and consequently
$$M(1) = |f(-1)| = |f(1)| = 5\,.$$
It may easily be shown that the set $A(1)$ consisting of the roots of the equation $u'(\theta) = 0$ contains only $0$ and $\pi$. We find
$$f'(z) = 3z^2+8z-1\,, \quad f'(-1) = 6\,, \quad f'(1) = 10\,.$$
In view of remark 3 we have
$$M'_+(1) = \max_{z\in A(1)} |f'(x)|\,, \quad M'_-(1) = \min_{z\in A(1)} |f'(z)|\,.$$
Hence
$$M_+(1) = 10 \quad \text{and} \quad M'_-(1) = 6\,.$$

### References


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*Reçu par la Rédaction le 23. 3. 1960*

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POLONICI MATHEMATICI
X (1961)

# Remarks on the extremal functions of a certain class of analytical functions

by J. Zamorski (Wrocław)

Let us study the class $T$ of the analytic functions satisfying in the ring $0 < |z| < 1$ the differential equation

$$\text{(1)} \qquad \frac{zf'(z)}{f(z)} = \alpha p(z) + \beta$$

where $\alpha$ and $\beta$ are any complex numbers and $p(z) = 1 + a_1 z + \ldots$ satisfies the inequality $\operatorname{re} p(z) > 0$. We can easily see that the form of the functions of the class $T$ is as follows:

$$\text{(2)} \qquad f(z) = Cz^{\alpha+\beta}\exp\left\{\alpha\int_0^z \frac{p(s)-1}{s}\,ds\right\} = Cz^{\alpha+\beta}\left\{1+\sum_{k=1}^{\infty} a_k z^k\right\}, \qquad C = \text{const}\,.$$

Let $T_{\alpha,\beta}$ be a subclass of the class $T$ obtained by fixing the numbers $\alpha$ and $\beta$ and putting $C = 1$. In particular $T_{1,0}$ will be identical with the class of all regular starlike schlicht functions, and $T_{-1,0}$ will be identical with the class of all meromorphic starlike schlicht functions. If we put $\alpha = \varrho = 1/(1-ai)$, $\beta = 1-\varrho$ (real $a$) we obtain the class of Špaček spiral schlicht functions [2] and putting $\alpha = -\varrho$, $\beta = -1+\varrho$ we obtain the class of spiral meromorphic schlicht functions. Class $T_{p\varrho,p(1-\varrho)}$ is a certain subclass ($p$ integral) of the class of $p$-valent functions.

In my previous paper [3] it was proved that the following estimations are true:

(I) if $\operatorname{re}\alpha \geqslant 0$ then $\displaystyle |a_n| \leqslant \frac{1}{n!}\prod_{k=0}^{n-1}|2\alpha+k|$;

(II) if $-|\alpha|^2 < \operatorname{re}\alpha < 0$ then

(i) $\displaystyle |a_n| \leqslant \frac{1}{n!}\prod_{k=0}^{n-1}|2\alpha+k|$ for $n = 1, \ldots, N+1$,

(ii) $|a_n| < \frac{1}{nN!}\prod_{k=0}^{N}|2a+k|$ for $n = N+2, \ldots,$

where $N$ is the largest natural number for which

$$N\,\mathrm{re}\,a \geqslant -|a|^2;$$

(III) if $\mathrm{re}\,a \leqslant -|a|^2$ then $|a_n| \leqslant \frac{2|a|}{n}$.

Moreover, in the above mentioned paper there were presented some instances for functions giving equalities in the above estimations.

We are going to prove the following.

LEMMA. *The coefficients of the functions of the class $T_{a,\beta}$ satisfy the inequality*

$$n^2|a_n|^2 \leqslant 4|a|^2 + 4\sum_{k=1}^{n-1}\{k\,\mathrm{re}\,a + |a|^2\}|a_k|^2 .$$

Proof. The function

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = \sum_{k=1}^{\infty}\omega_k z^k \tag{3}$$

satisfies the inequality $|\omega(z)| < 1$ for $|z| < 1$.

