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R. LEIPUS and A. RAČKAUSKAS (Vilnius)

SECURITY PRICE MODELLING BY A BINOMIAL TREE

Abstract. We consider multidimensional tree-based models of arbitrage-free and path-independent security markets. We assume that no riskless investment exists. Contingent claims pricing and hedging problems in such a market are studied.

1. Introduction. A *binomial tree* (*tree*, for short) is a subset $\{s_0, s_{\varepsilon_1,\ldots,\varepsilon_n} : \varepsilon_i = \pm 1,\ n \ge 1\}$ of $(0,\infty)^d$. The starting point of our work is the observation that trees can be used as models for discrete time security markets. Indeed, let $\Omega = \{-1,1\}^{\mathbb{N}}$ ($\mathbb{N}$ is the set of positive integers) with its Borel σ-algebra $\mathcal{A}$ and the usual invariant probability $\mathbf{P}$. Let $\mathcal{A}_0$ denote the trivial σ-algebra $\{\emptyset, \Omega\}$, and let $\mathcal{A}_n$ ($n \ge 1$) be the σ-algebra generated by the first n coordinates $\varepsilon_1, \ldots, \varepsilon_n$ of the element $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots) \in \Omega$. The sequence $\{S_n,\ n = 0, 1, \ldots\}$ of random vectors on Ω is defined by

$$S_0 = s_0, \quad S_n(\varepsilon) = s_{\varepsilon_1,\ldots,\varepsilon_n}, \quad n \ge 1. \tag{1.1}$$

Then the collection

$$(\Omega, \mathcal{A}, \mathbf{P}, (S_n, \mathcal{A}_n)_{n=0,1,\ldots,N}) \tag{1.2}$$

serves as a discrete time security market model, where the trading takes place at times $0, 1, \ldots, N$ ($N < \infty$). This construction goes back to Cox, Ross, and Rubinstein (1979). Their CRR tree is a subset $\{s_0, s_{\varepsilon_1,\ldots,\varepsilon_n}\}$ of $(0,\infty)^2$ and is determined by a number $r > 0$ and a function $x : \{-1,1\} \to (0,\infty)$ as follows: if $s_0 = (1, s_0^1)$, then

$$s_{\varepsilon_1,\ldots,\varepsilon_n} = T(\varepsilon_n) s_{\varepsilon_1,\ldots,\varepsilon_{n-1}}, \quad n \ge 1,$$

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where $T(\varepsilon_n)$ is the diagonal matrix $\mathrm{diag}(1+r, x(\varepsilon_n))$. The corresponding market model contains a riskless security (bond) with price dynamics $S_n^0 = (1+r)^n$ and a risky security (stock) with price dynamics $S_n^1 = S_0^1 x(\varepsilon_1)\dots x(\varepsilon_n)$. Moreover, the market model is arbitrage-free if

$$0 < x(-1) < 1+r < x(+1). \tag{1.3}$$

Extensions of the binomial model to general discrete time arbitrage-free security markets were subsequently considered by many authors (see, for instance, Harrison and Pliska (1981), Jensen and Nielsen (1996), Lamberton and Lapeyre (1996), Jacod and Shiryaev (1998) and references therein).

Due to some element of abstraction of the riskless security, i.e., a security that permanently grows in value at the rate $1+r$, we suggest the alternative way of modelling an arbitrage-free market where the riskless investment does not exist or is unknown. The role of numeraire is thus played by the price of some fixed *benchmark* portfolio of the securities. As pointed out by Long, Jr. (1990) the notion of the numeraire portfolio (which slightly differs from ours) is related to several ideas, such as the behaviour of asset returns in efficient markets; pricing by risk-neutral valuation; growth-optimal portfolios; empirical definitions of abnormal returns.

Under the arbitrage-free market requirement the class of trees reduces to those with the following property, analogous to (1.3):

P1. *There exists* $a \in \mathbb{R}_+^d = [0,\infty)^d$, $a \neq 0$, *such that for each* $n \geq 1$,

$$\widetilde{s}_{\varepsilon_1,\dots,\varepsilon_{n-1}} \in (\widetilde{s}_{\varepsilon_1,\dots,\varepsilon_{n-1},-1}, \widetilde{s}_{\varepsilon_1,\dots,\varepsilon_{n-1},1}),$$

where $\widetilde{s}_{\varepsilon_1,\dots,\varepsilon_n} := s_{\varepsilon_1,\dots,\varepsilon_n}/\langle s_{\varepsilon_1,\dots,\varepsilon_n}, a\rangle$ ($\langle x,y\rangle$ *denotes the inner product in* $\mathbb{R}^d$) *and* $(x,y) = \{\alpha x + (1-\alpha)y : \alpha \in (0,1)\}$ *for* $x,y \in \mathbb{R}^d$.

Another feature of the CRR model is its path-independence, which means that an up movement of the price followed by a down movement leads to the same node as a down movement of the price followed by an up movement. This corresponds to the so-called recombining tree assumption, i.e.

P2. *For each* $n \geq 2$,

$$s_{\varepsilon_1,\dots,\varepsilon_{n-2},1,-1} = s_{\varepsilon_1,\dots,\varepsilon_{n-2},-1,1}. \tag{1.4}$$

The recombining tree assumption provides a much more computationally efficient model giving $n+1$ nodes in the nth time step instead of 2^n in the "bushy" tree.

Tree models under consideration have a recursive form:

$$s_{\varepsilon_1,\dots,\varepsilon_n} = T(\varepsilon_1,\dots,\varepsilon_n) s_{\varepsilon_1,\dots,\varepsilon_{n-1}},$$

where $T(\varepsilon_1,\dots,\varepsilon_n)$ is a $d\times d$ matrix (see also Motoczyński and Stettner (1998), where a multidimensional extension of the CRR model is studied). Conditions P1 and P2 are met by specifying the matrix $T(\varepsilon_1,\dots,\varepsilon_n)$.

