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ON SYSTEM OF SUBOBJECT FUNCTORS IN THE CATEGORY OF ORDERED SETS

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Let $\mathcal{S}$ be the category of all non-empty sets with mappings as morphisms, $\mathcal{U}$ the category of all non-empty (partially) ordered sets with isotone maps as morphisms.

Let Exp be the endofunctor $\mathcal{S} \to \mathcal{S}$ with

$$\operatorname{Exp} X = \{Y\colon\ Y \subset X,\ Y \neq \emptyset\}$$

and

$$[\operatorname{Exp} f](Y) = f(Y) = \{f(y)\colon\ y \in Y\}$$

for all sets $X \neq \emptyset$, $Y \subset X$ and all maps f. Defining $\eta_X(x) = \{x\}$ for $x \in X$ and $\mu_X(\mathcal{X}) = \bigcup_{Y \in \mathcal{X}} Y$ for $\mathcal{X} \in \operatorname{Exp}\operatorname{Exp} X$, we get a monad $(\operatorname{Exp}, \eta, \mu)$ (see [3], p. 138). We look now for such functors $T\colon \mathcal{U} \to \mathcal{U}$ for which the diagram

(a)
$$\begin{array}{ccc} \mathcal{U} & \xrightarrow{T} & \mathcal{U} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathcal{S} & \xrightarrow{\operatorname{Exp}} & \mathcal{S} \end{array}$$

(U is the forgetful functor) is commutative (so $T(A, \varrho) = (\operatorname{Exp} A,\ T(\varrho))$, where $T(\varrho)$ is a partial order on $\operatorname{Exp} A$ for each $(A, \varrho) \in \mathcal{U}$).

(b)
$$Y_1 \subset Y_2 \subset A \Rightarrow Y_1 T(\varrho) Y_2$$

as T is a functor, we get for any isotone mapping $f\colon (A, \varrho) \to (B, \sigma)$

$$Y_1\, T(\varrho)\, Y_2 \Rightarrow f(Y_1)\, T(\sigma)\, f(Y_2).$$

We shall call such T a *functor lifting* Exp and extending inclusion. A description of these liftings was considered in [4]. The system of all considered functors T will be denoted by $\boldsymbol{T}$. Having two elements T',

15 — Banach Center Publ. t. 9

$T'' \in \boldsymbol{T}$, we say that

$$T' \leqslant T'' \quad \text{iff} \quad T'(\varrho) \subseteq T''(\varrho)$$

for all (A, ϱ).

Now $\boldsymbol{T}_M$ will be the subsystem of $\boldsymbol{T}$ formed by these functors $T \in \boldsymbol{T}$ for which (T, η, μ) is a monad (denoted simply by T). Having $(A, \varrho) \in \mathscr{U}$, η_A and μ_A are now considered as morphisms in $\mathscr{U}$, i.e. they must be isotone. This is in fact the only condition on $T \in \boldsymbol{T}$ to yield such a monad as needed commutativity of involved diagrams follows obviously from the fact that T lifts Exp. $\boldsymbol{T}_M$ has the smallest element T_1 and the greatest element T_2 (see [5], Theorem 2) defined as follows:

Let $(A, \varrho) \in \mathscr{U}$. $T_1(\varrho)$ is the transitive hull of set-inclusion on $\operatorname{Exp} A$ and the relation $\{(\{a\}, \{b\}) \colon a \varrho b,\ a, b \in A\}$. $X\, T_2(\varrho)\, Y$ iff $X \subset Y$ or $X - Y = \{x\}$, $x = \sup X$ in (A, ϱ) and there exists $y \in Y$, for which $x \varrho y$ (here $X, Y \in \operatorname{Exp} A$). The present paper should contribute to the study of $\boldsymbol{T}$, especially with respect to $\boldsymbol{T}_M$. Next lemma picts up one special situation for $T \in \boldsymbol{T}_M$, which will repeatedly occur in the sequel.

LEMMA 1. *Let* $(T, \eta, \mu) \in \boldsymbol{T}_M$, *and* A *be a set*, $(\operatorname{Exp} A, \sigma) \in \mathscr{U}$, $\mathscr{Y}, \mathscr{X} \in \operatorname{Exp}\operatorname{Exp} A$, $\mathscr{X}\, T(\sigma)\, \mathscr{Y}$ *and* $\mu_A(\mathscr{X}) = \bigcup_{X \in \mathscr{X}} X \operatorname{non} \subset \bigcup_{Y \in \mathscr{Y}} Y = \mu_A(\mathscr{Y})$. *Then* $\mathscr{X} - \mathscr{Y} = \{X_0\}$, *there exists* $Y_0 \in \mathscr{Y}$ *with* $X_0 \sigma Y_0$, $X_0 = \sup \mathscr{X}$ *in* $(\operatorname{Exp} A, \sigma)$, *and* $X_0 \operatorname{non} \subset Y_0$.

