

## ANALYSES OF LANGUAGES ACCEPTED BY VARIETOR MACHINES IN CATEGORY

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### Introduction

Languages accepted by finite sequential machines are characterized by the well-known Kleene theorem: they are precisely rational languages, i.e. those obtained from finite languages by a finite succession of operations  $\cup$  (union),  $\cdot$  (concatenation) and  $*$  (iteration). How far can this beautiful result be extended? Thatcher and Wright [7] present a generalization for tree machines: here not one concatenation and one iteration are sufficient, but rather a system of concatenations and corresponding iterations is considered. The internal reason for this is that in the case of tree machines it is not sufficient (as for sequential ones) to study machines with one initial state, but more initial states must be taken into account, and each initial state plays an individual role: it is important which initial states were used for a given tree to be accepted.

In the present paper we consider a very general situation: machines in a category in the sense of Arbib and Manes [4]. We define acceptors and finite acceptors on the one hand, and rational languages on the other hand. The main result is that every language accepted by a finite acceptor is rational. There is no hope for the converse (rational  $\Rightarrow$  acceptable) to hold. In fact, even in the variety  $\mathcal{V}$  of groupoids, defined by the equation

$$x \cdot x = y \cdot y,$$

the Kleene theorem fails to hold for  $\mathcal{V}$ -machines (i.e., binary tree machines in  $\mathcal{V}$ ), see [8]. It remains open even for varieties of tree machines under which circumstances the Kleene theorem holds.

The definition of languages and their concatenation which we present might seem technical at first, but it is justified by the motivating example of sequential machines with several initial states. This motivation is exhibited in [2], where the present result was announced; therefore we skip it here and start with the definitions needed. Let us remark that, at least, it suffices to consider one concatenation and one iteration here, not as in the case of tree machines (where, on the other hand, the definition of these operations is very natural and not technical at all).

The present paper is a part of a broad program of investigating the languages of (deterministic and non-deterministic) variator machines, see [2], [3], [8], [9], [10].

### 1. Accepted and rational languages

**1.1.** Throughout the paper we assume that a category  $\mathcal{X}$  is given together with its factorization system  $(\mathcal{E}, \mathcal{M})$  and a functor  $F: \mathcal{X} \rightarrow \mathcal{X}$  subject to conditions C1–C4 below. Let us recall (e.g. from [4]) that an algebra of type  $F$ , shortly an  $F$ -algebra, is an arrow  $FQ \xrightarrow{\delta} Q$  and a homomorphism from  $FQ \xrightarrow{\delta} Q$  into  $FQ' \xrightarrow{\delta'} Q'$  is a morphism  $f: Q \rightarrow Q'$  in  $\mathcal{X}$  for which the following diagram commutes:

$$\begin{array}{ccc} FQ & \xrightarrow{\delta} & Q \\ \downarrow Ff & & \downarrow f \\ FQ' & \xrightarrow{\delta'} & Q' \end{array}$$

A free  $F$ -algebra, generated by an object  $X$ , is an  $F$ -algebra  $FX^{\#} \xrightarrow{\varphi} X$  together with an arrow  $X \xrightarrow{\gamma} X^{\#}$ , universal in the usual sense: for each algebra  $FQ \xrightarrow{\delta} Q$  and each morphism  $k: X \rightarrow Q$  there exists a unique “free extension” homomorphism  $k^*: (X^{\#}, \varphi) \rightarrow (Q, \delta)$  with  $k = \eta \cdot k^*$ .

The conditions we require throughout are as follows:

C1.  $F$  is a *variator*, i.e., every object generates a free algebra.

C2.  $F$  preserves  $\mathcal{E}$ , i.e.  $e \in \mathcal{E}$  implies  $Fe \in \mathcal{E}$ .

C3.  $\mathcal{X}$  has finite limits and countable colimits.

C4. Pullback condition: opposite an  $\mathcal{E}$ -epi in any pullback there is an  $\mathcal{E}$ -epi.