In Section 2 some necessary notions and definitions are collected. The tree based model is studied in detail in Section 3. In addition, several concrete examples of arbitrage-free and path-independent market models are provided. Section 4 deals with contingent claim pricing and hedging problems for the cases considered.

2. Some notions and definitions. Consider a security market where trading according to the rules given below takes place at times $0, 1, \ldots, N$. Let Ω be a set of market states (not necessarily finite); $\mathcal{F}$ be a σ-algebra of subsets of Ω; $\mathbb{F}$ be a class of σ-algebras $\mathcal{F}_0 = \{\emptyset, \Omega\} \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_N \subset \mathcal{F}$; and $\mathbf{P}$ be a probability measure on $(\Omega, \mathcal{F})$. Denote by $S_n^i > 0$ the time n price of the ith security. We assume that, for all i and n, S_n^i is an $\mathcal{F}_n$-measurable random variable. Set $S_n = (S_n^1, \ldots, S_n^d)$ and call the collection

$$(\Omega, \mathcal{F}, \mathbf{P}, (S_n, \mathcal{F}_n)_{0 \le n \le N}) \tag{2.1}$$

a *discrete-time security market.*

Assume that, at each time $n \le N$, the investor can buy/sell securities. Let Φ_n^i be the number of ith securities held at time n. We assume that for any i the process $\{\Phi_n^i,\ n = 0, 1, \ldots, N\}$ is predictable and we call the d-dimensional process $\Phi = \{\Phi_n \equiv (\Phi_n^1, \ldots, \Phi_n^d),\ n = 0, 1, \ldots, N\}$ a *strategy.* The *portfolio value* at time n is the random variable $V_n = \langle \Phi_n, S_n \rangle \equiv \sum_{i=1}^d \Phi_n^i S_n^i$.

Recall that the strategy Φ is called *self-financing* if $\langle \Phi_n, S_n \rangle = \langle \Phi_{n+1}, S_n \rangle$ for each $n = 0, 1, \ldots, N-1$. An *admissible strategy* is defined as a self-financing strategy for which $V_n \ge 0$ a.s. for each $n = 0, 1, \ldots, N$. An admissible strategy Φ is called an *arbitrage strategy* if the corresponding portfolio values satisfy $V_0 = 0$, $\mathbf{E}V_N > 0$. The market (2.1) is *arbitrage-free* if there is no arbitrage strategy.

We define

$$\widetilde{S}_n := S_n / \langle S_n, a \rangle, \qquad n = 0, \ldots, N,$$

where $a \in \mathbb{R}_+^d$, and set $\Delta\widetilde{S}_j := \widetilde{S}_j - \widetilde{S}_{j-1}$.

Standard arguments imply the following arbitrage-free criteria for the market (2.1).

PROPOSITION 2.1. (1) *If there exist $a \in \mathbb{R}_+^d$, $a \neq 0$, and a measure $\mathbf{P}_a$ equivalent to $\mathbf{P}$ ($\mathbf{P}_a \sim \mathbf{P}$) such that the process $\{\widetilde{S}_n,\ n = 0, \ldots, N\}$ is a martingale with respect to $\mathbf{P}_a$ then the market* (2.1) *is arbitrage-free.*

(2) *If the market* (2.1) *is arbitrage-free, then for all $a \in \mathbb{R}_+^d$, $a \neq 0$, there exists a probability measure $\mathbf{P}_a \sim \mathbf{P}$ such that the process $\{\widetilde{S}_n,\ n = 0, \ldots, N\}$ is a martingale with respect to $\mathbf{P}_a$.*

For convenience of the reader the proof of the proposition is provided in the appendix.

3. Binomial model of the security market. Consider the discrete time security market model $(\Omega, \mathcal{A}, \mathbf{P}, (S_n, \mathcal{A}_n)_{n=0,1,\ldots,N})$ corresponding to the tree $\{s_0, s_{\varepsilon_1,\ldots,\varepsilon_n} : \varepsilon_i = \pm 1,\ n \geq 1\}$ as defined in the introduction.

PROPOSITION 3.1. *If the tree* $\{s_0, s_{\varepsilon_1,\ldots,\varepsilon_n}\}$ *has property* P1 *then the market model* $(\Omega, \mathcal{A}, \mathbf{P}, (S_n, \mathcal{A}_n)_{n=0,1,\ldots,N})$ *is arbitrage-free.*

Proof. By the definition (1.1) of the process $\{S_n\}$ and property P1 of the tree it follows that the process $\{S_n/\langle S_n, a\rangle\}$ is a martingale with respect to the $\mathbf{P}$-equivalent measure

$$\mathbf{P}_a = \prod_{n=1}^{\infty} (\alpha_n \delta_1 + (1-\alpha_n)\delta_{-1}),$$

where $\alpha_n \in (0,1)$ satisfies

$$\widetilde{s}_{\varepsilon_1,\ldots,\varepsilon_{n-1}} = \alpha_n \widetilde{s}_{\varepsilon_1,\ldots,\varepsilon_{n-1},1} + (1-\alpha_n)\widetilde{s}_{\varepsilon_1,\ldots,\varepsilon_{n-1},-1}.$$

The result now follows by Proposition 2.1. ∎

Next we give rigorous definitions and notions needed to introduce our model. Fix $a \in \mathbb{R}^d$ such that $\langle a, a\rangle = 1$. Let $L_a = \{ta : t \in \mathbb{R}\}$ and let $\widetilde{e}_1, \ldots, \widetilde{e}_{d-1}$ be an orthonormal basis in

$$L_a^{\perp} = \{x \in \mathbb{R}^d : \langle x, a\rangle = 0\}.$$

For convenience set $\widetilde{e}_d = a$. For $x, y \in \mathbb{R}^d$ define a vector product

$$x \mathbin{\widetilde{\otimes}} y = \sum_{k=1}^{d} \langle x, \widetilde{e}_k\rangle \langle y, \widetilde{e}_k\rangle \widetilde{e}_k.$$

For $\varepsilon \in \{-1, 1\}$ set $\varepsilon' := (1-\varepsilon)/2$.

