint\limits_{[b;d]} g(s,t)\varphi(s,t)\,dsdt} \leqslant \sup_{(x,y)} H(x, y), \tag{3}$$

where the infimum and the supremum are taken over all $(x, y) \in \{b; d\}$.

2° *If, moreover, the functions* $f(s,t)$ *and* $g(s,t)$ *are continuous in rectangles* $[a, a+\delta; c, d]$ *and* $[a, b; c, c+\delta]$, *respectively, with* $\delta > 0$ *as small as we please, and if*

$$g(a, c) \neq 0, \qquad \int_a^x g(s,c)\,ds \neq 0, \qquad \int_c^y g(a,t)\,dt \neq 0 \tag{4}$$

$(a < x \leqslant b,\ c < y \leqslant d)$, *then there exists a point* $(\xi, \eta) \in [b; d]$ *such that*

$$\frac{\iint\limits_{[b;d]} f(s,t)\varphi(s,t)\,dsdt}{\iint\limits_{[b;d]} g(s,t)\varphi(s,t)\,dsdt} = H(\xi, \eta). \tag{5}$$

Proof. 1° Let (α, γ) be the point, lying inside $[a, b; c, d] = [b; d]$, at which the value of the function $\varphi(s,t)$ is positive. Taking the normal sequence of partitions

$$\pi_k \left\{ \begin{matrix} a = s_0^{(k)} < s_1^{(k)} < \ldots < s_{m_k-1}^{(k)} < s_{m_k}^{(k)} = b \\ c = t_0^{(k)} < t_1^{(k)} < \ldots < t_{n_k-1}^{(k)} < t_{n_k}^{(k)} = d \end{matrix} \right\} \qquad (k = 1, 2, \ldots) \tag{6}$$

of the rectangle $[b; d]$ such that $\alpha = s^{(k)}_{\mu_k}, \gamma = t^{(k)}_{\nu_k}$ $(0 < \mu_k < m_k, 0 < \nu_k < n_k)$, we denote by $I(\pi_k; f)$, $I_1(\pi_k; f)$ and $I_2(\pi_k; f)$ the sums

$$\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\iint\limits_{P_{ij}^{(k)}} f(s,t)\varphi(s,t)\,dsdt\,, \qquad \sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\iint\limits_{P_{ij}^{(k)}} f(s,t)[\varphi(s,t)-\varphi(s_{i-1}^{(k)}, t_{j-1}^{(k)})]\,dsdt\,,$$

$$\sum_{i=1}^{m_k}\sum_{j=1}^{n_k}\varphi(s_{i-1}^{(k)}, t_{j-1}^{(k)})\iint\limits_{P_{ij}^{(k)}} f(s,t)\,dsdt\,, \tag{7}$$

respectively, where $P_{ij}^{(k)}$ is the rectangle $[s_{i-1}^{(k)}, s_i^{(k)}; t_{j-1}^{(k)}, t_j^{(k)}]$. By $I(\pi_k; g)$, $I_1(\pi_k; g)$ and $I_2(\pi_k; g)$ we shall understand the sums (7) in which f is replaced by g. Let Q_1, Q_2 be the infimum and the supremum of the function H in $\{b; d\}$.

Evidently,

$$I(\pi_k; f) = \iint\limits_{[b;d]} f(s,t)\varphi(s,t)\,dsdt\,, \qquad I(\pi_k; f) = I_1(\pi_k; f)+I_2(\pi_k; f)\,.$$

From the well-known test of integrability it follows

$$\lim_{k\to\infty} I_1(\pi_k; f) = \lim_{k\to\infty} I_1(\pi_k; g) = 0\,,$$

whence

$$\lim_{k\to\infty} I_2(\pi_k; f) = \iint\limits_{[b;d]} f\varphi\,dsdt\,, \qquad \lim_{k\to\infty} I_2(\pi_k; g) = \iint\limits_{[b;d]} g\varphi\,dsdt\,. \tag{8}$$

