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THE SYMMETRIC PLURICOMPLEX GREEN FUNCTION

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1. Introduction. Let Ω be an open and connected set in $\mathbb{R}^n$; $x_0 \in \Omega$. Then the classical Green function $G_\Omega(x, x_0)$ is the solution to the Dirichlet problem

$$\begin{cases} G_\Omega(x, x_0) = 0, \quad \forall x \in \partial\Omega \\ \Delta_x G(x, x_0) = \delta_{x_0}. \end{cases}$$

In [7], Klimek introduced the pluricomplex Green function g_Ω, that can be defined as solution to

$$\begin{cases} g_\Omega(z, z_0) \in PSH(\Omega) \\ g_\Omega(z, z_0) = 0, \quad \forall z \in \partial\Omega \\ (dd_z^c g_\Omega(z, z_0))^n = (2\pi)^n \delta_{z_0} \end{cases}$$

where Ω is a domain (open, bounded, and connected set) in $\mathbb{C}^n$; $z_0 \in \Omega$.

An alternative definition of g_Ω for any domain Ω in $\mathbb{C}^n$, $z_0 \in \Omega$ is

$$g_\Omega(z, z_0) = \sup\{\varphi(z); \varphi \in PSH(\Omega), \varphi \le 0, \varphi(z) - \log|z - z_0| \text{ bounded above near } z = z_0\}.$$

It is well known that the classical Green function is symmetric: $G_\Omega(x, x_0) = G_\Omega(x_0, x)$. However, the pluricomplex Green function need not be symmetric.

It was shown by Bedford and Demailly [2] that there exists a strictly pseudoconvex smooth Ω such that $g_\Omega(z, z_0) \neq g_\Omega(z_0, z)$.

2. The symmetric pluricomplex Green function. In [3], we introduced the symmetric pluricomplex Green function $W_\Omega(z, \omega)$,

$$W_\Omega(z, \omega) = \sup\{\varphi(z, \omega) \in 2 - PSH(\Omega \times \Omega), \quad \varphi \le 0,$$
$$\varphi(z, \omega) \le \log|z - \omega| - \log\max[d(z, \complement\Omega), d(\omega, \complement\Omega)]\}.$$

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Here, $2-PSH(\Omega\times\Omega)$ denotes the subharmonic functions that are also separately plurisubharmonic. The purpose of this note is to consider some basic properties of W_Ω.

DEFINITION. A domain Ω is said to be *strongly hyperconvex* if $W_\Omega(z,\omega)$ is an exhaustion function for Ω for each fixed $\omega\in\Omega$.

Remark. Every strictly pseudoconvex set is strongly hyperconvex.

LEMMA 1. *Suppose φ is plurisubharmonic near zero and that $|\varphi(z)-\log|z||<K$ near zero for some constant K. Then $\mu(0)\geq(2\pi)^n$ where μ is the weak*-limit of $(dd^c\max[\varphi,t])^n, t\to-\infty$.*

Proof. Let $1>r>0$ so that φ is plurisubharmonic and so that

$$|\varphi(z)-\log(z)|\leq K \quad \text{on } B(0,r).$$

Given $0<\epsilon<1$, then

$$|z|>e^{\frac{-1}{\epsilon}}\Rightarrow \log|z|\geq\frac{-1}{\epsilon}\Rightarrow -\epsilon\log|z|<1\Rightarrow(\varphi-\log|z|>-K)\Rightarrow$$
$$\varphi-\log|z|>-(K+1)-\epsilon\log|z|\Rightarrow\varphi(z)>(1-\epsilon)\log|z|-(K+1).$$

Thus

$$\Omega_\epsilon=\{z\in B(0,r);\varphi(z)<(1-\epsilon)\log|z|-(K+1)\}$$

is a neighborhood of zero and relatively compact in $B(0,r)$ if $e^{-\frac{1}{\epsilon}}<r$.