Consider the tree $\{s_0, s_{\varepsilon_1,\ldots,\varepsilon_n}\}$ defined recursively by the equations

$$s_{\varepsilon_1,\ldots,\varepsilon_n} = [s_{\varepsilon_1,\ldots,\varepsilon_{n-1}} \mathbin{\widetilde{\otimes}} x_n(\varepsilon_n)] X_n\Big(\sum_{k=1}^{n-1} \varepsilon'_k\Big), \qquad n = 1, 2, \ldots, \tag{3.1}$$

where $x_n : \{-1, 1\} \to (0, \infty)^d$ and $X_n : \mathbb{N} \to (0, \infty)$.

The solution to (3.1) is

$$s_{\varepsilon_1,\ldots,\varepsilon_n} = [s_0 \mathbin{\widetilde{\otimes}} x_1(\varepsilon_1) \mathbin{\widetilde{\otimes}} \ldots \mathbin{\widetilde{\otimes}} x_n(\varepsilon_n)] X_1 X_2(\varepsilon'_1) \ldots X_n(\varepsilon'_1 + \ldots + \varepsilon'_{n-1}).$$

Our aim is to ensure that the functions x_n and X_n are defined in such a way that the tree $\{s_0, s_{\varepsilon_1,\ldots,\varepsilon_n}\}$ is recombining and has property P1.

Define

$$Z_n := \frac{X_n(1)}{X_n(0)}, \qquad b_{nj} := \frac{\langle x_n(-1), \widetilde{e}_j\rangle}{\langle x_n(1), \widetilde{e}_j\rangle}, \qquad n \geq 1,\ j = 1, \ldots, d.$$

PROPOSITION 3.2. *The tree given by* (3.1) *is recombining if and only if for any* $n \geq 1$,

(i) $X_n(i+1)/X_n(i)$ *does not depend on* $i = 0,1,\ldots,n-2$; *and*
(ii) $b_{nj} = Z_n \ldots Z_2 b_{1j},\ j = 1,\ldots,d$.

Proof. Condition (1.4) applied to the tree (3.1) reads

$$[x_{n-1}(1)\widetilde{\otimes} x_n(-1)]X_n\Big(\sum_{i=1}^{n-2}\varepsilon_i'\Big) = [x_{n-1}(-1)\widetilde{\otimes} x_n(1)]X_n\Big(\sum_{i=1}^{n-2}\varepsilon_i' + 1\Big). \tag{3.2}$$

This implies that the ratio $X_n(\sum_{i=1}^{n-2}\varepsilon_i' + 1)/X_n(\sum_{i=1}^{n-2}\varepsilon_i')$ does not depend on $\sum_{i=1}^{n-2}\varepsilon_i'$. Thus

$$Z_n = \frac{X_n(\sum_{i=1}^{n-2}\varepsilon_i' + 1)}{X_n(\sum_{i=1}^{n-2}\varepsilon_i')}, \quad n = 2,3,\ldots \tag{3.3}$$

Taking the scalar product of both sides of (3.2) with $\widetilde{e}_j$, we find that for $n = 2,3,\ldots$ and $j = 1,\ldots,d$,

$$\frac{\langle x_n(-1),\widetilde{e}_j\rangle}{\langle x_n(1),\widetilde{e}_j\rangle} = Z_n \ldots Z_2\, \frac{\langle x_1(-1),\widetilde{e}_j\rangle}{\langle x_1(1),\widetilde{e}_j\rangle}. \tag{3.4}$$

Obviously, (i) and (ii) imply (3.2). ■

It is easy to see that the tree $\{s_0, s_{\varepsilon_1,\ldots,\varepsilon_n}\}$ defined by (3.1) satisfies P1 provided

$$\widetilde{1} \in \bigg(\frac{x_n(-1)}{\langle x_n(-1),a\rangle}, \frac{x_n(1)}{\langle x_n(1),a\rangle}\bigg),$$

where $\widetilde{1} \in \mathbb{R}^d$ is such that $\langle \widetilde{1}, \widetilde{e}_j\rangle = 1$ for each $j = 1,\ldots,d$.

COROLLARY 3.1. *Let $a \in \mathbb{R}^d_+$, $a \neq 0$, and let the price process $\{S_n\}$ be given by* (1.1), (3.1). *Assume that for any $n \geq 1$,*

(i) $X_n(i+1)/X_n(i)$ *does not depend on* $i = 0,\ldots,n-2$;
(ii) $b_{nj} = Z_n \ldots Z_2 b_{1j},\ j = 1,\ldots,d$;
(iii) *for each $n = 1,\ldots,N$ and $j = 1,\ldots,d-1$, there exists $\alpha_n \in (0,1)$ such that*

$$\frac{\langle x_n(1),\widetilde{e}_j\rangle}{\langle x_n(1),a\rangle} = \frac{1}{\alpha_n + (1-\alpha_n)c_j}, \quad \textit{where} \quad c_j = b_{1j}/b_{1d}. \tag{3.5}$$

Then the market model (1.2) *is arbitrage-free and path-independent.*

Proof. From (ii) and (iii) we have

$$\frac{\langle x_n(-1),\widetilde{e}_j\rangle}{\langle x_n(-1),a\rangle} = \frac{c_j}{\alpha_n + (1-\alpha_n)c_j}. \tag{3.6}$$