Proof. $\mu_A(\mathscr{X}) \operatorname{non} \subset \mu_A(\mathscr{Y})$ implies $\mathscr{X} \operatorname{non} \subset \mathscr{Y}$. The assertion of the lemma follows immediately from the fact that $T \leqslant T_2$ and from the definition of T_2.

In Propositions 1–4 some constructions are described which applied to subsystems of $\boldsymbol{T}$ ($\boldsymbol{T}_M$ resp.) or to an element of $\boldsymbol{T}$ ($\boldsymbol{T}_M$ resp.) yield again an element of $\boldsymbol{T}$ ($\boldsymbol{T}_M$ resp.).

PROPOSITION 1. *Let* $\boldsymbol{T}' \subset \boldsymbol{T}$ ($\boldsymbol{T}' \subset \boldsymbol{T}_M$ *resp.*). *Put* $\varrho_{\boldsymbol{T}'}(A, \varrho) = \bigcap_{T \in \boldsymbol{T}'} T(\varrho)$. *Let* $F(A, \varrho) = (\operatorname{Exp} A, \varrho_{\boldsymbol{T}'}(A, \varrho))$, $F(f) = \operatorname{Exp} f$. *Then* $F \in \boldsymbol{T}$ ($F \in \boldsymbol{T}_M$ *resp.*).

Proof. For example, let us consider the case $\boldsymbol{T}' \subset \boldsymbol{T}_M$. It is clear that $\varrho_{\boldsymbol{T}'}(A, \varrho)$ is an order on $\operatorname{Exp} A$ and that $F \colon \mathscr{U} \to \mathscr{U}$ is a functor. η_A is an isotone mapping from (A, ϱ) to $F(A, \varrho)$ as it is isotone map from (A, ϱ) to $(\operatorname{Exp} A, T(\varrho))$ for all $T \in \boldsymbol{T}'$.

Isotonicity of μ_A. Let $\mathscr{X}, \mathscr{Y} \in \operatorname{Exp}\operatorname{Exp} A$, $\mathscr{X}\, \varrho_{\boldsymbol{T}'}(\operatorname{Exp} A, \varrho_{\boldsymbol{T}'}(A, \varrho))\, \mathscr{Y}$. Then, for $\mathscr{X}\, T(T(\varrho))\, \mathscr{Y}$, $T \in \boldsymbol{T}'$, we have $\bigcup_{X \in \mathscr{X}} X\, T(\varrho) \bigcup_{Y \in \mathscr{Y}} Y$. Hence $(\mu_A(\mathscr{X}) =)\ \bigcup_{X \in \mathscr{X}} X\, \varrho_{\boldsymbol{T}'}(A, \varrho) \bigcup_{Y \in \mathscr{Y}} Y\ (= \mu_A(\mathscr{Y}))$. So μ_A is isotone mapping

from $\big(\operatorname{Exp}\operatorname{Exp} A, \varrho_{\boldsymbol{T}'}(\operatorname{Exp} A, \varrho_{\boldsymbol{T}'}(A, \varrho))\big)$ to $\big(\operatorname{Exp} A, \varrho_{\boldsymbol{T}'}(A, \varrho)\big)$.

PROPOSITION 2. *Let* m *be an infinite cardinal*, $T \in \boldsymbol{T}$ ($T \in \boldsymbol{T}_M$ *resp.*), $(A, \varrho) \in \mathscr{U}$. *Put for* $X, Y \in \operatorname{Exp} A$

$$X\, \varrho_1(A, \varrho)\, Y \equiv X \subset Y \text{ or } X\, T(\varrho)\, Y, \quad \operatorname{card} X \leqslant m,$$

$$X\, \varrho_2(A, \varrho)\, Y \equiv X \subset Y \text{ or } X\, T(\varrho)\, Y, \quad \operatorname{card} X < m.$$

Let $F_i(A, \varrho) = (\operatorname{Exp} A, \varrho_i(A, \varrho))$, $F_i(f) = \operatorname{Exp} f$, $i = 1, 2$. *Then* $F_i \in \boldsymbol{T}$ ($F_i \in \boldsymbol{T}_M$ *resp.*).