Let $G(x,y) = \iint\limits_{[x;y]} g(s,t)\,dsdt$. Suppose that $G(x,y) > 0$ for every $(x,y) \in \{b; d\}$. Then, there exists a constant $l > 0$ such that $G(x,y) \geqslant l$ in the rectangle $[\alpha, b; \gamma, d]$. Applying the transformation (2), we obtain

$$\begin{aligned} I_2(\pi_k; g) \geqslant \sum_{i=\mu_k}^{m_k-1}\sum_{j=\nu_k}^{n_k-1} & G(s_i^{(k)}, t_j^{(k)})[\varphi(s_{i-1}^{(k)}, t_{j-1}^{(k)}) - \varphi(s_{i-1}^{(k)}, t_j^{(k)}) - \varphi(s_i^{(k)}, t_{j-1}^{(k)}) + \\ & + \varphi(s_i^{(k)}, t_j^{(k)})] + \sum_{i=\mu_k}^{m_k-1} G(s_i^{(k)}, t_{n_k}^{(k)})[\varphi(s_{i-1}^{(k)}, t_{n_k-1}^{(k)}) - \varphi(s_i^{(k)}, t_{n_k-1}^{(k)})] + \\ & + \sum_{j=\nu_k}^{n_k-1} G(s_{m_k}^{(k)}, t_j^{(k)})[\varphi(s_{m_k-1}^{(k)}, t_{j-1}^{(k)}) - \varphi(s_{m_k-1}^{(k)}, t_j^{(k)})] + \\ & + G(s_{m_k}^{(k)}, t_{n_k}^{(k)})\varphi(s_{m_k-1}^{(k)}, t_{n_k-1}^{(k)}) \geqslant l\varphi(\alpha, \gamma)\,. \end{aligned}$$

Hence

$$I_2(\pi_k; g) \neq 0 \quad \text{and} \quad \iint\limits_{[b;d]} g(s,t)\varphi(s,t)\,dsdt \neq 0\,.$$

The last two inequalities hold also in the case $G(x,y) < 0$.

In virtue of lemma 1,

$$Q_1 \leqslant \frac{I_2(\pi_k; f)}{I_2(\pi_k; g)} \leqslant Q_2\,, \tag{9}$$

and by (8) we obtain (3).

2° If the functions f and g are continuous in $[a+\delta; d]$ and $[b; c+\delta]$, respectively, and if the conditions (4) are satisfied, then, it is easy to verify that the function H is continuous in the rectangle $[b; d]$. From (9), (8) and the Darboux theorem the equality (5) follows.

Corollary 1. Replacing the function f in the inequality (3) by pf, where $p = p(s,t)$ is positive and Riemann-integrable in $[b; d]$, and the function g by p, we obtain, under the assumption

$$\frac{\iint\limits_{[b;d]} p(s,t)f(s,t)\,dsdt}{\iint\limits_{[b;d]} p(s,t)\,dsdt} = \inf_{(x,y)\in\{b;d\}} \frac{\iint\limits_{[x;y]} p(s,t)f(s,t)\,dsdt}{\iint\limits_{[x;y]} p(s,t)\,dsdt}\,,$$

the two-dimensional analogue of the Tchebyshev inequality (compare [1], pp. 128-129)

$$\iint\limits_{[b;d]} pf\,dsdt \cdot \iint\limits_{[b;d]} p\varphi\,dsdt \leqslant \iint\limits_{[b;d]} p\,dsdt \cdot \iint\limits_{[b;d]} pf\varphi\,dsdt;$$

it is easy to observe that the assumption that $\varphi(s,t)$ is non-negative can be omited.