Therefore by (3.5), (3.6),

$$\alpha_n \frac{\langle x_n(1),\widetilde{e}_j\rangle}{\langle x_n(1),a\rangle} + (1-\alpha_n)\frac{\langle x_n(-1),\widetilde{e}_j\rangle}{\langle x_n(-1),a\rangle} = 1$$

for all n. This obviously implies property P1, thus Propositions 3.1 and 3.2 yield the assertion. ∎

REMARK 3.1. One can easily verify that conditions (ii) and (iii) of Corollary 3.1 are equivalent to

(ii′) $b_{nd} = Z_n \dots Z_2 b_{1d}$;

(iii′) for each $n = 1, \dots, N$ and $j = 1, \dots, d-1$, there exists $\alpha_n \in (0,1)$ such that

$$\frac{\langle x_n(1), \widetilde{e}_j\rangle}{\langle x_n(1), a\rangle} = \frac{1}{\alpha_n + (1-\alpha_n)c_j}, \qquad \frac{\langle x_n(-1), \widetilde{e}_j\rangle}{\langle x_n(-1), a\rangle} = \frac{c_j}{\alpha_n + (1-\alpha_n)c_j}.$$

REMARK 3.2. Under the conditions of Corollary 3.1 we have

$$X_n(i) = X_n(0) Z_n^i \tag{3.7}$$

and

$$\frac{\langle x_n(\varepsilon_n), \widetilde{e}_j\rangle}{\langle x_n(\varepsilon_n), a\rangle} = \frac{c_j^{\varepsilon'_n}}{\alpha_n + (1-\alpha_n)c_j}.$$

Thus $s_{\varepsilon_1,\dots,\varepsilon_n}$ can be rewritten as follows:

$$\begin{aligned} s_{\varepsilon_1,\dots,\varepsilon_n} &= \sum_{j=1}^{d} \langle s_{\varepsilon_1,\dots,\varepsilon_n}, \widetilde{e}_j\rangle \widetilde{e}_j \\ &= \sum_{j=1}^{d} \langle s_0, \widetilde{e}_j\rangle \Big[\prod_{k=1}^{n} \langle x_k(\varepsilon_k), \widetilde{e}_j\rangle X_k(\varepsilon'_1 + \dots + \varepsilon'_{k-1})\Big] \widetilde{e}_j \\ &= \sum_{j=1}^{d} \langle s_0, \widetilde{e}_j\rangle \\ &\quad \times \Bigg[\prod_{k=1}^{n} \frac{c_j^{\varepsilon'_k}}{\alpha_k + c_j(1-\alpha_k)} \langle x_k(\varepsilon_k), a\rangle X_k(\varepsilon'_1 + \dots + \varepsilon'_{k-1})\Bigg] \widetilde{e}_j \\ &= \prod_{k=1}^{n} [\langle x_k(\varepsilon_k), a\rangle X_k(0) Z_k^{\varepsilon'_1 + \dots + \varepsilon'_{k-1}}] \sum_{j=1}^{d} \langle s_0, \widetilde{e}_j\rangle \tau_j^{(1,n)} \widetilde{e}_j, \end{aligned} \tag{3.8}$$

where

$$\tau_j^{(m,n)} = \tau_j^{(m,n)}(\varepsilon_1, \dots, \varepsilon_n) := \prod_{k=m}^{n} \frac{c_j^{\varepsilon'_k}}{\alpha_k + c_j(1-\alpha_k)}.$$

Equivalently, (3.8) can be rewritten in the following recursive form:

$$s_{\varepsilon_1,\dots,\varepsilon_n} = s_{\varepsilon_1,\dots,\varepsilon_{n-1}} \mathbin{\widetilde{\otimes}} Y_{\varepsilon_1,\dots,\varepsilon_n}, \tag{3.9}$$

where

$$Y_{\varepsilon_1,\dots,\varepsilon_n} = \langle x_n(\varepsilon_n), a\rangle X_n(0) Z_n^{\varepsilon'_1+\dots+\varepsilon'_{n-1}} \sum_{j=1}^{d} \tau_j^{(n,n)} \widetilde{e}_j.$$

It is easy to see that

$$\langle x_n(\varepsilon_n), a\rangle X_n(\varepsilon'_1+\dots+\varepsilon'_{n-1}) = \langle s_{\varepsilon_1,\dots,\varepsilon_n}, a\rangle / \langle s_{\varepsilon_1,\dots,\varepsilon_{n-1}}, a\rangle. \tag{3.10}$$

Next we consider some particular cases of the model (3.1).

Example 3.1. Assume that

$$s_{\varepsilon_1,\dots,\varepsilon_n} = s_{\varepsilon_1,\dots,\varepsilon_{n-1}} \mathbin{\widetilde{\otimes}} x(\varepsilon_n)\delta_n,$$

where $\delta_n > 0$ for all $n \ge 1$ and $x : \{-1,1\} \to (0,\infty)$. Thus, conditions (i) and (ii) of Corollary 3.1 are automatically satisfied and condition (iii) on the values of $x(1)$ and $x(-1)$ and vectors $a, \widetilde{e}_1, \dots, \widetilde{e}_{d-1}$ becomes

$$\alpha^{(1)} = \dots = \alpha^{(d-1)} = \alpha \in (0,1),$$

where

$$\alpha^{(j)} = \frac{1 - \dfrac{\langle x(-1), \widetilde{e}_j\rangle}{\langle x(-1), a\rangle}}{\dfrac{\langle x(1), \widetilde{e}_j\rangle}{\langle x(1), a\rangle} - \dfrac{\langle x(-1), \widetilde{e}_j\rangle}{\langle x(-1), a\rangle}} \tag{3.11}$$

for $j = 1, \dots, d-1$.