Proof. We shall prove Proposition 2 for $T \in \boldsymbol{T}_M$ and F_2.

(a) $\varrho_2(A, \varrho)$ is an order on $\operatorname{Exp} A$. Reflexivity and antisymmetry are clear. Transitivity (put ϱ_2 instead of $\varrho_2(A, \varrho)$): Let $X \varrho_2 Y$, $Y \varrho_2 Z$. So $X\, T(\varrho)\, Y$, $Y\, T(\varrho)\, Z$. If $X \subset Y$, $Y \subset Z$, then $X \subset Z$, so $X \varrho_2 Z$. If $X \varrho_2 Y$, $X \operatorname{non} \subset Y$, then $\operatorname{card} X < m$ and $X \varrho_2 Z$. If $X \subset Y$, $Y \varrho_2 Z$, $Y \operatorname{non} \subset Z$, then $\operatorname{card} Y < m$, so $\operatorname{card} X < m$ and $X \varrho_2 Z$.

(b) It is evident that F_2 is an endofunctor in $\mathscr{U}$.

(c) η_A is an isotone mapping from (A, ϱ) into $F_2(A, \varrho)$. This is clear $a_1 \varrho a_2 \Rightarrow \{a_1\}\, T(\varrho)\, \{a_2\}$, and $\operatorname{card}\{a_1\} < m$.

(d) Isotonicity of μ_A for F_2. Let $\mathscr{X}, \mathscr{Y} \in \operatorname{Exp}\operatorname{Exp} A$, $\mathscr{X}\, \varrho_2\big(\operatorname{Exp} A, \varrho_2(A, \varrho)\big)\, \mathscr{Y}$. The only case, which needs a consideration, is $\mu_A(\mathscr{X}) \operatorname{non} \subset \mu_A(\mathscr{Y})$. As $\mathscr{X}\, T\big(\varrho_2(A, \varrho)\big)\, \mathscr{Y}$, it is $\mathscr{X} - \mathscr{Y} = \{X_0\}$, there exists $Y_0 \in \mathscr{Y}$ such that $X_0\, \varrho_2(A, \varrho)\, Y_0$, $X_0 = \sup \mathscr{X}$ in $\big(\operatorname{Exp} A, \varrho_2(A, \varrho)\big)$, $X_0 \operatorname{non} \subset Y_0$ (see Lemma 1). Then $\operatorname{card} X_0 < m$. As $X_0 = \sup \mathscr{X}$ in $\big(\operatorname{Exp} A, \varrho_2(A, \varrho)\big)$, it is $X \subset X_0$ or $\operatorname{card}(X - X_0) = 1$ for all $X \in \mathscr{X}$. Therefore, for all these X's we have $\operatorname{card} X \leqslant \operatorname{card} X_0$. As $\mathscr{X} \operatorname{non} \subset \mathscr{Y}$ we have $\operatorname{card} \mathscr{X} < m$. Hence $\operatorname{card} \mu_A \mathscr{X} \leqslant \operatorname{card} \mathscr{X} . \operatorname{card} X_0 < m$. It is $\mathscr{X}\, T\big(\varrho_2(A, \varrho)\big)\, \mathscr{Y}$. As T is a functor and identity mapping of $\big(\operatorname{Exp} A, \varrho_2(A, \varrho)\big)$ to $\big(\operatorname{Exp} A, T(\varrho)\big)$ is isotone, we have $\mathscr{X}\, T\big(T(\varrho)\big)\, \mathscr{Y}$. So $\mu_A(\mathscr{X})\, T(\varrho)\, \mu_A(\mathscr{Y})$ (notice that $T \in \boldsymbol{T}_M$ and μ_A is isotone for T). By this the proof of isotonicity of μ_A for F_2 is accomplished.

DEFINITION 1. Let m, T, F_1, F_2 be as in Proposition 2. We put

$$r_1^m(T) = F_1, \quad r_2^m(T) = F_2.$$

DEFINITION 2. Let $(A, \varrho) \subset \mathscr{U}$. Put $t(A, \varrho)$ (briefly $t(A)$) $= \sup\{\operatorname{card} X \colon X \subset A,\ X \text{ is an antichain in } (A, \varrho)\}$ (one-point set is taken as an antichain).