Introducing (3) into (1) we obtain

$$\omega(z)\Big[\sum_{k=0}^{\infty}(k+2a)a_k z^k\Big] = \sum_{k=0}^{\infty} k a_k z^k .$$

Hence

$$\omega(z)\Big[\sum_{k=0}^{n-1}(k+2a)a_k z^k\Big] = \sum_{k=0}^{n} k a_k z^k + \sum_{k=n+1}^{\infty} c_k z^k \tag{4}$$

where $c_k$ are certain complex numbers. Computing on the circle $|z| = r < 1$ the mean value of square of modulus of both sides (4) and tending from $r$ to 1 and taking into account $|\omega(z)| < 1$ we obtain

$$\sum_{k=0}^{n-1}|k+2a|^2|a_k|^2 \geqslant \sum_{k=1}^{n} k^2|a_k|^2 .$$

The last inequality gives us the assertion of the lemma. This lemma was proved for $a = -1$ by J. Clunie [1].

Further we will use two theorems included in my previous paper [3].

THEOREM A. *A function of the class $T_{a,\beta}$ for which the functional $E(f) = E(a_1, \ldots, a_n)$ attains its extremal value is of the form*

$$f(z) = z^{a+\beta}\Big[\prod_{k=1}^{n}(1-\sigma_k z)^{-\beta_k}\Big]^a$$



*where*

$$|\sigma_k| = 1, \quad \beta_k > 0, \quad \sum_{k=1}^{n}\beta_k = 2 .$$

THEOREM B. *A function of the class $T_{a,\beta}$ extremal with respect to the functional $E(f) = E(a_1, \ldots, a_n)$ satisfies the following differential equations:*

$$\frac{zf'(z)-\beta f(z)}{f(z)}P_i(z) = Q_i(z), \quad i = 1, 2$$

*where*

$$P_1(z) = \sum_{j=1}^{n}\frac{jA_j}{z^j} - \sum_{j=1}^{n} j\bar{A}_j z^j ,$$

$$P_2(z) = \sum_{j=1}^{n}\frac{A_j}{z^j} - \mathrm{re}\sum_{k=1}^{n} a_k A_k + \sum_{j=1}^{n}\bar{A}_j z^j ,$$

$$Q_1(z) = a\Big[\sum_{j=1}^{n}\frac{1}{z^j}\sum_{k=0}^{n-j}a_k(k+j)A_{k+j} + \sum_{k=1}^{n} k a_k A_k + \sum_{j=1}^{n} z^j\sum_{k=0}^{n-j}\bar{a}_k(k+j)\bar{A}_{k+j}\Big],$$

$$Q_2(z) = a\Big[\sum_{j=1}^{n}\frac{1}{z_j}\sum_{k=0}^{n-j}a_k A_{k+j} + i\,\mathrm{im}\sum_{k=1}^{n}a_k A_k - \sum_{j=1}^{n} z^j\sum_{k=0}^{n-j}\bar{a}_k\bar{A}_{k+j}\Big],$$

$$A_j = \frac{a}{j}\sum_{k=0}^{n-j}a_k\Big\{\frac{\partial E}{\partial x_{k+j}} - i\frac{\partial E}{\partial y_{k+j}}\Big\}, \quad x_k + iy_k = a_k ,$$

$$a_k = \frac{(-1)^{k-1}}{a}\begin{vmatrix} a_1 & 1 & 0 & \ldots & 0 \\ 2a_2 & a_1 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ ka_k & a_{k-1} & a_{k-2} & \ldots & a_1 \end{vmatrix}, \quad a_0 = 1 .$$

On the basis of the above we will prove the following.