Corollary 2. If $g(s,t) = 1$ in (3), we have

$$\begin{aligned} \inf_{(x,y)} \frac{\iint\limits_{[x;y]} f\,dsdt}{(x-a)(y-c)} \cdot \iint\limits_{[b;d]} \varphi\,dsdt & \\ \leqslant \iint\limits_{[b;d]} f\varphi\,dsdt & \leqslant \sup_{(x;y)} \frac{\iint\limits_{[x;y]} f\,dsdt}{(x-a)(y-c)} \cdot \iint\limits_{[b;d]} \varphi\,dsdt\,, \end{aligned}$$

whence

$$\Big|\iint\limits_{[b;d]} f\varphi\,dsdt\Big| \leqslant \sup_{(x,y)\in\{b;d\}} \frac{\Big|\iint\limits_{[x;y]} f\,dsdt\Big|}{(x-a)(y-c)} \cdot \iint\limits_{[b;d]} \varphi\,dsdt\,. \tag{10}$$

It is easy to verify that (10) remains true for functions f having finite improper Riemann integral $\iint\limits_{[a+,b;c+,d]} f(s,t)\,dsdt$, if the integrals $\iint\limits_{[b;d]} f\varphi\,dsdt$ and $\iint\limits_{[x;y]} f\,dsdt$ are understood as improper (with respect to $\rangle b, d\langle$) in the Riemann sense.

3. An analogue of a Natanson theorem.

We shall prove that the inequality (10) is true for any Lebesgue-integrable function f if integrals $\iint\limits_{[b;d]} f\varphi\,dsdt$ and $\iint\limits_{[x;y]} f\,dsdt$ are considered in the Lebesgue sense. Of course, it is interesting only the case when the supremum in (10) is finite. For the sake of brevity we assume $a = c = 0$, $b = d = 1$ and denote by L^* the class of functions $f(s,t)$, Lebesgue-integrable in the square $[0,1;0,1] = [1;1]$, such that

$$\left|\frac{1}{h_1 h_2}\iint\limits_{[h_1;h_2]} f(s,t)\,dsdt\right| \leqslant M(f) \quad \text{for} \quad 0 < h_i \leqslant 1 \quad (i = 1, 2),$$

where $M(f)$ is a constant depending only on f.

Lemma 2. *For every function $f \in L^*$ and for every positive ε there is a function $f_\varepsilon \in L^*$ continuous in any closed domain included in $[1;1]$ which does not contain the line $\rangle 1;1 \langle$, eventually with the exception of a finite number of segments parallel to the axes x, y, such that the improper (with respect to $\rangle 1;1 \langle$) Riemann integral $\iint\limits_{[0+,1;0+,1]} |f_\varepsilon(s,t)|\,dsdt$ is finite and*

$$\sup_{0<h_1,h_2\leqslant 1} \frac{1}{h_1 h_2}\iint\limits_{[0,h_1;0,h_2]} |f(s,t) - f_\varepsilon(s,t)|\,dsdt \leqslant \varepsilon. \tag{11}$$

Proof. Let h_1 and h_2 be two fixed positive numbers less than or equal to 1 and let n_1 and n_2 denote positive integers such that

$$2^{-n_1} < h_1 \leqslant 2^{1-n_1}, \qquad 2^{-n_2} < h_2 \leqslant 2^{1-n_2}.$$

Divide the square $[0,1;0,1]$ into rectangles by straight lines, parallel to axes of coordinates, passing through points $2^{1-\nu}$ $(\nu = 1, 2, \ldots)$ on both axes.

In view of the two-dimensional analogue of the Weierstrass theorem, in every rectangle $P_{\mu\nu} = [2^{-\mu}, 2^{1-\mu}; 2^{-\nu}, 2^{1-\nu}]$ there exists an algebraic polynomial $W_{\mu\nu}(s,t)$ of two variables such that

$$\iint\limits_{P_{\mu\nu}} |f(s,t) - W_{\mu\nu}(s,t)|\,dsdt < \frac{\varepsilon}{2^{\mu+\nu+2}} \qquad (\mu, \nu = 1, 2, \ldots).$$