PROPOSITION 3. *Let* m *be an infinite cardinal*, $T \in \boldsymbol{T}$ ($T \in \boldsymbol{T}_M$ *resp.*), $(A, \varrho) \subset \mathscr{U}$. *Put for* $X, Y \in \operatorname{Exp} A$

$$X\, \varrho_3\, Y \equiv X \subset Y \text{ or } X\, T(\varrho)\, Y, \quad t(X) \leqslant m,$$

$$X\, \varrho_4\, Y \equiv X \subset Y \text{ or } X\, T(\varrho)\, Y, \quad t(X) < m.$$

Let $F_i(A,\varrho) = \big(\mathrm{Exp}\,A,\ \varrho_i(A,\varrho)\big)$, $F_i(f) = \mathrm{Exp}\,f$ *for* $i = 3, 4$. *Then* $F_i \in \boldsymbol{T}$ ($F_i \in \boldsymbol{T}_M$ *resp.*).

Proof. Take again the case $T \in \boldsymbol{T}_M$ and prove $F_4 \in \boldsymbol{T}_M$. The proof runs along the same lines as the proof of Proposition 2. We can concentrate ourselves to the proof of the isotonicity of μ_A for F_4.

Let $\mathscr{X}, \mathscr{Y} \in \mathrm{Exp}\,\mathrm{Exp}\,A$, $\mathscr{X}\, F_4\big(\varrho_4(A,\varrho)\big)\, \mathscr{Y}$. Suppose $\mu_A(\mathscr{X})$ non $\subset \mu_A(\mathscr{Y})$. Then $\mathscr{X} - \mathscr{Y} = \{X_0\}$, there exists $Y_0 \in \mathscr{Y}$ such that $X_0\ \varrho_4(A,\varrho)\ Y_0$, $X_0 = \sup \mathscr{X}$ in $\big(\mathrm{Exp}\,A,\ \varrho_4(A,\varrho)\big)$ and X_0 non $\subset Y_0$. Therefore $t(X_0) < m$ ($t(X_0)$ calculated in (A,ϱ)). As $\mathscr{X}$ non $\subset \mathscr{Y}$, it is $t(\mathscr{X}) < m$ in $\big(\mathrm{Exp}\,A,\ \varrho_4(A,\varrho)\big)$. Take $X \in \mathscr{X}$. It is $X \subset X_0$ or $X - X_0 = \{x\}$. Take an antichain $Z \subset \mu_A(\mathscr{X})$ in (A,ϱ) and suppose Z infinite, $\mathrm{card}\,Z > t(\mathscr{X})$, $\mathrm{card}\,Z > t(X_0)$. We can take Z so that $Z \cap X_0 = \emptyset$. For every $x_i \in Z$ we can choose $X_i \in \mathscr{X}$ with $X_i - X_0 = \{x_i\}$, $x_i = \sup X_i$ in (A,ϱ). For $i \neq j$ we have x_i non $\varrho\ y_i$, therefore X_i non $\varrho_4(A,\varrho)\, X_j$. The system of all X_i's forms an antichain in $\big(\mathrm{Exp}\,A,\ \varrho_4(A,\varrho)\big)$ of cardinality of Z so greater than $t(\mathscr{X})$, a contradiction. So $t\big(\mu_A(\mathscr{X})\big) \leqslant t(\mathscr{X}) + t(X_0) < m$. As $\mathscr{X}\, T\big(T(\varrho)\big)\, \mathscr{Y}$, we have $\mu_A(\mathscr{X})\ T(\varrho)\ \mu_A(\mathscr{Y})$, hence $\mu_A(\mathscr{X})\ \varrho_4(A,\varrho)\ \mu_A(\mathscr{Y})$ and μ_A is isotone for F_4.

DEFINITION 3. Let m, T, F_3, F_4 be as in Proposition 3. We put

$$r_3^m(T) = F_3, \qquad r_4^m(T) = F_4.$$

DEFINITION 4. Let $(A,\varrho) \in \mathscr{U}$. Put $s(A,\varrho)$ (briefly $s(A)$) $= \min\{\mathrm{card}\,I\colon$ there exist chains K_i in (A,ϱ), $i \in I$, such that $A = \bigcup_{i\in I} K_i\}$.