THEOREM 1. *If* $\mathrm{re}\,a \geqslant 0$, *the functions of the form*

$$f(z) = z^{a+\beta}(1+\eta z)^{-2a}, \quad |\eta| = 1$$

*are the only functions giving the equality*

$$|a_n| = \frac{1}{n!}\sum_{k=0}^{n-1}|2a+k| .$$

Proof. From the lemma we see that if

$$|a_n| = \frac{1}{n!}\sum_{k=0}^{n-1}|2a+k|$$

then

$$|a_j| = \frac{1}{j!}\prod_{k=0}^{j-1} |2a+k|, \quad j = 1, 2, \ldots, n-1.$$

Since, if $|a_n|$ attains its maximal value for a certain function, then for the same function the maximal value is also attained by $|a_1|$. Turning the plane $z$ and choosing a suitable constant $C$ for formula (2) we can seek the maximum $\operatorname{re} a_1$ instead of seeking the maximum $|a_1|$. The functional $\operatorname{re} a_1$ satisfies the assumption of theorem A and thus the function giving its maximum is of the form

(5) $$f(z) = z^{a+\beta}(1+\eta z)^{-2a},$$

and this proves the theorem.

Note 1. This theorem is also true if $-|a|^2 < \operatorname{re} a < 0$. Of course function (5) gives then the maximal value of $|a_n|$ only for the first $N+1$ coefficients. For subsequent coefficients the sharp estimations and the form of the function giving the maximal value of $|a_n|$ remain open.

THEOREM 2. *If* $\operatorname{re} a < -|a|^2$ *the functions of the form*

$$f(z) = z^{a+\beta}(1+\eta z^n)^{-2a/n}$$

*are the only functions giving the equality*

$$|a_n| = \frac{2}{n}|a|.$$

Proof. Since $k \operatorname{re} a + |a|^2 < 0$ for every natural $k$ then from lemma we can easily see that if $|a_n| = 2|a|/n$, then all previous coefficients equal zero. It follows from (1) that for coefficients of the corresponding function $p(z) = 1 + a_1 z + \ldots$, $\operatorname{re} p(z) > 0$ we have $|a_n| = 2$, and all previous coefficients equal zero. Thus a function with a positive real part corresponding to the extremal function for $|a_n|$ of the class $T_{a,\beta}$ is of the form

$$p(z) = 1 + 2e^{i\varphi}z^n + a_{n+1}z^{n+1} + \ldots$$

The function

$$\omega(z) = \frac{p(z)-1}{p(z)+1} = e^{i\varphi}z^n + \ldots$$

obviously satisfies the inequality $|\omega(z)| < 1$ for $|z| < 1$. Thus, taking into account the Schwarz lemma, we have

$$\omega(z) = e^{i\varphi}z^n$$

and hence

(6) $$p(z) = \frac{1+e^{i\varphi}z^n}{1-e^{i\varphi}z^n}.$$

Putting (6) into (2) we obtain the theorem.



THEOREM 3. *If* $\operatorname{re} a = -|a|^2$ *the functions of the form*

$$f(z) = z^{a+\beta}(1+\eta z^n)^{-2a/n}, \quad |\eta| = 1, \quad n \neq 2$$

*are the only functions giving the equality*

$$|a_n| = 2|a|/n.$$

Note 2. For $n = 2$ the equality $|a_2| = |a|$ may be satisfied also by other functions, as is shown by the example of the function

$$\frac{1}{z}(1+\mu z + z^2), \quad -2 \leqslant \mu \leqslant 2$$

belonging to the class $T_{-1,0}$, which realizes the above equality and is of different form from that given in theorem 3.