Let $f_\varepsilon(s,t)$ be defined as $W_{\mu\nu}(s,t)$ inside each rectangle $P_{\mu\nu}$, and zero at the remaining points of the square $[0,1;0,1]$. Then

$$\frac{1}{h_1 h_2}\iint\limits_{[0,h_1;0,h_2]} |f(s,t) - f_\varepsilon(s,t)|\,dsdt \leqslant 2^{n_1+n_2} \sum_{\mu=n_1,\,\nu=n_2}^{\infty} \iint\limits_{P_{\mu\nu}} |f(s,t) - W_{\mu\nu}(s,t)|\,dsdt$$
$$\leqslant \frac{\varepsilon}{4}\sum_{\mu=0,\,\nu=0}^{\infty} \frac{1}{2^{\mu+\nu}} = \varepsilon.$$

So we have proved the inequality (11) from which it follows that the Lebesgue integral $\iint\limits_{[0,1;0,1]} |f_\varepsilon(s,t)|\,dsdt$ is finite, i.e. the Riemann

improper integral $\iint\limits_{[0+,1;0+,1]} |f_\varepsilon(s,t)|\,dsdt$, being equal to it, is finite also, and $f_\varepsilon \in L^*$.

Now, an analogue of the Natanson lemma ([4], pp. 243-245) for double integrals will be given.

Theorem 2. *If*

$$K = \sup_{0<h_1,h_2\leqslant 1} \left|\frac{1}{h_1 h_2}\iint\limits_{[h_1;h_2]} f(s,t)\,dsdt\right| < \infty,$$

then

$$\left|\iint\limits_{[1;1]} f(s,t)\varphi(s,t)\,dsdt\right| \leqslant K \iint\limits_{[1;1]} \varphi(s,t)\,dsdt. \tag{12}$$

Proof. Let f_ε be the same as above; from (11) there follows

$$\left|\frac{1}{h_1 h_2}\iint\limits_{[h_1;h_2]} f_\varepsilon(s,t)\,dsdt\right| \leqslant K + \varepsilon \qquad (0 < h_i \leqslant 1,\ i = 1, 2),$$

hence, by corollary 2, § 2,

$$\left|\iint\limits_{[1,1]} f(s,t)\varphi(s,t)\,dsdt\right| \leqslant \left|\iint\limits_{[1,1]} f_\varepsilon \varphi\,dsdt\right| + \left|\iint\limits_{[1;1]} (f - f_\varepsilon)\varphi\,dsdt\right|$$
$$\leqslant (K + \varepsilon)\iint\limits_{[1;1]} \varphi(s,t)\,dsdt + \varphi(0,0)\varepsilon,$$

and $\varepsilon > 0$ being arbitrary, we obtain the inequality (12).

Remark. One may also consider unbounded functions φ, satisfying the conditions given in §1 in the square $\{0,1;0,1\}$, the points of unboundedness which lie on coordinate axes. In this case the inequality (12) remains true if both Lebesgue integrals in (12) are finite, or, if the second integral is finite and the function f is non-negative. This result may be obtained by theorem 2 and a suitable approximation of the function φ by bounded functions of two variables.

Applying the inequality (12), the two theorems of Romanovski and Faddeev type ([4], pp. 245-247) concerning the convergence of singular integrals

$$I_n(f) = \iint\limits_{[1;1]} f(s,t)\varphi_n(s,t)\,dsdt$$

may be easily obtained for the classes L_1^* and L_2^* of Lebesgue-integrable functions f in the square $[0,1;0,1] = [1;1]$ such that

$$\lim_{\substack{h_1\to 0+\\ h_2\to 0+}} \frac{1}{h_1 h_2}\iint\limits_{[h_1;h_2]} f(s,t)\,dsdt = f(0,0),$$

$$\lim_{\substack{h_1\to 0+\\ h_2\to 0+}} \frac{1}{h_1 h_2}\iint\limits_{[h_1;h_2]} |f(s,t) - f(0,0)|\,dsdt = 0.$$