Note that the condition $\alpha^{(j)} \in (0,1)$, where $\alpha^{(j)}$ is defined by (3.11), is equivalent to the following one: either

$$\langle x(-1), \widetilde{e}_j\rangle < \langle x(-1), a\rangle, \qquad \langle x(1), a\rangle < \langle x(1), \widetilde{e}_j\rangle$$

or

$$\langle x(-1), a\rangle < \langle x(-1), \widetilde{e}_j\rangle, \qquad \langle x(1), \widetilde{e}_j\rangle < \langle x(1), a\rangle.$$

In particular, consider the case where the function $x = (x_1, \dots, x_d)$ *is known*. Then we can choose the portfolio a as follows: first check whether there exist k_0, $1 \le k_0 \le d$, and $J \subset \{1, \dots, d\}$ such that

$$\begin{cases} x_j(-1) < x_{k_0}(-1),\ x_{k_0}(1) < x_j(1),\ \forall j \in J,\ j \ne k_0, \\ x_{k_0}(-1) < x_j(-1),\ x_j(1) < x_{k_0}(1),\ \forall j \notin J,\ j \ne k_0. \end{cases} \tag{3.12}$$

If such a k_0 exists, put $a = e_{k_0}$, where $e_1, \dots, e_d$ are the standard orthonormal basis vectors.

If such a k_0 does not exist, then one has to change the standard basis to another one, say $\widetilde{e}_1, \dots, \widetilde{e}_d$, and then to check condition (3.12) with $x_j = \langle x, \widetilde{e}_j\rangle$. It is easy to construct examples where condition (3.12) is not valid for the standard basis but valid for another basis.

EXAMPLE 3.2. Assume that in the model (3.1), $e_1, \dots, e_d$ is the standard orthonormal basis in $\mathbb{R}^d$ and let for each $n \geq 1$,

$$\langle x_n(\varepsilon_n), e_d \rangle = 1 + r, \qquad r > 0.$$

Then $c_j = \langle x_1(-1), e_j \rangle / \langle x_1(1), e_j \rangle$, $Z_n \equiv 1$ and

$$\langle x_n(\varepsilon_n), e_j \rangle = \frac{(1+r) c_j^{\varepsilon'_n}}{\alpha_n + (1-\alpha_n) c_j}.$$

The tree (3.1) reduces to

$$s_{\varepsilon_1,\dots,\varepsilon_n} = s_{\varepsilon_1,\dots,\varepsilon_{n-1}} \widetilde{\otimes} x_n(\varepsilon_n)$$

or, equivalently,

$$s_{\varepsilon_1,\dots,\varepsilon_n} = T_n(\varepsilon_n) s_{\varepsilon_1,\dots,\varepsilon_{n-1}},$$

where $T_n(\varepsilon_n) = \operatorname{diag}(x_n^1(\varepsilon_n), \dots, x_n^{d-1}(\varepsilon_n), 1+r)$, $x_n^j := \langle x_n(\varepsilon_n), e_j \rangle$. This model corresponds to the CRR model with one dth riskless security and $d-1$ risky securities, where jumps at each time n are non-identically distributed.

EXAMPLE 3.3. Assume that the model (3.1) is such that

$$\langle x_n(1), a \rangle \langle x_n(-1), a \rangle = 1, \qquad n \geq 1. \tag{3.13}$$

Together with condition (ii′) of Remark 3.1, this implies that for any $n \geq 1$,

$$\langle x_n(-1), a \rangle = (Z_n \dots Z_2 b_{1d})^{1/2}, \qquad \langle x_n(1), a \rangle = (Z_n \dots Z_2 b_{1d})^{-1/2},$$

or more concisely

$$\langle x_n(\varepsilon_n), a \rangle = (Z_n \dots Z_2 b_{1d})^{-\varepsilon_n/2}, \qquad n \geq 1.$$

Therefore, by (3.9), we have

$$s_{\varepsilon_1,\dots,\varepsilon_n} = s_{\varepsilon_1,\dots,\varepsilon_{n-1}} \widetilde{\otimes} Y_{\varepsilon_1,\dots,\varepsilon_n}, \qquad n \geq 1,$$

where

$$\begin{aligned} Y_{\varepsilon_1,\dots,\varepsilon_n} &= (Z_n \dots Z_2 b_{1d})^{\varepsilon_n/2} X_n(0) Z_n^{\varepsilon'_1+\dots+\varepsilon'_{n-1}} \sum_{j=1}^d \frac{c_j^{\varepsilon'_n}}{\alpha_n + c_j(1-\alpha_n)} \widetilde{e}_j \\ &= X_n(0) b_{1d}^{\varepsilon'_n - 1/2} Z_n^{\varepsilon'_n+\varepsilon'_1+\dots+\varepsilon'_{n-1}-1/2} \prod_{k=2}^{n-1} Z_k^{\varepsilon'_n-1/2} \sum_{j=1}^d \frac{c_j^{\varepsilon'_n}}{\alpha_n + c_j(1-\alpha_n)} \widetilde{e}_j. \end{aligned}$$

4. Contingent claim pricing and hedging. Consider the discrete time security market model $(\Omega, \mathcal{A}, \mathbf{P}, (S_n, \mathcal{A}_n)_{n=0,1,\dots,N})$ corresponding to the tree $\{s_0, s_{\varepsilon_1,\dots,\varepsilon_n} : \varepsilon_i = \pm 1,\ n \geq 1\}$ as in (3.1). Assume that $f(S_N)$ is a contingent claim, where $S_n \equiv (S_n^1, \dots, S_n^d)$ and $f : \mathbb{R}^d \to (0,\infty)$ is a measurable function. Let the assumptions of Corollary 3.1 be satisfied.