**1.2.** Modifying the definition of machines of type  $F$ , due to Arbib and Manes, we define an *acceptor* of type  $F$ . This is an  $F$ -algebra with two subobjects (of initial and terminal states, respectively). Thus, an acceptor is a 6-tuple

$$A = (Q, \delta, I, i, T, t),$$

where  $Q$  is an object (of states) in  $\mathcal{X}$ ,  $\delta: FQ \rightarrow Q$  is a morphism (“next-state map”), and  $i: I \rightarrow Q$ ,  $t: T \rightarrow Q$  are morphisms in  $\mathcal{M}$ . We can extend  $i$  freely to a homomorphism  $i^*: (I^{\#}, \varphi) \rightarrow (Q, \delta)$ , called the *run map* of  $A$ :

$$\begin{array}{ccc} FI^{\#} & \xrightarrow{\varphi} & I^{\#} \\ \downarrow Fi^* & \nearrow i & \downarrow i^* \\ FQ & \xrightarrow{\delta} & Q \end{array}$$

The language accepted by  $A$  is defined as the preimage of the terminal-state subobject under the run map, i.e., the subobject  $\lambda: L \rightarrow I^{\#}$  in the following pullback:

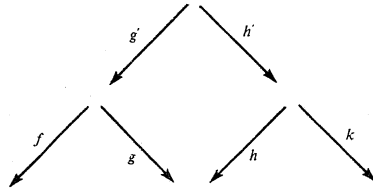
$$\begin{array}{ccc} L & \xrightarrow{\lambda} & I^{\#} \\ \downarrow j & & \downarrow i^* \\ T & \xrightarrow{t} & Q \end{array}$$

(since  $t \in \mathcal{M}$  implies  $\lambda \in \mathcal{M}$ , this is indeed a subobject of  $I^{\#}$ ).

**1.3.** In the present paper we shall make an extensive use of the calculus of relations in a category, see [5]. A relation from  $A$  to  $B$  ( $A, B$  objects of  $\mathcal{X}$ ) is any  $\mathcal{M}$ -subobject of  $A \times B$ . A relation is determined by a pair of arrows  $T \xrightarrow{f} A$ ,  $T \xrightarrow{g} B$  if the image of the induced arrow  $T \rightarrow A \times B$  is this relation. We then write  $[f, g]: A \rightarrow B$ . Every relation has a number of determining pairs. For example, given  $e: T_1 \rightarrow T$  in  $\mathcal{E}$ , we have

$$[f, g] = [e \cdot f, e \cdot g]: A \rightarrow B.$$

The composition of relations  $[f, g]: A \rightarrow B$  and  $[h, k]: B \rightarrow C$  is defined via the pullback of  $g$  and  $h$ ,



as follows:

$$[f, g] \circ [h, k] = [g' \cdot f, h' \cdot k]: A \rightarrow C.$$

The pullback condition (C4 in 1.1) guarantees that this definition is correct, i.e. independent of the choice of the representatives  $f, g$  and  $h, k$ , see [5]. Evidently, it is associative.

**1.4.** Given two objects  $I, J$  in  $\mathcal{K}$ , an *extended language* (or just language) from  $I$  to  $J$  is a relation  $[f, g]: I^\# \rightarrow J$ . We denote it by

$$[f, g]: I \mapsto J.$$

**1.5.** Let  $A = (Q, \delta, I, i, T, t)$  and  $\lambda, j$  be as in 1.2.

**DEFINITION.** The *extended language of an acceptor*  $A$  is

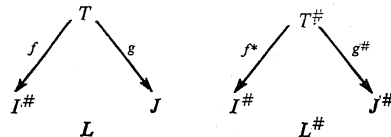
$$L(A) = [\lambda, j]: I \mapsto T.$$

**1.6.** For each morphism  $f: X \rightarrow Y$  in  $\mathcal{K}$  there exists a free extension of  $f \cdot \eta_X: X \rightarrow Y^\#$  to a homomorphism, denoted by

$$f^\#: (X^\#, \varphi_X) \rightarrow (Y^\#, \varphi_Y).$$

Thus,  $f^\# = (f \cdot \eta_Y)^\#$ . Given an extended language  $[f, g]: I \mapsto J$ , we define an extended language

$$[f, g]^\# = [f^\#, g^\#]: I \mapsto J^\#.$$



## 1.7. Rational operations on extended languages

A) *Union*  $[a_1, \beta_1] \cup [a_2, \beta_2]$  is defined whenever these languages have a joint domain  $I$  and range  $J$ ; this is the union of  $\mathcal{M}$ -subobjects of  $I^\# \times J$ .