PROPOSITION 4. *Let m be an infinite cardinal,* $T \in \boldsymbol{T}$ ($T \in \boldsymbol{T}_M$ *resp.*), $(A,\varrho) \in \mathscr{U}$. *Put for* $X, Y \in \mathrm{Exp}\,A$

$$X\ \varrho_5(A,\varrho)\ Y \equiv X \subset Y \ \text{or}\ X\,T(\varrho)\,Y, \quad s(X) \leqslant m,$$

$$X\ \varrho_6(A,\varrho)\ Y \equiv X \subset Y \ \text{or}\ X\,T(\varrho)\,Y, \quad s(X) < m.$$

Let $F_i(A,\varrho) = \big(\mathrm{Exp}\,A,\ \varrho_i(A,\varrho)\big)$, $F_i(f) = \mathrm{Exp}\,f$ *for* $i = 5, 6$. *Then* $F_i \in \boldsymbol{T}$ ($F_i \in \boldsymbol{T}_M$ *resp.*).

Proof. Take again the case $T \in \boldsymbol{T}_M$ and prove $F_6 \in \boldsymbol{T}_M$. The facts that F_6 is an endofunctor in $\mathscr{U}$ and that η_A is isotone are easy to prove.

Isotonicity of μ_A for F_6. Let $\mathscr{X}, \mathscr{Y} \in \mathrm{Exp}\,\mathrm{Exp}\,A$, $\mathscr{X}\ \varrho_6\big(\mathrm{Exp}\,A,\ \varrho_6(A,\varrho)\big)\ \mathscr{Y}$. Suppose $\mu_A(\mathscr{X})$ non $\subset \mu_A(\mathscr{Y})$. As $\mathscr{X}$ non $\subset \mathscr{Y}$, it is $s(\mathscr{X}) < m$ in $\big(\mathrm{Exp}\,A,\ \varrho_6(A,\varrho)\big)$. There exists $\{\mathscr{X}_i\colon i \in I,\ \mathrm{card}\,I < m,\ \mathscr{X}_i$ a chain in $\big(\mathrm{Exp}\,A,\ \varrho_6(A,\varrho)\big)\}$ such that $\bigcup_{i\in I} \mathscr{X}_i = \mathscr{X}$. The further data are following: $\mathscr{X} - \mathscr{Y} = \{X_0\}$, there exists $Y_0 \in \mathscr{Y}$ such that $X_0\ \varrho_6(A,\varrho)\ Y_0$, $X_0 = \sup \mathscr{X}$ in $\big(\mathrm{Exp}\,A,\ \varrho_6(A,\varrho)\big)$ and X_0 non $\subset Y_0$. Put $f(X) = X - X_0$ for $X \in \mathscr{X}$. So $f(X) = \emptyset$ or $f(X) = \{x\}$ for $x = \sup X$ in (A,ϱ). Let $\mathscr{X}' = \{X \in \mathscr{X}\colon f(X) \neq \emptyset\}$. Let $X_1, X_2 \in \mathscr{X}'$, $f(X_1) = \{x_1\}$, $f(X_2) = \{x_2\}$, $X_1\,\varrho_6(A,\varrho)\,X_2$. Then $X_1 \subset X_2$ (so $x_1\ \varrho\ x_2$) or $X_1 - X_2 = \{x_1\}$ and again $x_1\ \varrho\ x_2$. Put $\mathscr{X}' \cap \mathscr{X}_i = \mathscr{X}_i'$. Then $\{x\colon \{x\} = f(X),\ X \in \mathscr{X}_i'\}$ is a chain in (A,ϱ). As $\mathrm{card}\,I < m$, $s(X_0) < m$ and $\mu_A(\mathscr{X}) = X_0 \cup \bigcup_{X\in\mathscr{X}'} f(X)$, it is $s\big(\mu_A(\mathscr{X})\big) < m$. $\mathscr{X}\ \varrho_6\big(\mathrm{Exp}\,A,\ \varrho_6(A,\varrho)\big)\ \mathscr{Y}$ implies $\mathscr{X}\,T\big(T(\varrho)\big)\,\mathscr{Y}$, we get $\mu_A(\mathscr{X})\ T(\varrho)\ \mu_A(\mathscr{Y})$. So $\mu_A(\mathscr{X})\ \varrho_6(A,\varrho)\ \mu_A(\mathscr{Y})$ and μ_A is isotone for F_6.

