tional $T \in \mathcal{O}'(\tilde{S}^{n-1})$ such that (6.9) and that $F(\zeta) = \mathscr{P}_\lambda T(\zeta)$, which proves the surjectivity of (7.14). The rest of the proof is same as above. ∎

Remark. The linear topological isomorphism (7.13) and (7.14) are very special cases of the Ehrenpreis–Palamodov fundamental principle.

References

[1] L. Drużkowski, *Effective formula for the cross norm in the complexified unitary spaces*, Zeszyty Nauk. Uniw. Jagiell., Prace Mat. 16 (1974), 47–53.

[2] M. Hashizume, A. Kowata, K. Minemura and K. Okamoto, *An integral representation of the Laplacian on the Euclidean space*, Hiroshima Math. J. 2 (1972), 535–545.

[3] M. Hashizume, K. Minemura and K. Okamoto, *Harmonic functions on Hermitian hyperbolic spaces*, ibid. 3 (1973), 81–108.

[4] S. Helgason, *Eigenspaces of the Laplacian; Integral representations and irreducibility*, J. Functional Analysis 17 (1974), 328–353.

[5] L.K. Hua, *Harmonic Analysis of Functions of Several Complex Variables in Classical Domains* (Russian), Moscow 1959. Translations of Math. Monographs, 6, AMS, 1963.

[6] C.O. Kiselman, *Prolongement des solutions d'une équation aux dérivées partielles à coefficients constants*, Bull. Soc. Math. France 97 (1969), 329–356.

[7] J.L. Lions and E. Magenes, *Problèmes aux limites non-homogènes VII*, Ann. Math. Pura Appl. 63 (1963), 201–224.

[8] A. Martineau, *Equations différentielles d'ordre infini*, Bull. Soc. Math. France 95 (1967), 109–154.

[9] M. Morimoto, *A generalization of the Fourier–Borel transformation for the analytic functions with non convex carrier*, Tokyo J. Math. 2 (1979), 301–322.

[10] C. Müller, *Spherical Harmonics*, Lecture Notes in Math. 17, Springer, 1966.

[11] R.T. Seeley, *Spherical harmonics*, AMS Monthly 73, part II, 4 (1966), 115–121.

[12] —, *Eigenfunction expansions of analytic functions*, Proc. Amer. Math. Soc. 21 (1969), 734–738.

[13] J. Siciak, *Holomorphic continuation of harmonic functions*, Ann. Pol. Math. 29 (1974), 67–73.

[14] E.M. Stein and G. Weiss, *Introduction to Fourier Analysis on Euclidean Spaces*, Princeton, New Jersey, 1971.

[15] N.Ja. Vilenkin, *Special Functions and the Theory of Group Representations*, Translation of Math. Monographs, 22, AMS, 1968.

Presented to the Semester
COMPLEX ANALYSIS
February 15–May 30, 1979

COMPLEX ANALYSIS
BANACH CENTER PUBLICATIONS, VOLUME 11
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1983

COMPLEX METHODS IN NON-LINEAR ANALYSIS

JOSEF NAAS

Institut für Mathem. d. Akad. d. Wiss. d. DDR
DDR-1086 Berlin, Mohrenstr. 39, DDR

WOLFGANG TUTSCHKE

Sektion Mathematik der Universität, Halle, DDR-401, Universitätsplatz 8/9, DDR

One of the most important questions of analysis is the investigation of functional dependences using the concept of the limit. With it, on the one hand, many conclusions are valid under rather weak assumptions (they hold, for instance, for mappings of topological spaces). This fact may give an impression that the possibilities of the complex function theory (which starts from the consideration of complex-valued functions of complex variables) are contained in an abstract mapping theory of topological (or some more general) spaces.

On the other hand it is important to take into account that more specific assumptions permit a richer theory. The functions regarded within the complex function theory lead to the concept of holomorphy. Holomorphic functions have various specific properties. Their local behaviour determines, for instance, their global behaviour. Such properties of holomorphic functions cause that the complex function theory is an autonomous theory describing the general concept of holomorphy.

From this, however, it is not yet possible to conclude that a boundless development of the concept of holomorphy gives the unique end of a general "complex analysis". In our opinion from this the possibility of too affected generalization started indeed (as again in the case of other mathematical theories). For some generalizations of the concept of holomorphy, for instance, the applicability seems to be not satisfactory at all.