Let $t<\inf_{z\in\partial\Omega_\epsilon}(1-\epsilon)\log|z|-(K+1)=\delta<0$ and define $\varphi_{2t}=\max[\varphi,2t]$.

Then $\Omega_\epsilon^t=\{z\in B(0,r);\ \varphi_{2t}<\max[(1-\epsilon)\log|z|-(K+1),t]\}$ is a neighborhood of zero and relatively compact in $B(0,r)$. Thus

$$\int\limits_{\Omega_\epsilon^t}(dd^c\varphi_t)\geq\int\limits_{\Omega_\epsilon^t}(dd^c\max[(1-\epsilon)\log|z|-K+1,t])^n=(2\pi)^n(1-\epsilon)^n$$

and since $\Omega_\epsilon^t\subset\Omega_\epsilon\subset B(0,e^{-\frac{1}{\epsilon}})$,

$$\int\limits_{B(0,e^{-\frac{1}{\epsilon}})}(dd^c\varphi_{2t})^n\geq(2\pi)^n(1-\epsilon)^n.$$

So if μ is the weak*-limit of $(dd^c\varphi_{2t})^n$, $t\to-\infty$, then $\mu(0)\geq(2\pi)^n$ which proves the lemma. ■

THEOREM 1. *Suppose Ω is strongly hyperconvex. Then $g_\Omega\geq W_\Omega$ with equality if and only if*

$$\tau(z)=\int\limits_\Omega(dd^c_\xi\max[W_\Omega(z,\xi),-1])^n=(2\pi)^n,\quad\forall z\in\Omega.$$

Proof (cf. [3, Prop. VII:2). Note first that $\int_\Omega(dd^c_z\max[g_\Omega(z,\xi),t])^n=(2\pi)^n$,

$\forall t < 0$, $\forall z \in \Omega$ and that

$$\tau(z) = \int_{\Omega} (dd^c_\xi \max[W_\Omega(z,\xi), t])^n$$

is independent of t for all negative t. Also, it follows from definitions that $g_\Omega \geq W_\Omega$. It follows from Lemma 1 that $\tau(z) \geq (2\pi)^n$ with equality if $W_\Omega \equiv g_\Omega$. ∎

On the other hand, assume $\tau(z) \equiv (2\pi)^n$. Again, by Lemma 1, $(dd^c W_\Omega(z,\xi))^n = 0$ on $z \neq \xi$. Let $\xi \in \Omega$ be given and consider for $0 < \epsilon < 1$, $(1-\epsilon)W_\Omega(z,\xi)$. Then $(1-\epsilon)W_\Omega(z,\xi) = 0$ on $\partial\Omega, (1-\epsilon)W_\Omega(z,\xi) \geq g_\Omega(z,\xi)$ for z near ξ. Since $(dd^c_z(1-\epsilon)W_\Omega(z,\xi))^n = 0$ outside ξ, $(1-\epsilon)W_\Omega(z,\xi) \geq g_\Omega(z,\xi)$ on Ω. Letting $\epsilon \searrow 0$, we find that $W_\Omega = g_\Omega$.

LEMMA 2. *Let $\Omega_1 \subset \mathbb{C}^n, \Omega_2 \subset \mathbb{C}^n$ be two open and connected sets. Then*

$$\max(W_{\Omega_1}, W_{\Omega_2}) \leq W_{\Omega_1 \times \Omega_2}.$$

Proof.

$$\begin{aligned} 0 &\geq \max[W_{\Omega_1}(z_1,\omega_1), W_{\Omega_2}(z_2,\omega_2)] \\ &\leq \max(\log|z_1-\omega_1| - \log\max[d(z_1, \complement\Omega_1), d(\omega_1, \complement\Omega_1)], \\ &\qquad \log|z_2-\omega_2| - \log\max[d(z_2, \complement\Omega_2), d(\omega_2, \complement\Omega_2)]) \\ &\leq \log|(z_1,z_2)-(\omega_1,\omega_2)| - \log\min[d(z_1,\complement\Omega_1), d(\omega_1,\complement\Omega_1), d(z_2,\complement\Omega_1), d(\omega_2,\complement\Omega_2)] \end{aligned}$$

so the inequality now follows from the definition of $W_{\Omega_1\times\Omega_2}$ via [3, Cor. VII:1]. ∎