Then the value of $f(S_N)$ at time n is

$$V_n = \langle S_n, a\rangle \mathbf{E}_a\left(\frac{f(S_N)}{\langle S_N, a\rangle}\,\Big|\, \mathcal{F}_n\right), \tag{4.1}$$

where the conditional expectation $\mathbf{E}_a$ is taken with respect to the measure

$$\mathbf{P}_a = \prod_{n=1}^{\infty}(\alpha_n\delta_1 + (1-\alpha_n)\delta_{-1}).$$

THEOREM 4.1. *Let the assumptions of Corollary* 3.1 *be satisfied. Then the time n value of the contingent claim $f(S_N)$ is*

$$V_n = F_n(\varepsilon_1, \ldots, \varepsilon_n), \tag{4.2}$$

where

$$\begin{aligned} F_n(i_1, \ldots, i_n) &\\ = \sum_{i_{n+1},\ldots,i_N=\pm 1} & f(s_{i_1,\ldots,i_n} \widetilde{\otimes}\, t_n(i_1, \ldots, i_N)) \\ \times \prod_{k=n+1}^{N} & [\langle x_k(i_k), a\rangle^{-1} X_k^{-1}(0) Z_k^{-(i'_1+\ldots+i'_{k-1})}(1-\alpha_k)^{i'_k}\alpha_k^{1-i'_k}] \end{aligned} \tag{4.3}$$

and

$$t_n(i_1, \ldots, i_N) := \prod_{k=n+1}^{N}[\langle x_k(i_k), a\rangle X_k(0) Z_k^{i'_1+\ldots+i'_{k-1}}] \sum_{j=1}^{d} \tau_j^{(n+1,N)}(i_{n+1}, \ldots, i_N)\, \widetilde{e}_j$$

($i'_k := (1-i_k)/2,\ i_k = \pm 1$).

Proof. By (3.10),

$$\langle S_N, a\rangle = \langle S_n, a\rangle \prod_{k=n+1}^{N} \langle x_k(\varepsilon_k), a\rangle X_k(\varepsilon'_1 + \ldots + \varepsilon'_{k-1}).$$

Thus, (4.1) yields

$$V_n = \mathbf{E}_a\Big(f(S_N) \prod_{k=n+1}^{N} \langle x_k(\varepsilon_k), a\rangle^{-1} X_k^{-1}(\varepsilon'_1 + \ldots + \varepsilon'_{k-1})\,\Big|\, \mathcal{F}_n\Big). \tag{4.4}$$

Observe that

$$S_N(\varepsilon) \equiv s_{\varepsilon_1,\ldots,\varepsilon_N} = s_{\varepsilon_1,\ldots,\varepsilon_n} \widetilde{\otimes}\, t_n(\varepsilon_1, \ldots, \varepsilon_N),$$

where

$$t_n(\varepsilon_1,\ldots,\varepsilon_N)=Y_{\varepsilon_1,\ldots,\varepsilon_{n+1}}\,\widetilde{\otimes}\ldots\widetilde{\otimes}\,Y_{\varepsilon_1,\ldots,\varepsilon_N}$$
$$=\prod_{k=n+1}^{N}\langle x_k(\varepsilon_k),a\rangle X_k(\varepsilon_1'+\ldots+\varepsilon_{k-1}')\sum_{j=1}^{d}\tau_j^{(n+1,N)}\widetilde{e}_j.$$

Since $\varepsilon_{n+1},\ldots,\varepsilon_N$ are independent of the σ-algebra $\mathcal{F}_n$, applying (3.7) we obtain from (4.4),

$$\begin{aligned}V_n=&\sum_{i_{n+1},\ldots,i_N=\pm1}f(s_{\varepsilon_1,\ldots,\varepsilon_n}\,\widetilde{\otimes}\,t_n(\varepsilon_1,\ldots,\varepsilon_n,i_{n+1},\ldots,i_N))\\&\times\prod_{k=n+1}^{N}[\langle x_k(i_k),a\rangle^{-1}X_k^{-1}(\varepsilon_1'+\ldots+\varepsilon_n'+i_{n+1}'+\ldots+i_{k-1}')]\\&\times\prod_{l=n+1}^{N}(1-\alpha_l)^{i_l'}\alpha_l^{1-i_l'}\\\equiv\ &F_n(\varepsilon_1,\ldots,\varepsilon_n).\ \blacksquare\end{aligned}$$

Example 4.1. Consider the model given in Example 3.1 and assume that the conditions of Corollary 3.1 hold. Let $b:=b_{1d}\equiv\langle x(-1),a\rangle/\langle x(1),a\rangle$.

Since $\langle x(i_k),a\rangle=b^{i_k'}\langle x(1),a\rangle$, by Theorem 4.1 we obtain

$$\begin{aligned}(4.5)\qquad &F_n(i_1,\ldots,i_n)\\&=\sum_{i_{n+1},\ldots,i_N=\pm1}f\biggl(\prod_{k=1}^{N}(\langle x(i_k),a\rangle\delta_k)\sum_{j=1}^{d}\langle s_0,\widetilde{e}_j\rangle\frac{c_j^{i_1'+\ldots+i_N'}}{(\alpha+c_j(1-\alpha))^N}\,\widetilde{e}_j\biggr)\\&\quad\times\prod_{k=n+1}^{N}(\langle x(i_k),a\rangle^{-1}\delta_k^{-1}(1-\alpha)^{i_k'}\alpha^{1-i_k'})\\&=\sum_{k=0}^{N-n}\binom{N-n}{k}f\biggl(b^{i_1'+\ldots+i_n'+k}\gamma_{1,N}\sum_{j=1}^{d}\langle s_0,\widetilde{e}_j\rangle\frac{c_j^{i_1'+\ldots+i_n'+k}}{(\alpha+c_j(1-\alpha))^N}\,\widetilde{e}_j\biggr)\\&\quad\times b^{-k}\gamma_{n+1,N}^{-1}\Bigl(\frac{1-\alpha}{\alpha}\Bigr)^k\alpha^{N-n},\end{aligned}$$

where

$$\gamma_{m,N}:=\langle x(1),a\rangle^{N-m+1}\delta_m\ldots\delta_N,\qquad m\le N.$$