There are, for sure, many immediate applications of complex analysis (for instance those connected with the approximation theory of one or several complex variables). Fundamental applications of complex analysis, however, are connected with the theory of partial differential equations. This is true not only in the case of Cauchy–Riemann systems and the Laplace equation (holomorphic functions are, as it is well-known, connected immediately with these partial differential equations). For solving more general real partial differential equations it is needed to expand the methodical resources of the complex function theory (this is needed, already, for solving linear uniformly elliptic systems with help of complex methods; the last observation leads to the generalized analytic functions in the sense of I. V. Vekua and to the pseudo-analytic functions in the sense of L. Bers).

The necessary development of methods of complex analysis is realized with help of the T_G- and the Π_G-operators defined by

$$T_G f(z) = -\frac{1}{\pi}\iint\limits_G \frac{f(\zeta)}{\zeta - z}\, d\xi d\eta,$$

$$\Pi_G f(z) = -\frac{1}{\pi}\iint\limits_G \frac{f(\zeta)}{(\zeta - z)^2}\, d\xi d\eta,$$

where $z = x+iy$, $z^* = x-iy$, $\zeta = \xi+i\eta$. These operators are connected with the inhomogeneous Cauchy–Riemann equation

$$\frac{\partial w}{\partial z^*} = f,$$

for which the function

$$w_0 = T_G f$$

is a special solution (this solution satisfies the additional relation $\partial w_0/\partial z = \Pi_G f$). The inhomogeneous Cauchy–Riemann equation is important not only for solving general partial differential equations with help of complex methods, but it leads to new points of view on the theory of holomorphic functions itself (cf. L. Hörmander [7]).

Further, applicability of complex-analytical methods is not only restricted to linear partial differential equations. With help of complex methods it is also possible to solve non-linear partial differential equations (cf. [12]). This is true also for such equations, for which hitherto existing methods of real analysis are not applicable. In addition, complex analysis proves to be a general method of non-linear analysis. With its help it is possible, for example, to obtain explicite representations for implicit functions (applications of the complex analysis to the theory of implicit functions).

The present paper collects possibilities of complex analysis for solving different problems of non-linear analysis. Especially we are going to describe results of the research group "Partial complex differential equations" (Mathematical Institute, University of Halle).

B. Bojarski and T. Iwaniec regarded geometrical aspects [3] of the problem (strictly speaking, they considered quasiconformal mappings satisfying non-linear elliptic systems of partial differential equations of first order). For numerical methods (including those for non-linear partial differential equations) based on complex methods we refer to W. Wendland [24].

1. Investigation of qualitative properties of partial differential equations and partial differential inequalities

It is well-known that holomorphic functions possess certain important properties. The new methods of complex analysis permit to extend many of those properties to the case of more general complex-valued functions. It is sufficient for instance, that the function be a solution of a linear or non-linear complex partial differential equation which generalizes Cauchy–Riemann's equations. It is also sufficient that the function in question be a solution of a complex differential inequality.

The reduction of properties of generalized analytic functions to properties of holomorphic functions is based on representation theorems for generalized analytic functions. There are two principal possibilities of representing complex-valued functions by means of holomorphic functions, namely additive and multiplicative representations (using composite functions one can combine the two ways).

Denote by k the derivative with respect to z^* of a given function w. Then

$$\Phi = w - T_G k$$

is holomorphic, and so w admits the additive representation

$$w = \Phi + T_G k.$$

In order to get an analogous multiplicative representation we define another auxiliary function, namely (see I. N. Vekua [22])

$$g = \begin{cases} k/w, & \text{if } w \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Assume that exists a Lebesgue integrable function $K = K(z)$ such that w fulfils the inequality

$$(1) \qquad |k| \leqslant K(z)|w|.$$

Then the function g proves integrable (w and k are assumed to be continuous). Define $\omega = T_G g$ and $\Phi = we^{-\omega}$. Then the function Φ is holomorphic at all points where $w \neq 0$. If we assume additionally that

$$(2) \qquad w = 0 \quad \text{implies} \quad k = 0,$$

then the function Φ is holomorphic everywhere and w has the representation

$$w = \Phi e^{\omega}.$$

Both assumptions (1) and (2) are fulfilled, if (1) holds with a continuous $K = K(z)$.