EXAMPLE. Denote by C_Ω the Carathéodory pseudodistance on Ω. We give an example of a bounded pseudoconvex set Ω, such that

$$\log\tanh C_\Omega \neq W_\Omega \neq g_\Omega.$$

Let $\Omega_1 = \{z \in \mathbb{C}; \frac{1}{2} < |z| < 1\}$ and let Ω_2 be any strictly pseudoconvex domain where $W_{\Omega_2}(z_2^0, \omega_2^0) < g_{\Omega_2}(z_2^0,\omega_2^0)$ for a point $(z_2^0,\omega_2^0) \in \Omega_2 \times \Omega_2$ (by [2], such a set exists). Note first that $W_{\Omega_1} = g_{\Omega_1}$ and that

$$W_{\Omega_1}(z_1,\omega_1) > \log\tanh C_\Omega(z_1, w_1), \ \forall z_1 \neq \omega_1 \in \Omega_1$$

(cf. Klimek [7], p. 234–235]).

Then $\log\tanh C_{\Omega_1\times\Omega_2}((z_1,z_2),(\omega_1,z_2)) < W_{\Omega_1}(z_1,\omega_1) \leq \max[W_{\Omega_1}(z_1,w_1), W_{\Omega_2}(z_2,z_2)] \leq W_{\Omega_1\times\Omega_2}((z_1,z_2),(\omega_1,z_2))$ by Lemma 2. Thus $\log\tanh C_{\Omega_1\times\Omega_2} \neq W_{\Omega_1\times\Omega_2}$; it remains to prove that $W_{\Omega_1\times\Omega_2} \neq g_{\Omega_1\times\Omega_2}$. Suppose $W_{\Omega_1\times\Omega_2} \equiv g_{\Omega_1\times\Omega_2}$. Since Ω_1 and Ω_2 are pseudoconvex, it follows from Theorem 9. 6 in [6] that $g_{\Omega_1\times\Omega_2} = \max[g_{\Omega_1}, g_{\Omega_2}]$ so $W_{\Omega_1\times\Omega_2}((z_1,z_2),(z_1,\omega_2)) = g_{\Omega_2}(z_2,\omega_2)$ is plurisubharmonic in ω_2 which is a contradiction to the assumption

$$W_{\Omega_2}(z_2^0,\omega_2^0) < g_{\Omega_2}(z_2^0,\omega_2^0)$$

by Proposition VII:2 in [3].

3. Some estimates. If Ω is a domain in $\mathbb{R}^n$, regular for the classical Dirichlet problem, then for every function φ, subharmonic near $\overline{\Omega}$ we have the Riesz representation formula:

$$\varphi(\omega) = \int_{\Omega} G(\xi,\omega)\Delta\varphi(\xi) + \int_{\partial\Omega} \varphi(\xi) d\sigma_\omega(\xi), \quad \omega \in \Omega$$

where G is the Green function for Ω and $d\sigma_\omega$ is the harmonic measure relatively Ω and ω.