Pricing of a European call option. Assume that the contingent claim $f(S_N)$ is the payoff of the European call option on "index" $\langle S_N,w\rangle$ with exercise price K, where $w\in[0,1]^d$, $\langle w,w\rangle=1$:

$$f(S_N)=(\langle S_N,w\rangle-K)^+.$$

For the price dynamics corresponding to (3.8) the value of a European call option at time n, by Theorem 4.1, is given by (4.2) with

$$F_n(i_1,\dots,i_n) = \sum_{i_{n+1},\dots,i_N=\pm 1} (\langle s_{i_1,\dots,i_N}, w\rangle - K)^+ \times \prod_{k=n+1}^{N} [\langle x_k(i_k), a\rangle^{-1} X_k^{-1}(0) Z_k^{-(i'_1+\dots+i'_{k-1})} (1-\alpha_k)^{i'_k} \alpha_k^{1-i'_k}].$$

Note that

$$\langle s_{i_1,\dots,i_N}, w\rangle = \prod_{k=1}^{N} [\langle x_k(i_k), a\rangle X_k(0) Z_k^{i'_1+\dots+i'_{k-1}}] \times \sum_{j=1}^{d} \langle s_0, \widetilde{e}_j\rangle \tau_j^{(1,N)}(i_1,\dots,i_N)\langle w, \widetilde{e}_j\rangle. \tag{4.6}$$

If the benchmark portfolio a coincides with the weights $w = (w_1,\dots,w_d)$ (or any orthonormal basis vector in $L_w^\perp$), i.e. $a = w$, then (4.6) simplifies to

$$\langle s_{i_1,\dots,i_N}, w\rangle = \langle s_0, w\rangle \prod_{k=1}^{N} \langle x_k(i_k), w\rangle X_k(0) Z_k^{i'_1+\dots+i'_{k-1}}. \tag{4.7}$$

If $w = a$, in the case of Example 4.1 with $b > 1$, one sees by (4.5) and (4.7) that the time n value of the payoff $(\langle S_N, w\rangle - K)^+$ corresponding to the European call option equals

$$\begin{aligned} F_n(\varepsilon_1,\dots,\varepsilon_n) &= \sum_{k=0}^{N-n} \binom{N-n}{k} (b^{\varepsilon'_1+\dots+\varepsilon'_n+k}\gamma_{1,N}\langle s_0, w\rangle - K)^+ \\ &\quad \times b^{-k}\gamma_{n+1,N}^{-1}(1-\alpha)^k \alpha^{N-n-k} \\ &= b^{\varepsilon'_1+\dots+\varepsilon'_n}\gamma_{1,n}\langle s_0, w\rangle \varphi(k_0, N-n, 1-\alpha) \\ &\quad - K\gamma_{n+1,N}^{-1} b^{-(N-n)}(1-\alpha+\alpha b)^{N-n}\varphi(k_0, N-n, \overline{\alpha}), \end{aligned}$$

where

$$\begin{aligned} \varphi(k,n,p) &:= \sum_{j=k+1}^{n} \binom{n}{j} p^j (1-p)^{n-j}, \\ k_0 &:= \left[\frac{\log K - \log\gamma_{1,N} - \log\langle s_0, w\rangle}{\log b}\right] - \varepsilon'_1 - \dots - \varepsilon'_n \\ \overline{\alpha} &:= \frac{1-\alpha}{1-\alpha+\alpha b}. \end{aligned}$$

Hedging. To find the *hedging* strategy $\{\Phi_n,\ n=0,\dots,N\}$ for the contingent claim $f(S_N)$, we have to solve the equations

$$\langle \Phi_n, s_{\varepsilon_1,\dots,\varepsilon_n}\rangle = F_n(\varepsilon_1,\dots,\varepsilon_n)$$

for each $n=0,1,\dots,N$, where $F_n(i_1,\dots,i_n)$ is given in (4.3). Since $\{\Phi_n,\ n=0,\dots,N\}$ is a predictable sequence and therefore Φ_n does not depend on ε_n, we have

$$\begin{aligned}\langle \Phi_n, s_{\varepsilon_1,\dots,\varepsilon_{n-1},-1}\rangle &= F_n(\varepsilon_1,\dots,\varepsilon_{n-1},-1),\\ \langle \Phi_n, s_{\varepsilon_1,\dots,\varepsilon_{n-1},1}\rangle &= F_n(\varepsilon_1,\dots,\varepsilon_{n-1},1).\end{aligned}$$

Note that in the case $d=2$ one easily obtains the unique solution

$$\Phi_n^1 = \frac{F_n(\varepsilon_1,\dots,\varepsilon_{n-1},-1)s^2_{\varepsilon_1,\dots,\varepsilon_{n-1},1} - F_n(\varepsilon_1,\dots,\varepsilon_{n-1},1)s^2_{\varepsilon_1,\dots,\varepsilon_{n-1},-1}}{s^1_{\varepsilon_1,\dots,\varepsilon_{n-1},-1}s^2_{\varepsilon_1,\dots,\varepsilon_{n-1},1} - s^1_{\varepsilon_1,\dots,\varepsilon_{n-1},1}s^2_{\varepsilon_1,\dots,\varepsilon_{n-1},-1}},$$

$$\Phi_n^2 = \frac{F_n(\varepsilon_1,\dots,\varepsilon_{n-1},-1)s^1_{\varepsilon_1,\dots,\varepsilon_{n-1},1} - F_n(\varepsilon_1,\dots,\varepsilon_{n-1},1)s^1_{\varepsilon_1,\dots,\varepsilon_{n-1},-1}}{s^2_{\varepsilon_1,\dots,\varepsilon_{n-1},-1}s^1_{\varepsilon_1,\dots,\varepsilon_{n-1},1} - s^2_{\varepsilon_1,\dots,\varepsilon_{n-1},1}s^1_{\varepsilon_1,\dots,\varepsilon_{n-1},-1}}.$$