Analogous representation theorems are valid also in the case of several complex variables. Denote by $k_{i_1 \ldots i_\lambda}$ the partial complex derivative

$$\frac{\partial^\lambda w}{z^*_{i_1} \ldots z^*_{i_\lambda}}$$

of order $\lambda \leqslant n$; it is sufficient to consider derivatives with respect to variables z^*_i

different from each other. For polycylindric domains $G = G_1 \times \ldots \times G_n$ in C^n we define

$$\Phi = w - \sum_{\lambda} (-1)^{\lambda+1} \sum_{i_1,\ldots,i_\lambda}^{*} T_{G_{i_1}} \ldots T_{G_{i_\lambda}} k_{i_1\ldots i_\lambda}, \tag{3}$$

where $\sum^*$ denotes summation over multiindices $i_1, \ldots, i_\lambda$ with distinct i's. Since Φ is holomorphic, equation (3) yields an additive representation of w by holomorphic functions.

In order to get a multiplicative representation one defines the following auxiliary functions (see [19])

$$g_{i_1\ldots i_\lambda} = \begin{cases} \dfrac{\partial}{\partial z^*_{i_\lambda}} \cdots \dfrac{\partial}{\partial z^*_{i_2}} \left(\dfrac{k_{i_1}}{w}\right), & \text{if} \quad w \neq 0, \\ 0 \quad \text{otherwise.} \end{cases}$$

Analogously to the case of one complex variable we assume the existence of integrable functions $K_{i_1\ldots i_\lambda} = K_{i_1\ldots i_\lambda}(z_1, \ldots, z_n)$, such that

$$|k_{i_1\ldots i_\lambda}| \leqslant K_{i_1\ldots i_\lambda}(z_1, \ldots, z_n)|w|. \tag{4}$$

Then the function Φ which we now define is holomorphic at all points where $w \neq 0$:

$$\Phi = we^{-\omega}, \quad \omega = \sum_{\lambda} (-1)^{\lambda+1} \sum_{i_1,\ldots,i_\lambda}^{*} T_{G_{i_1}} \ldots T_{G_{i_\lambda}} g_{i_1\ldots i_\lambda}.$$

In analogy to (2) we assume, moreover, that

$$w = 0 \quad \text{implies} \quad k_1 = 0, \ldots, k_n = 0. \tag{5}$$

Then Φ is holomorphic everywhere and w is representable as the product

$$w = \Phi e^{\omega}.$$

Both assumptions (4) and (5) are fulfilled, if (4) holds with continuous multipliers $K_{i_1\ldots i_\lambda}$.

The additive and multiplicative representation just proved imply a priori estimates, theorems of Phragmén–Lindelöf's type (see H. Malonek [11]) and, finally, the fact that the zeros of a function w fulfilling (4), (5) form an analytic set (this means, in the case of (1), (2), that the zeros are isolated or the function w vanishes identically).

2. Complex methods in the theory of implicit functions

The simplest implicit equation that has been solved by complex methods is the algebraic equation

$$\sum_{\nu=0}^{n} a_\nu z^\nu = 0,$$

where a_ν are constants. As a result one gets the fundamental theorem of algebra. A generalized fundamental theorem of algebra concerns the equation

$$\sum_{\nu,\mu=0}^{n} a_{\nu\mu} z^\nu z^{*\mu} = 0 \tag{6}$$

and was proved by M. B. Balk [1]. Such an equation possesses, in general, both isolated and non-isolated solutions. In order to determine the isolated solutions Heinz [6] developped the following method:

Denote the left-hand side of equation (6) by $G(z, z^*)$. Without restriction of generality it may be assumed, that the degree of G with respect to z is less or equal the degree of G with respect to z^*. Now consider the Euclidean algorithm for G and G^* with respect to z^*. As a result of the algorithm one gets a polynomial $h(z)$ with the property that every (isolated) solution of the equation $G(z, z^*) = 0$ is a solution of the equation $h(z) = 0$. In this way one reduces the problem of finding the (isolated) zeros of the non-holomorphic function $G(z, z^*)$ to that of finding the zeros of the holomorphic function $h = h(z)$.