Stokes theorem gives a similar formula for plurisubharmonic functions (cf. Demailly [4], [5] and Kołodziej [10]). Suppose V, φ and $\psi \in PSH(\Omega) \cap L^\infty(\Omega)$ and define

$$s(r) = \{z \in \Omega; \varphi(z) = r\}; \quad B(r) = \{z; \varphi(z) < r\}.$$

We assume that $B(r) \subset\subset \Omega \quad \forall r < 0$. Consider

$$\begin{aligned}
&\int_{S(r)} V d^c\varphi \wedge (dd^c\psi)^{k-1} = \text{(Stokes)} \\
&= \int_{B(r)} dV \wedge d^c\varphi \wedge (dd^c\psi)^{n-1} + \int V dd^c\varphi \wedge (dd^c\psi)^{n-1} \\
&= \int_{B(r)} d(\varphi - r) \wedge d^cV \wedge (dd^c\psi)^{n-1} + \int_{B(r)} V(dd^c\varphi) \wedge (dd^c\psi^{n-1} = \text{(Stokes)} \\
&= -\int_{B(r)} (\varphi - r) dd^cV \wedge (dd^c\psi)^{n-1} + \int_{B(r)} V dd^c\varphi \wedge (dd^c\psi)^{k-1}.
\end{aligned}$$

Hence

$$\begin{aligned}
(1) \qquad \int_{B(r)} V(dd^c\varphi) \wedge (dd^c\psi)^{n-1} = &\int_{B(r)} (\varphi - r) dd^cV \wedge (dd^c\psi)^{n-1} \\
&+ \int_{S(r)} V d^c\varphi \wedge (dd^c\psi)^{n-1}.
\end{aligned}$$

We now claim that $d^c\varphi \wedge (dd^c\psi)^{n-1}$ is a positive measure on $S(r)$.

For let $0 \le h \in C^\infty$ be given. Let $\epsilon > 0$ and define $\varphi_\epsilon = \max\{\varphi, r - \epsilon\}$. Then

$$\begin{aligned}
\int_{S(r)} h d^c\varphi \wedge (dd^c\psi)^{n-1} &= \int_{S(r)} h(d^c\varphi_\epsilon) \wedge (dd^c\psi)^{n-1} \\
&= \int_{S(r)} d\varphi_\epsilon \wedge d^ch \wedge (dd^c\psi)^{n-1} + \int_{B(r)} h(dd^c\varphi_\epsilon) \wedge (dd^c\psi)^{n-1} \\
&= \int_{r-\epsilon \le \varphi < r} d\varphi_\epsilon \wedge d^ch \wedge (dd^c\psi)^{n-1} + \int_{r-\epsilon \le \varphi < r} h dd^c\varphi_\epsilon \wedge (dd^c\psi)^{n-1}
\end{aligned}$$

$$= \int_{r-\epsilon\le\varphi<r} d(\varphi_\epsilon-(r-\epsilon))\wedge d^ch\wedge(dd^c\psi)^{n-1} + \int_{r-\epsilon\le\varphi<r} hdd^c\varphi_\epsilon\wedge(dd^c\psi)^{n-1}$$

$$= \int_{S(r)} (\varphi_\epsilon-(r-\epsilon))d^ch\wedge(dd^c\psi)^{n-1} - \int_{r-\epsilon\le\varphi<r} (\varphi_\epsilon-(r-\epsilon))dd^ch\wedge(dd^c\psi)^{n-1}$$

$$+ \int_{r-\epsilon\le\varphi<r} hdd^c\varphi_\epsilon\wedge(dd^c\psi)^{n-1}.$$

Here, the last term is nonnegative so

$$\int_{S(r)} hd^c\varphi\wedge(dd^c\psi)^{n-1}$$

$$\ge -\epsilon\Big|\int_{B(r)} dd^ch\wedge(dd^c\psi)^{n-1}\Big| - \epsilon\int |dd^ch\wedge(dd^c\psi)^{n-1}| \to 0, \quad \epsilon\to 0$$

which proves the claim.