DEFINITION 5. Let m, F_5, F_6 be as in Proposition 4. Then we put

$$r_5^m(T) = F_5, \qquad r_6^m(T) = F_6.$$

r_i^m, $i = 1, 2, \ldots, 6$ defined in Definitions 1, 3, 5 are mappings of $\boldsymbol{T}$ in $\boldsymbol{T}$ carrying $\boldsymbol{T}_M$ into $\boldsymbol{T}_M$. Taking all infinite cardinals m and T_2 ($T_2 = \max \boldsymbol{T}_M$), we get a class of liftings of the type $r_i^m(T_2)$ as there holds

LEMMA 2. *Let* (m, i, n, j) non $\in \{(m, 6, m, 4),\ (m, 4, m, 6)\}$. *Then*

$$r_i^m(T_2) = r_j^n(T_2) \equiv m = n, \quad i = j.$$

Proof. Clearly, $m \neq n$ implies $r_i^m(T_2) \neq r_i^n(T_2)$ for all i. Also for $m \neq n$ we get evidently

$$r_1^m(T_2) \neq r_i^m(T_2) \quad \text{for} \quad i = 2, \ldots, 6,$$

$$r_2^m(T_2) \neq r_i^n(T_2) \quad \text{for} \quad i = 3, \ldots, 6,$$

$$r_3^m(T_2) \neq r_4^n(T_2),$$

$$r_5^m(T_2) \neq r_6^n(T_2).$$

It remains to prove

(a) $r_3^m(T_2) \neq r_5^m(T_2)$,

(b) $r_3^m(T_2) \neq r_6^m(T_2)$,

(c) $r_4^m(T_2) \neq r_5^m(T_2)$.

The proof of assertions (a), (b) consists in constructing a set (A,ϱ) such that $t(A,\varrho) \leqslant m < s(A,\varrho)$, where m is a given infinite cardinal. This construction is a generalization of an example due to D. Kurepa ([2], 3.1). We shall proceed as follows. Take the smallest number m_1 with the property $2^{m_1} > m$. We have $m_1 \leqslant m$. Let B be a set with $\mathrm{card}\,B = m_1$ and let us order the set B by a well-order of the corresponding initial type. This type will be denoted by β. We can put $B = \beta$ and consider $A = 2^\beta$, where 2^β is the system of all maps of β in the set $\{0, 1\}$ ($0 < 1$) ordered lexicographically (this ordering is denoted as $\leqslant_1$). Let $\gamma < \beta$, γ an ordinal. Then $2^{\mathrm{card}\,\gamma} \leqslant m$. As $\gamma \subset \beta$, we can suppose that 2^γ is a subset of 2^β (e.g. the maps from 2^γ are extended to those of 2^β by assigning 0 to the elements of $\beta - \gamma$). The set $D = \bigcup_{\gamma<\beta} 2^\gamma$ is dense in A, i.e. for $a, b \in A$, $a <_1 b$ there exists $c \in D$ such that $a \leqslant_1 c \leqslant_1 b$. As $\mathrm{card}\,2^\gamma \leqslant m$ for $\gamma < \beta$, we get $\mathrm{card}\,D \leqslant m$.

Now, order A by a well-order $\leqslant_2$. Put $\varrho = \leqslant_1 \cap \leqslant_2$. Let $Z \subset (A, \varrho)$ be a chain. This chain must be well-ordered, say $Z = \{z_0, z_1, z_2, \ldots\}$, and this chain is also a chain in $(A, \leqslant_1)$. As D is a dense set in $(A, \leqslant_1)$ and $\operatorname{card} D \leqslant m$, the cardinality of Z is $\leqslant m$. As $\operatorname{card} A > m$, it follows that $s(A, \varrho) > m$.

Let Z be an antichain in (A, ϱ), $Z = \{z_0, z_1, z_2, \ldots\}$, $z_0 <_2 z_1 <_2 z_2 <_2 \ldots$ For $i < j$ we must have $z_j \leqslant_1 z_i$. For the same reason as before, $\operatorname{card} Z \leqslant m$. So $t(A, \varrho) \leqslant m$.

Proof of (a). Let (A, ϱ) be the set just constructed and take a, b non $\in A$. Put $(E, \varrho) = (A, \varrho) \oplus \{a\} \oplus \{b\}$ ($\oplus$ means the ordinal sum). Put $G_1 = A \cup \cup \{a\}$, $G_2 = A \cup \{b\}$. It is $G_1\, r_3^m(T_2)(\varrho)\, G_2$, G_1 non $r_5^m(T_2)(\varrho)\, G_2$. The proof of (b) is similar.