Another type of implicit equations solved by complex method is the case of algebraic functions. Such a function is defined as a solution $w = w(z)$ of $G(z, w) = 0$, where G is a polynomial with respect to z and w.

Combining Heinz' method described above and the theory of algebraic functions, P. Czerner [4] solved the implicit equation $G(z, z^*, w, w^*) = 0$ by complex methods (the left-hand side is a polynomial with respect to z, z^*, w and w^*). His result is the following:

There exist a finite number of critical curves decomposing the plane into a finite number of domains G_j. In each G_j there exist a finite number of functions algebraic with respect to z, z^*, which are denoted by $w_\mu = w_\mu(z, z^*)$, $\mu = 1, \ldots, m_j$ (these functions are solutions of an equation of type $\tilde{G}(z, z^*, w) = 0$, where $\tilde{G}$ is a polynomial with respect to z, z^*, w; the theory of such equations is very similar to the ordinary theory of algebraic functions). Some of the branches $w_\mu(z, z^*)$ are solutions of the regarded equation

$$G(z, z^*, w, w^*) = 0.$$

Since the $w_\mu(z, z^*)$ are representable locally by power series with respect to z and z^*, one can calculate w in each of the domains G_j by extension of the power series representation.

Instead of polynomials $G(z, z^*)$ and $G(z, z^*, w, w^*)$ one can consider power series (also with respect to several variables z_ν, z^*_ν). As regards the zeros of real power series, G. Schopf [14] proved, for instance, the following fact:

Let $G(z, z^*)$ be a power series with respect to both z and z^*. Then the zeros are isolated points or isolated analytic curves.

3. Theorems on the extension of solutions

Suppose that a solution of a differential equation in a domain G_1 is given. In order to extend this solution to a larger domain G_2, there are two different possibilities. The first one is the reflection principle (for linear partial differential equations of second order see e.g. D. Colton [5]). The second one is based on representations by power series. This method is applicable also to real power series, i.e. for power series with respect to z and z^*. In order to apply this method it is needed to assume that the right-hand side of the differential equation under consideration admits itself a representation by power series. On the other hand, extension of solutions with the help of power series can be used in the case of nonlinear differential equations. Already in the classical complex function theory this method is applied in the case of nonlinear differential equations. One of the most important theorems based on this method is Kaplan's theorem [8]:

Let $w = w(z)$ be a local solution of a complex differential equation (in the ordinary sense)

$$\frac{dw}{dz} = f(z, w).$$

Assume that the right-hand side $f(z, w)$ is an entire function; then the following assertion holds: With the exception of a set of capacity zero it is possible to extend the given solution to the whole complex plane.

The proof is based on Painleve's and Meier's theorems.

H. Mächler in his thesis [10] proved the following generalization to partial differential equations:

Let w be a given solution of a system

$$\frac{\partial w}{\partial z} = f_1(z, w),$$

$$\frac{\partial w}{\partial z^*} = f_2(z, w)$$

in a domain G_1. Assume that the right-hand sides f_1, f_2 are defined in a larger domain G_2 containing G_1 in its interior; then the given solution w can be extented to almost every boundary point of G_1.

In order to prove his theorem H. Mächler generalizes Painleve's theorem to the case of real power series. Meier's theorem is also needed in the proof of Mächler's theorem. An application of Meier's theorem is possible after a suitable change of coordinates, defined with the help of Beltrami's equation; its application requires, of course, the assumption that the system under consideration is uniformly elliptic. The last assumption is equivalent to the inequality $|f_2| \leqslant q_0 \cdot |f_1|$, $0 < q_0 < 1$. In the new coordinate system the given solution is holomorphic. And thus the basic idea of Mächler's proof is a reduction of the general case to the case of holomorphic functions.

Mächler's theorem formulated above and similar results (see [10]) show the way in which complex methods allow one to prove the existence of extensions of solutions of nonlinear partial differential equations.

We remark that in the case of several complex variables there is also another kind of extension theorems. For holomorphic functions of several complex variables the following well-known theorem, for instance, is valid. If G is a given domain in C^n and $G \setminus K$ is connected (where K is a compact subset of G), then every holomorphic in $G \setminus K$ function can be extended to a holomorphic function in K. Analogous theorems on solutions of Vekua's equation in C^n,

$$\frac{\partial w}{\partial z_j^*} = A_j w + B_j w^*$$

were proved by Le hung Son in his papers [16], [17].