Since  $\mathcal{K}$  has finite sums, it has finite unions of  $\mathcal{M}$ -subobjects; the union of subobjects, represented by

$$m_1: U_1 \rightarrow V \quad \text{and} \quad m_2: U_2 \rightarrow V \quad \text{in } \mathcal{M},$$

is the image of the induced morphism  $U_1 + U_2 \rightarrow V$ .

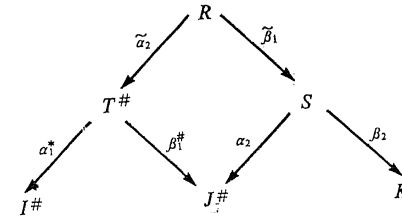
B) *Concatenation*  $[a_1, \beta_1] \odot [a_2, \beta_2]$  is defined whenever the range of  $[a_1, \beta_1]$  equals the domain of  $[a_2, \beta_2]$ . Thus, let

$$[a_1, \beta_1]: I \mapsto J \quad \text{and} \quad [a_2, \beta_2]: J \mapsto K$$

be languages; we then define

$$[a_1, \beta_1] \odot [a_2, \beta_2] = [a_1, \beta_1]^\# \odot [a_2, \beta_2]: I \mapsto K,$$

$$[a_1, \beta_1]^\# \odot [a_2, \beta_2] = [\tilde{a}_2 \cdot a_1^*, \tilde{\beta}_1 \cdot \beta_2].$$



C) *Iteration*  $[a, \beta]^*$  is defined whenever the domain and range of  $[a, \beta]$  are equal. Given  $[a, \beta]: I \mapsto I$ , define  $[a, \beta]^*: I \mapsto I$  by

$$[a, \beta]^* = A \cup [a, \beta] \cup ([a, \beta] \odot [a, \beta]) \cup (([a, \beta] \odot [a, \beta]) \odot [a, \beta]) \dots$$

where  $A = [\eta_I, 1_I]: I \mapsto I$ .

Again, since  $\mathcal{K}$  has countable colimits,  $[a, \beta]^*$  is well defined.

**1.8.** Let  $\mathcal{F}$  be a fixed class of objects of  $\mathcal{K}$ ; we call its elements *finite objects*. In the present context no hypothesis on  $\mathcal{F}$  is needed.

An acceptor  $A = (Q, \delta, I, i, T, t)$  is called *finite* if its state object  $Q$  as well as objects  $I$  and  $T$  are finite. A language  $I \mapsto J$  is finite if it is determined by a pair

$$f: M \rightarrow I^\#, \quad g: M \rightarrow J$$

with a finite  $M$ .

**1.9. DEFINITION.** An extended language is said to be *rational* if it can be expressed by rational operations  $\cup, \odot, *$  used finitely many times on finite extended languages.

## 2. The main theorem

**2.1. Algorithmic variators.** In a number of situations, free  $F$ -algebras  $(X^\#, \varphi)$  can be obtained by a natural construction, called the *free-algebra algorithm* in [1]. First, define a sequence of objects  $W_n$  and morphisms  $k_n: W_n \rightarrow W_{n+1}$  by induction

- (a)  $W_0 = X$ ,  $W_1 = X + FX$  and  $k_0: X \rightarrow X + FX$  is canonical;
- (b)  $W_{n+1} = X + FW_n$  and  $k_{n+1} = 1_X + Fk_n$

$$\begin{array}{c} X \xrightarrow{k_0} X + FX \xrightarrow{1_X + Fk_0} X + F(X + FX) \xrightarrow{1_X + F(1_X + Fk_0)} \dots \rightarrow X^\# \\ \uparrow \varphi_0 \quad \quad \quad \uparrow \varphi \\ FX \xrightarrow{Fk_0} F(X + FX) \xrightarrow{F(1_X + Fk_0)} \dots \rightarrow FX^\# \end{array}$$

Denote by  $X^\#$  and  $w_n: W_n \rightarrow X^\#$  ( $n = 0, 1, 2, \dots$ ) the colimit of this sequence. Since we have a canonical injection  $FW_n \rightarrow W_{n+1} = X + FW_n$ , we obtain a natural morphism  $\varphi: FX^\# \rightarrow X^\#$  provided that also  $FX^\#$  and  $Fw_n: FW_n \rightarrow FX^\#$  is a colimit of the sequence  $\{Fk_n\}_{n=0}^\infty$ . Put  $\eta = w_0: X \rightarrow X^\#$ ; then  $(X^\#, \varphi)$  is a free  $F$ -algebra, see [1].