Proof of (c). Let M be an antichain of the cardinality m, a, b non $\in M$. Put $(A, \varrho) = M \oplus \{a\} \oplus \{b\}$. Put $G_1 = M \cup \{a\}$, $G_2 = M \cup \{b\}$. It is $G_1\, r_5^m(T_2)$ $(\varrho)\, G_2$, G_1 non $r_4^m(T_2)(\varrho)\, G_2$.

Remark. Result of Dilworth [1] implies $r_4^{\aleph_0}(T_2) = r_6^{\aleph_0}(T_2)$.

One of the needed information on $\boldsymbol{T}$ or $\boldsymbol{T}_M$ is the answer to the question whether they are classes or hyperclasses. Proposition 5 relates to this question.

PROPOSITION 5. *Let $\mathcal{U}^*$ be the full subcategory of $\mathcal{U}$ consisting of all ordered finite non-empty sets, $\boldsymbol{T}^*$, $\boldsymbol{T}_M^*$ be the system defined for $\mathcal{U}^*$ in the same way as $\boldsymbol{T}$ and $\boldsymbol{T}_M$ for $\mathcal{U}$. Then* $\operatorname{card} \boldsymbol{T}^* = 2^{\aleph_0}$, $\operatorname{card} \boldsymbol{T}_M^* = 2$.

Proof. $\operatorname{card} \boldsymbol{T}_M^* = 2$ can be proved along the same lines as Proposition 3 in [5]. Let us prove that $\operatorname{card} T^* = 2^{\aleph_0}$. Exp is now considered as an endofunctor in $\mathcal{U}^*$. Let

(1) $$m_1, m_2, \ldots, m_k, \ldots,$$

(2) $$s_1, s_2, \ldots, s_k, \ldots$$

be sequences of positive integers with

(3) $$m_1 < s_1 < \binom{s_1}{m_1} < m_2 < s_2 < \binom{s_2}{m_2} < \ldots < \binom{s_k}{m_k} < m_{k+1} < s_{k+1} < \ldots$$

Put $t_k = \binom{s_k}{m_k}$.

Let C_k be a set with $2 + s_k + t_k$ elements $b_k, a_k, b_k^1, \ldots, b_k^{t_k}, a_k^1, \ldots \ldots, a_k^{s_k}$. For this set α_k is a mapping from the system D_k of all subsets with m_k elements from the set $\{a_k^1, \ldots, a_k^{s_k}\}$ onto $\{b_k^1, \ldots, b_k^{t_k}\}$. Let σ_k be the ordering of C_k generated by pairs (a_k^j, b_k^i), where $a_k^j \in D \in D_k$, $\alpha_k(D) = b_k^i$, (b_k^j, a_k) for all j, (a_k, b_k) (see Fig. 1). Put $A_k = \{a_k, a_k^1, \ldots, a_k^{s_k}\}$, $B_k = C_k - \{a_k\}$. By the same symbol (and by C_k, as well) also the corresponding ordered sets with the restrictions of σ_k to these sets will be denoted.

Let P be a non-empty subset of the set $\{s_1, s_2, \ldots, s_k, \ldots\}$. Let (M, σ) be any finite ordered set. We shall define the order $\varrho_P(M, \sigma)$ on Exp M in the following way.

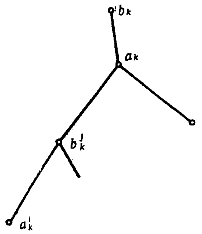

Fig. 1

For $X, Y \subset M$, $X \neq \emptyset \neq Y$ we put $X\, \varrho_P(M, \sigma)\, Y$ iff $X \subset Y$ or there exist numbers $s_{i_1}, \ldots, s_{i_r} \in P$ and isotone maps $h_j \colon C_{i_j} \to (M, \sigma)$, $j = 1, \ldots, r$ such that

(4) $$X \subset h_1(A_{i_1}), \quad h_1(B_{i_1}) \subset h_2(A_{i_2}), \quad h_2(B_{i_2}) \subset \ldots$$
$$\ldots, h_{r-1}(B_{i_{r-1}}) \subset h_r(A_{i_r}), \quad h_r(B_{i_r}) \subset Y.$$

It is easy to prove that $\varrho_P(M, \sigma)$ is an order on Exp M and that $F_P(M, \sigma) = \big(\operatorname{Exp} M, \varrho_P(M, \sigma)\big)$, $F_P(f) = \operatorname{Exp} f$ is an endofunctor in $\mathcal{U}^*$. In proving these facts it is sufficient to observe, in which case the cardinality of $h_j(A_{i_j})$ and $h_{j-1}(B_{i_{j-1}})$ or of $h_j(A_{i_j})$ and $h_j(B_{i_j})$ resp. are the same and to use this observation for proving antisymmetry for $\varrho_P(M, \sigma)$. Anyway, one can use [4], Theorem 1 as e.g. $\varrho_P(M, \sigma) \subset \pi^1(M, \sigma)$.