4. Existence theorems for nonlinear differential equations

In order to construct a solution of a non-linear elliptic first order system

$$H_j\left(x, y, u_1, \ldots, u_{2n}, \frac{\partial u_1}{\partial x}, \ldots, \frac{\partial u_{2n}}{\partial y}\right) = 0, \quad j = 1, \ldots, 2n, \tag{7}$$

in a given domain G by means of complex methods it is needed to write (7) in complex normal form

$$\frac{\partial w}{\partial z^*} = F\left(z, w, \frac{\partial w}{\partial z}\right), \tag{8}$$

where $w = (w_1, \ldots, w_n)$, $w_j = u_j + iu_{n+j}$. (W. Rüprich [13] deals with the implicit partial complex differential equation

$$F\left(z, w, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial z^*}\right) = 0.)$$

Denote $\partial w/\partial z$ by h. Then the equation (8) is equivalent to the system

$$w = \Phi + T_G F(\cdot, w, h),$$
$$h = \Phi' + \Pi_G F(\cdot, w, h),$$

where Φ is a holomorphic function (see [12], [20]). In order to solve a boundary problem or in order to construct a solution of (8) fulfilling a side condition we solve an analogous problem for Φ and define an operator $(w, h) \mapsto (W, H)$ by

$$W = \Phi + T_G F(\cdot, w, h), \quad H = \Phi' + \Pi_G F(\cdot, w, h) \tag{9}$$

(we remark that the function Φ depends on (w, h)). If (w, h) is a fixed point of the operator defined by (9) then w is solution of (9) which fulfils the imposed boundary or side condition. In this way it is possible, for instance, to solve Dirichlet's problem (see [20]) or to construct a solution fulfilling a condition of linear conjugation along an interior curve (see [21]). In all quoted papers the boundary values are assumed to be Hölder-continuously differentiable. In order to weaken the assumptions about the boundary values it is needed to regard the Slobodeckij space $W_p^s(\partial G)$ with $s = 1-1/p$. Then the solution w of (8) belongs to the Sobolev space $W_p^1(G)$. This is a result of A. Seif [15].

Complex methods permit one also to solve nonlinear differential equations for functions w depending on time t. Let

$$\frac{\partial w}{\partial z^*} = F\left(z, t, w, \frac{\partial w}{\partial z}, \frac{\partial w}{\partial t}\right) \tag{9}$$

be a differential equation describing an instationary process in the plane. Using Rothe's method we replace this equation by the elliptic equation

$$\frac{\partial w_l}{\partial z^*} = F\left(z, t_l, w_l, \frac{\partial w_l}{\partial z}, \frac{w_l - w_{l-1}}{t_l - t_{l-1}}\right), \tag{10}$$

which is solvable with the help of complex methods. In this way one reduces the differential equation (9) to a finite number of elliptic equations of type (10) (as starting function w_0 one uses the given initial values).

Another generalization of equation (8) is the equation

$$\frac{\partial w}{\partial z^*} = F\left(z, w, \frac{\partial w}{\partial z}, u\right), \tag{11}$$

in which the right-hand side depends on a control u. If one examines a boundary value problem for this equation, then the following problem arises: Find a characterization for u in the case in which the solution w of the boundary value problem minimizes a given linear or non-linear functional depending on the solution w and the control u. In the case of Dirichlet's boundary value problem for (11) a necessary condition for u was given by H.-K. Klink [9].

Finally, we remark that the methods described are applicable also to generalized analytic functions in C^n or R^n.

References

[1] М. Б. Балк, *Основная теорема алгебры для полианалитических полиномов*, Литовский мат. сборник 8 (1968), 401–404.

[2] L. Bers and L. Nirenberg, *On a representation theorem for linear elliptic systems with discontinuous coefficients and its applications*, Convegno intern. sulle equaz. alle deriv. parz., Triest, (1954), 111–140.

[3] B. Bojarski and T. Iwaniec, *Quasiconformal mappings and nonlinear elliptic systems in two variables I, II*, Bull. Acad. Pol. Sci. Ser. sci. math., astr. et phys. 22 (1974), 473–478 and 479–484.