Example. Let $0\le V\in PSH(\Omega), \psi\in PSH\cap L^\infty(\Omega)$ and

$$\varphi_t = \max[W(z,\xi),-t].$$

Then (1) gives

$$\int_{B(r)} V(dd^c\varphi_t)\wedge(dd^c\psi)^{n-1} = \int_{B(r)} (\varphi_t-r)dd^cV\wedge(dd^c\psi)^{n-1}$$

$$+ \int_{s(r)} Vd^c\varphi_t\wedge(dd^c\psi)^{n-1}.$$

Letting $r\to 0$ we get

$$\int_\Omega -\varphi_t dd^cV\wedge(dd^c\psi)^{n-1} \le \int_\Omega Vdd^c\varphi_t\wedge(dd^c\psi)^{n-1}$$

$$\le \sup_\Omega V \int_\Omega dd^c\varphi_t\wedge(dd^c\psi)^{n-1}$$

so if we choose $\psi=\sum_{j=1}^n |z_i|^2$ then

(i) $$\int_\Omega -W(z,\xi)\Delta V \le \|V\|_{L^\infty}\int_\Omega \Delta_z W(z,\xi),$$

if we choose $\psi=V$, then

(ii) $$\int_\Omega -W(z,\xi)(dd^cV(z))^n \le \|V\|_{L\infty}\int_\Omega dd^c_zW(z,\xi)(dd^cV)^{n-1},$$

and finally if $\psi=\varphi_t$,

$$\int -\varphi_t dd^cV\wedge(dd^c\varphi_t)^{n-1} \le \int V(dd^c\varphi_t)^n$$

and so

(iii) $$\int -W(z,\xi)dd^cV \wedge (dd^cW)^{n-1} \le \int V(dd^c_\xi W(z,\xi))^n.$$

4. Integrability of plurisubharmonic functions. Suppose μ is a positive measure on Ω. How do we know there is a $\varphi \in PSH \cap L^\infty_{\mathrm{loc}}(\Omega)$ with $(dd^c\varphi)^n = \mu$? Here is a necessary condition.

PROPOSITION 1. *Let $R > 1$ fixed, B the unit ball. Then there exists a constant c such that*

$$\int_{\bar{B}} -\varphi(dd^cu)^n \le c\int_B -\varphi dV$$

for all $0 \ge \varphi \in PSH(RB)$ and $-1 \le u \le 0$, $u \in PSH(RB)$.

Proof. See [3, Prop. VI:2]. ■

Let now Ω be hyperconvex with exhaustion function ψ. Let μ be a positive measure and assume $0 \ge V \in PSH \cap L^\infty(\Omega), (dd^cV)^n = \mu$. For $m > 0$, define $V_m = \max(V, m\psi)$. Then, by (1),

$$\int V_m(dd^c\psi)^n = \int \psi dd^cV_m \wedge (dd^c\psi)^{n-1} \le \frac{1}{m}\int V_m dd^cV_m \wedge (dd^c\psi)^{n-1} \le \ldots \le \frac{1}{m^{n-1}}\int \psi(dd^cV_m)^n.$$

so

$$0 \le \int -\psi(dd^cV_m)^n \le m^{n-1}\int -V_m(dd^c\psi)^n \le m^{n-1}(\sup_{z\in\Omega} -V(z))\int (dd^c\psi)^n.$$

If $\tau(z) \equiv (2\pi)^n$, we take $\psi(\xi) = W(z,\xi)$ and get

$$(2\pi)^n m^{n-1}V_m(z) \le \int W(z,\xi)(dd^cV_m\xi)^n \le 0.$$

If $\operatorname{supp}\mu$ is compact, then $V_m = V$ near the support of μ for m large enough and therefore

$$0 \le \int -W(z,\xi)d\mu(\xi) \le m^{n-1}\int -V(\xi)(dd^cW(z,\xi))^n \le m^{n-1}(\sup_{\xi\in\Omega} -W)\tau(z)).$$

We are thus led to consider the pluricomplex potential $\Omega \ni z \mapsto \int W(z,\xi)d\mu(\xi)$ for positive measures μ. We have just proved

THEOREM 2. *Suppose $-1 \le u \le 0, u \in PSH(\Omega)$ and that Ω is strongly hyperconvex. Then*

$$0 \le -\int W_\Omega(z,\xi)(dd^c\max[u(\xi), mW_\Omega(\eta,\xi)])^n$$
$$\le -m^{n-1}\int \max[u(\xi), mW_\Omega(\eta,\xi)](dd^c_\xi W_\Omega(z,\xi))^n \le m^{n-1}\tau(z), \quad z \in \Omega,\ \eta \in \Omega.$$

5. A metric defined by W. It is known that g_Ω gives rise to an infinitesimal Finsler pseudometric, cf. [1], [8] and [9]. We show here that W_Ω also defines an infinitesimal Finsler pseudometric.