Let us now prove one auxiliary statement.

LEMMA 3. *Let $s_k \notin P$. Then A_k non $\varrho_P(C_k)\, B_k$.*

Proof. Suppose $A_k \subset h_1(A_{i_1}), \ldots, h_r(B_{i_r}) \subset B_k$ is a sequence of type (4). First of all we prove

(5) $$A_k \subset h_r(B_{i_r}).$$

It is

(6) $$h_1^{-1}(a_k) \neq \{a_{i_1}\}.$$

Suppose (6) does not hold. So $h_1^{-1}(a_k) = \{a_{i_1}\}$. We have $h_1(a_{i_1}) = a_k$ and

(7) $$h_1(b_{i_1}^i)\, \sigma_k a_k, \quad h_1(b_{i_1}^i) \neq a_k \quad \text{for all } i.$$

We have $k < i_1$ and $m_{i_1} > t_k$. For any choice of $a_{i_1}^{ji} \in h_1^{-1}(a_k^i)$, $i = 1, \ldots, s_k$ (such choice clearly exists) there exists $b_{i_1}^j$ so that $a_{i_1}^{ji}\sigma_{i_1}b_{i_1}^j$. Therefore $a_k^i\sigma_k h_1(b_{i_1}^j)$, which gives $h_1(b_{i_1}^j) = a_k$ or b_k. This is a contradiction to (7). Therefore (6) is valid and hence $A_k \subset h_1(B_{i_1})$. By induction we get (5). (5) together with $h_r(B_{i_r}) \subset B_k$ implies $A_k \subset B_k$ which is a contradiction to the definitions of A_k and B_k.

From Lemma 3 we can deduce

LEMMA 4. $P_1 \neq P_2 \Rightarrow F_{P_1} \neq F_{P_2}$.

The proof is immediate, as by Lemma 3 $s_k \in P_2 - P_1$ $(s_k \in P_1 - P_2)$ implies $\varrho_{P_2}(C_k, \sigma_k) \neq \varrho_{P_1}(C_k, \sigma_k)$. $\mathrm{card}\,\boldsymbol{T}^* = 2^{\aleph_0}$ follows obviously from Lemma 4, the definition of P and from the evident upper bound $\mathrm{card}\,\boldsymbol{T}^* \leqslant 2^{\aleph_0}$.

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HOMOMORPHISMS OF GROUP RINGS

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Introduction

Homomorphisms of group rings with the ring of integers as the coefficient ring and torsion-free group were first investigated by Higman [1]. These investigations were continued among others, by Smirnov [13]. Parmenter and Sehgal considered automorphisms of the group ring $A[G]$ for infinite cyclic group G and arbitrary ring of coefficients A ([7], [8]). Lantz [4] described automorphisms of group rings of free abelian groups of finite rank with commuting coefficients.

The aim of this paper is to present a new method of investigation of a group of units and homomorphisms of group rings. For this purpose we shall investigate in § 1 properties of some subgroups of a group $U(C[G])$ of units of a group ring $C[G]$, where C is a commutative ring. In § 2 a structure of the group $U(C[G])$ is described in the case where G is a u.p. group. In § 3 we introduce 4 classes of homomorphisms of group rings related to subgroups defined in § 1. They are called G_i-homomorphisms $(i = 0, 1, 2, 3)$ and it is shown that in the case of u.p.-groups every homomorphism is a G_3-homomorphism. In § 4 structure of G_0- and G_1-homomorphisms is described. In § 5 we investigate properties of G_2- and G_3-homomorphisms using in the essential way results concerning G_1-homomorphisms. In § 4 and § 5 some criteria for a homomorphisms to be an injection, a surjection or an automorphism are given. In § 6 our results are applied to the description of the structure of group of automorphisms and hopficity and cohopficity of group rings of u.p.-groups.

The paper is written in such a way that it is possible to extend all the results on u.p.-groups to the arbitrary torsion-free group after showing the triviality of the group of units of group algebras of such groups over fields.