5. Appendix. To prove Proposition 2.1 we will need the following lemma.

LEMMA 5.1. *Let $a\in\mathbb{R}^d_+$ and $\{\Phi'_n,\ n=0,1,\dots,N\}$ be a d-dimensional predictable process. Then for an arbitrary number V_0 there exists a unique one-dimensional predictable process $\{\phi_n,\ n=0,\dots,N\}$ such that $\{\Phi_n=\Phi'_n+\phi_n a,\ n=0,1,\dots,N\}$ is a self-financing strategy and the initial value of the new portfolio is V_0.*

Proof. Let $\widetilde{V}_n := V_n/\langle S_n, a\rangle$ and define

$$\phi_n = \widetilde{V}_0 + \sum_{j=1}^{n}\langle \Phi'_j, \Delta\widetilde{S}_j\rangle - \langle \Phi_n, \widetilde{S}_n\rangle. \tag{5.1}$$

It follows immediately from (5.1) that $\{\phi_n,\ n=0,\dots,N\}$ is predictable.

On the other hand, the self-financing assumption is equivalent to

$$\widetilde{V}_n = \widetilde{V}_0 + \sum_{j=1}^{n}\langle \Phi_j, \Delta\widetilde{S}_j\rangle \tag{5.2}$$

(see Proposition 1.1.2 in Lamberton and Lapeyre (1996)). Substituting $\Phi_j = \Phi'_j + \phi_j a$ to (5.2) we obtain

$$\langle \Phi'_n, \widetilde{S}_n\rangle + \phi_n\langle a, \widetilde{S}_n\rangle = \widetilde{V}_0 + \sum_{j=1}^{n}\langle \Phi'_j, \Delta\widetilde{S}_j\rangle + \sum_{j=1}^{n}\phi_j\langle a, \Delta\widetilde{S}_j\rangle. \tag{5.3}$$

Since $\langle a, \widetilde{S}_n \rangle = 1$ and hence $\langle a, \Delta\widetilde{S}_n \rangle = 0$ for any n, (5.3) implies that

$$\phi_n = \widetilde{V}_0 + \sum_{j=1}^{n} \langle \Phi'_j, \Delta\widetilde{S}_j \rangle - \langle \Phi'_n, \widetilde{S}_n \rangle$$

and therefore $\{\phi_n\}$ is a unique predictable process such that $\{\Phi_n\}$ is a self-financing strategy. ∎

Proof of Proposition 2.1. (1) Let $a \in \mathbb{R}^d_+$ and suppose the sequence $\{\widetilde{S}_n, n = 0, \dots, N\}$ is a martingale with respect to the measure $\mathbf{P}_a \sim \mathbf{P}$. Assume that $\{\Phi_n\}$ is an admissible strategy such that $V_0 = 0$. Let

$$\widetilde{V}_n = V_n / \langle S_n, a \rangle, \qquad n = 1, \dots, N.$$

Since for every self-financing strategy $\{\Phi_n\}$ equality (5.2) holds, it follows that $\{\widetilde{V}_n\}$ is a martingale with respect to $\mathbf{P}_a$. Thus

$$\mathbf{E}_a \widetilde{V}_N = \mathbf{E}_a \widetilde{V}_0 = \mathbf{E}_a (V_0 / \langle S_0, a \rangle) = 0, \tag{5.4}$$

where $\mathbf{E}_a$ denotes the expectation with respect to the measure $\mathbf{P}_a$.

Since $V_N \geq 0$ and $\mathbf{P} \sim \mathbf{P}_a$, equality (5.4) implies that $\mathbf{E} V_N = 0$, i.e., the market is arbitrage-free.

(2) For any d-dimensional predictable process $\{\Phi'_n\}$ set

$$v_n := \sum_{j=1}^{n} \langle \Phi'_j, \Delta\widetilde{S}_j \rangle, \qquad n = 0, 1, \dots, N.$$

According to Theorem 1.1 in Kabanov and Kramkov (1994), we have to prove that $v_n \geq 0$ for each $n = 0, 1, \dots, N$ implies $v_N = 0$ a.s. By Lemma 5.1 there exists a unique process $\{\phi_n\}$ such that $\{\Phi_n = \Phi'_n + \phi_n a\}$ is a self-financing strategy and $V_0 = 0$. Then for the self-financing strategy $\{\Phi_n\}$ we have

$$\widetilde{V}_n = \widetilde{V}_0 + \sum_{j=1}^{n} \langle \Phi_j, \Delta\widetilde{S}_j \rangle = \sum_{j=1}^{n} \langle \Phi'_j, \Delta\widetilde{S}_j \rangle = v_n. \tag{5.5}$$

If $v_n \geq 0$ for all $n = 0, 1, \dots, N$, then from (5.3) and the definition of the arbitrage-free market it follows that $\mathbf{E} v_N = 0$. ∎

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Remigijus Leipus
Department of Mathematics
Vilnius University
Naugarduko 24
Vilnius 2600, Lithuania
E-mail: Remigijus.Leipus@maf.vu.lt

Alfredas Račkauskas
Institute of Mathematics and Informatics
Akademijos 4
Vilnius 2600, Lithuania
E-mail: alfredas@ieva.maf.vu.lt

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