We shall formulate one of these theorems, namely

THEOREM 3. *If the functions* $\varphi_n(s,t)$ $(n=1,2,\ldots)$ *subject to the same assumption as* $\varphi(s,t)$ *(given in* §1*), in* [1; 1], *and if*

$$\lim_{n\to\infty}\iint_{[\alpha;\gamma]}\varphi_n(s,t)\,dsdt=1$$

and

$$\overline{\lim_{n\to\infty}}\varphi_n(\alpha,0)<\infty\,,\qquad \overline{\lim_{n\to\infty}}\varphi_n(0,\gamma)<\infty\,,$$

with any positive α *and* γ $(\alpha,\gamma\leqslant 1)$, *then for every function* $f\in L_1^*$ *we have*

$$\lim_{n\to\infty}I_n(f)=f(0,0)\,.$$

Similar theorems hold when φ (or φ_n) is non-negative and non-decreasing in $[b;d]$ with respect to each variable separately, and such that the difference $\varphi(s,t')-\varphi(s,t'')$ is non-decreasing with respect to s for any pair $t'<t''$; in this case the function φ (or φ_n) attains its maximum at the point (b,d) (formerly at (a,c)). It is also evident how to formulate theorems of this type, when the maximum-points of φ (or φ_n) are the remaining vertices of the rectangle $[a,b;c,d]$.

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Démonstration du théorème de Carathéodory par la méthode des points extrémaux

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1. Les éléments frontières. Rappelons brièvement la théorie des éléments frontières, en principe sous sa forme classique (Carathéodory [1]). Soit D un domaine plan simplement connexe, dont la frontière $D^{\cdot}$ contient plus d'un point. Pour chaque $n=0,1,2,\ldots$ soit q_n une *coupure* de D, c'est-à-dire: 1) un arc simple contenu dans D sauf ses extrémités, situées sur $D^{\cdot}$, ou bien 2) une courbe fermée, contenue dans D, peut-être à l'exception d'un point. Une coupure partage D en deux domaines; soit g_n l'un d'eux. On dit que les g_n forment une *chaîne*, si pour $n=0,1,2,\ldots$

$$g_{n+1}\subset g_n\,,$$

$$q_{n+1} \text{ et } q_n \text{ sont disjoints, extrémités comprises}\,.$$

Une chaîne $\{g'_n\}$ est *plus fine* que $\{g_n\}$, si pour tout n il existe un k tel que $g'_k\subset g_n$. Deux chaînes, dont chacune est plus fine que l'autre, sont *équivalentes*; une chaîne qui équivaut à toute chaîne plus fine est dite *élémentaire*.

Soit $\{g_n\}$ une chaîne élémentaire. La classe des chaînes équivalentes est dite — ou bien: elle définit — un *élément sur* D, noté G. On dit que $\{g_n\}$ représente G, ce qu'on écrit: $g_n\to G$. Si $G\neq H$, $g_n\to G$, $h_n\to H$, les g_n et h_n sont disjoints pour n suffisamment grands.

L'ensemble de tous les éléments sur D sera noté D^*.

Soit $g_n\to G$. Considérons, pour un n fixé quelconque, l'ensemble V des éléments H tels que pour chaque $h_k\to H$ il existe un $h_{k'}\subset g_n$. V est dit *voisinage* de G. D^* devient ainsi une espace connexe de Hausdorff.

À tout élément G on peut faire correspondre l'ensemble $G'=\bigcap\bar{g}_n$, où $g_n\to G$. G' sera dit projection de G (fig. 1). La projection est une application continue (1); il peut pourtant arriver que les projections de deux éléments distincts ne soient pas disjointes. G' peut d'ailleurs être un

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(1) C'est-à-dire, si $G'\subset B$ et B est un ensemble ouvert, il existe un voisinage U de G tel que $H'\subset B$ pour $H\subset U$.