DEFINITION. Let $w \in \Omega, \xi \in \mathbb{C}^n$. We define

$$T(\omega,\xi) = \overline{\lim_{\substack{|l|\to 0\\ l\in\mathbb{C}}}}\, W_\Omega(w + l\xi, \omega) - \log|l|.$$

PROPOSITION 2. *$T(\omega,\xi)$ is upper semicontinuous on $\Omega\times\mathbb{C}^n$.*

Proof. Note that $(\omega,\xi,l)\mapsto W_\Omega(\omega+l\xi,\omega)-\log|l|$, $l\neq 0$ is upper semicontinuous and subharmonic in l, for $\omega,\omega+l\xi\in\Omega$. Also, for ω, ξ fixed $W_\Omega(\omega+l\xi,\omega)\le c+\log|l|$. Therefore $W_\Omega(\omega+l\xi,\omega)-\log|l|$ has a uniquely determined subharmonic extension over $l=0$. Also

$$T(\omega,\xi) = \overline{\lim_{|l|\to 0}}\, W_\Omega(\omega+l\xi,\omega) - \log|l| = \overline{\lim_{r\searrow 0}}\, W_\Omega(\omega+r\xi,\omega) - \log r.$$

By the mean value property for subharmonic functions,

$$\frac{1}{2\pi}\int_0^{2\pi} [W_\Omega(\omega+re^{i\theta}\xi,\omega) - \log r]d\theta \searrow T(\omega,\xi), \qquad r\searrow 0.$$

Note that for $r>0$ fixed the left hand side is upper semicontinuous in (ω,ξ) and since it decreases in r, the proposition follows.

Furthermore, $\exp T(\omega,t\xi) = |t|\exp T(\omega,\xi)$, $t\in\mathbb{C}$ so $\exp T(\omega,\xi)$ defines an infinitesimal Finsler pseudometric.

References

[1] K. Azukawa, *The invariant pseudometric related to negative plurisubharmonic function*, Kodai Math. J. 10 (1987), 83–92.

[2] E. Bedford and J. P. Demailly, *Two counterexamples concerning the pluri-complex Green function in $\mathbb{C}^n$*, Indiana Univ. Math. J. 37 (1988), 865–867.

[3] U. Cegrell, *Capacities in Complex Analysis. Aspects of Mathematics*, 14, Vieweg, 1988.

[4] J. P. Demailly, *Mesures de Monge-Ampère et mesures pluri-sousharmoniques*, Math. Z. 194 (1987), 519–564.

[5] —, *Mesures de Monge-Ampère et caractérisation géométrique des variétés algébriques affines*, Mém. Soc. Math. France 19 (1985), 1–125.

[6] M. Jarnicki and P. Pflug, *Invariant Distances and Metrics in Complex Analysis*, Walter de Gruyter & Co., 1993.

[7] M. Klimek, *Extremal plurisubharmonic functions and invariant pseudodistances*, Bull. Soc. Math. France 113 (1985), 231–240.

[8] —, *Infinitesimal pseudometrics and the Schwarz lemma*, Proc. Amer. Math. Soc. 105 (1989), 134–140.

[9] —, *Pluripotential Theory*, Oxford Science Publications, 1991.

[10] S. Kołodziej, *The logarithmic capacity in $\mathbb{C}^n$*, Ann. Polon. Math. 48 (1988), 253–267.