Proof. For $n = 1$ the theorem 3 follows directly from theorem A applied to the functional $\operatorname{re} a_1$. Now let $n \geqslant 3$. From the lemma we see that in order to have $|a_n| = 2|a|/n$ we require that all previous coefficients except $a_1$ should equal zero. The value $a_1$ is not defined by the lemma. We can then determine by theorem B applied to the functional $\operatorname{re} a_n$. Obviously for the extremal function $\operatorname{re} a_n = 2|a|/n$. Theorem B gives us the equality

$$P_1(z)Q_2(z) = P_2(z)Q_1(z).$$

In this equality we take into account what we already know about the extremal function and we compare the coefficients of $z^{1-n}$ and $z^{2-n}$. By this comparison we obtain the equality

(7) $$a a_1 i \operatorname{im}\left[\frac{(-1)^n}{n(n-1)}a_1^n + \frac{2|a|}{n}\right] + a\frac{(-1)^n}{n(n-1)}\bar{a}_1^{n-1} = a a_1 \frac{2|a|}{n-1} - (a+1)a_1 \operatorname{re}\left[\frac{(-1)^n}{n(n-1)}a_1^n + \frac{2|a|}{n}\right],$$

(8) $$\bar{a}_1^{n-2}(|a_1|^2 - 1) = 0$$

(these equalities are true only for $n \geqslant 3$). From equality (8) we infer that either $|a_1| = 1$ or $a_1 = 0$. If $|a_1| = 1$ then from (7) we obtain

$$2|a|(n-1-a) + \frac{2a+1}{2}(-1)^n(a_1^n + \bar{a}_1^n) = 0$$

and writing $a_1 = \cos\varphi + i\sin\varphi$ we have

(9) $$\cos n\varrho = (-1)^n \frac{2|a|}{2a+1}(a+1-n).$$

This equality is possible only when $a$ is a real number. But then $a$ satisfies the equation $a = -a^2$ and thus $a = -1$. We can easily see that for

$a = -1$ equality (9) is impossible. Hence $a_1 = 0$. So $a_1 = a_2 = \ldots = a_{n-1} = 0$ for the extremal function and we end the proof similarly to the proof of theorem 2.

Note 3. The case here studied is particulary interesting as it concerns the classes of meromorphic starlike schlicht functions and meromorphic spiral schlicht functions.

Note 4. The theorems here proved were known for regular starlike schlicht functions and for starlike meromorphic schlicht functions satisfying the additional assumption $a_1 = 0$ [1].

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*Reçu par la Rédaction le 15. 6. 1960*

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X (1961)

# Note on abstract differential inequalities and Chaplighin method

by W. Mlak (Kraków)

We are interested in this paper in an abstract treatment of the Chaplighin method [1], [9], [10], [11] for the equation

$$\frac{dx}{dt} = Ax(t) + f\big(t, x(t)\big). \tag{1}$$

$A$ is an infinitesimal generator of a semi-group of linear bounded operators of class $(C_0)$ in the Banach space $E$. The essential moment in the Chaplighin method is the fact that $f(t, x)$ is convex in $x$. The second feature of that method is the use differential inequalities. The purpose of the present paper is the investigation of the Chaplighin method by using methods which are closely related to the Hille-Yosida semigroups theory (see [4]). We make use of some theorems concerning ordinary differential and integral inequalities. In section 1 we give a brief outline of the notation and definitions. We also discuss some geometric properties of positive cones. Section 2 presents some results concerning abstract linear differential inequalities. In sections 3 and 4 we examine almost linear differential inequalities. Sections 5, 6 and 7 are devoted to the main object of this paper. Three principal questions are considered. The first one is the question of existence of the Chaplighin sequence on a common interval. Next we discuss the problem of uniform boundedness and convergence of the Chaplighin sequence. We then use some assumptions imposed on the relationship between the partial ordering and metric properties. Following R. Kalaba [6] we introduce the concept of Newtonian sequences. Finally we present some results which concern the estimation of the norm of the difference between the exact solution of (1) and the approximate one. The last section deals with the uniform boundedness of Newtonian sequences.

**1. Preliminaries.** Let $E$ be a real Banach space. The norm of $x \in E$ is denoted by $|x|$. The norm of bounded linear operators is also denoted by simple bars. The function $f(t, x)$ is defined on $\langle 0, a\rangle \times E$