[4] P. Czerner, *Eine zweistufige Reduktionstheorie für algebraische Gleichungen der Form $G(z, \bar{z}, w, \bar{w}) = 0$*, Math. Nachr. 87 (1979), 109–119.

[5] D. Colton, *Partial differential equations in the complex domain*, London, San Francisco, Melbourne 1976.

[6] W. Heinz, *Ein Verfahren zur Bestimmung der isolierten Nullstellen von Polynomen in z und $\bar{z}$*, Math. Nachr. 65 (1975), 219–222.

[7] L. Hörmander, *Introduction to Complex Analysis in Several Variables*, Princeton, Toronto, London 1967.

[8] W. Kaplan, *Analytic ordinary differential equations in the large*, Proc. U. S.–Jap. Seminar on Differential and Functional equations, Minneapolis, 1967.

[9] H.-K. Klink, *Über komplexe Methoden in der Kontrolltheorie: Ein Dirichletproblem*, Math. Nachr. 96 (1980), 53–81.

[10] H. Mächler, *Sätze vom Kaplanschen Typ für partielle Differentialgleichungssysteme erster Ordnung in der Ebene*, Dissertationsschrift, Universität Halle 1979.

[11] Х. Малонек, *К вопросу о справедливости теорем типа Фрагмена–Линделёфа для решений некоторых дифференциальных неравенств*, ИАН Арм. ССР, серия Матем., Ереван 14.3 (1979), 192–200.

[12] J. Naas und W. Tutschke, *Komplexe Analysis und ihr Allgemeinheitsgrad*, Math. Nachr. 75 (1976), 127–131.

[13] W. Rüprich, *Das Rand-Sprungwertproblem für implizite partielle komplexe Differentialgleichungssysteme in der Ebene*, ibid. 101 (1981), 309–319.

[14] G. Schopf, *Das Nullstellengebilde von Potenzreihen in z und $\bar{z}$*, Math. Nachr. 78 (1977), 319–326.

[15] A. Seif, *Konstruktion von Lösungen nichtlinearer elliptischer Differentialgleichungssysteme erster Ordnung in der Ebene durch komplexe Methoden im Sobolev–Raum $W_p^1(G)$ (insbesondere Lösung des Dirichletschen Randwertproblems mit Randwerten aus dem Slobodeckij-Raum $W_p^s(\partial G)$, $s = 1-1/p$)*, Dissertationsschrift, Universität Halle 1979.

[16] Le hung Son, *Fortsetzungssätze vom Hartogsschen Typ für verallgemeinerte analytische Funktionen mehrerer komplexer Variabler*, Math. Nachr. 93 (1979), 177–186.

[17] —, *Verfahren zur Fortsetzung verallgemeinerter analytischer Funktionen mehrerer komplexer Variabler*, Beitrage zur Analysis 13 (1979), 89–109.

[18] W. Tutschke, *Die neuen Methoden der komplexen Analysis und ihre Anwendung auf nichtlineare Differentialgleichungssysteme*, Sitzungsber. d. Akad. d. Wiss. d. DDR, Heft 17N, 1976.

[19] —, *Partielle komplexe Differentialgleichungen*, Berlin 1977.

[20] —, *Vorlesungen über partielle Differentialgleichungen*, Teubner-Text zur Mathematik, Leipzig 1978.

[21] —, *The reduction of the problem of linear conjugation for non-linear elliptic first order systems in the plane to the analogous problem for holomorphic functions*, Proceedings of the Conference on Analytic Functions, Kozubnik 1979, Lecture Notes in Mathematics 798, Springer-Verlag, Berlin–Heidelberg–New York 1980, pp. 446–455.

[22] И. Н. Векуа, *Об одном свойстве решений обобщенной системы уравнений Коши–Римана*, Сообщ. Акад. Наук Груз. ССР 14 (1953), 449–453.

[23] —, *Обобщенные аналитические функции*, Москва 1959.

[24] W. L. Wendland, *Elliptic systems in the plane*, London, San Francisco, Melbourne 1979.

Presented to the Semester
COMPLEX ANALYSIS
February 15–May 30, 1979