**DEFINITION.** A functor  $F$  is called an *algorithmic variator* if it preserves the colimit of the above sequence  $\{k_n\}_{n=0}^\infty$  for each object  $X$ .

**EXAMPLES.** Given a type  $\Omega = \{\omega_i\}_{i \in I}$  of algebras, i.e. a collection of cardinals, denote by  $F_\Omega$  the functor with  $F_\Omega X = \prod_{i \in I} X^{\omega_i}$  (where  $X^{\omega_i}$  denotes the product of  $\omega_i$  copies of  $X$ ), defined naturally on morphisms,  $F_\Omega f = \prod_{i \in I} f^{\omega_i}$ . In sets,  $F_\Omega$ -algebras are precisely universal algebras of type  $\Omega$ , and homomorphisms also agree. Then  $F_\Omega$  is an algorithmic variator iff  $\Omega$  is a finitary type.

In contrast, (a) every endofunctor of the category of countable sets and mappings is an algorithmic variator [2]; (b) every functor  $F_\Omega$  on the category of  $\omega$ -complete posets and  $\omega$ -continuous maps is an algorithmic variator [6].

**2.2. Convention.** Given morphisms  $f: A \rightarrow C$  and  $g: B \rightarrow C$ , we denote by  $f + g: A + B \rightarrow C$  the obvious induced morphism.

Let  $F$  be an algorithmic variator. Given an  $F$ -algebra  $FQ \xrightarrow{\delta} Q$  and a morphism  $f: X \rightarrow Q$ , its free extension to a homomorphism  $f^*: (X^\#, \varphi) \rightarrow (Q, \delta)$  can be described as follows. Put

$$f_0 = f: W_0 \rightarrow Q;$$

given  $f_n: W_n \rightarrow Q$ , define

$$f_{n+1} = f + Ff_n \cdot \delta(X + FW_n = W_{n+1} \rightarrow Q).$$

Then the (unique) morphism  $f^*: X^\# \rightarrow Q$  with

$$w_n \cdot f^* = f_n \quad (n = 0, 1, 2, \dots)$$

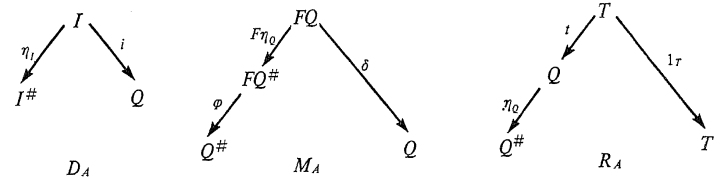
is a homomorphism and, of course,  $\eta \cdot f^* (= w_0 \cdot f^*) = f$ . See [1].

**2.3.** To formulate the main result of our paper, we shall introduce the following languages for any acceptor  $A = (Q, \delta, I, i, T, t)$ :

$$D_A = [\eta_I, i]: I \mapsto Q,$$

$$M_A = [F\eta_Q \cdot \varphi, \delta]: Q \mapsto Q,$$

$$R_A = [t \cdot \eta_Q, 1_T]: Q \mapsto T.$$



Let us mention that if  $A$  is a finite acceptor, then  $D_A, R_A$  are finite languages and  $M_A$  is finite whenever  $FQ$  is finite, which is so e.g. whenever  $F$  preserves finite objects (i.e.  $Q \in \mathcal{F}$  implies  $FQ \in \mathcal{F}$ ).

**2.4. Theorem on analyses.** Let  $F$  be an algorithmic variator. For any acceptor  $A = (Q, \delta, I, i, T, t)$ , the language  $L(A)$  accepted by  $A$  fulfils

$$L(A) = D_A \odot ((A \cup M_A)^* \odot R_A).$$

**2.5. COROLLARY.** If  $F$  is an algorithmic variator preserving finite objects, then every language accepted by a finite acceptor is rational.

The rest of our paper is devoted to the proof of 2.4.

## 3. Auxiliary propositions

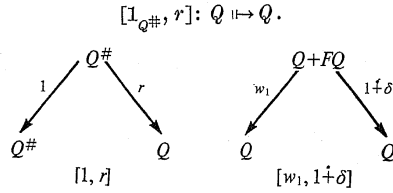
**3.1. Analyses of iteration of any language  $[\alpha, \beta]: I \mapsto I$  containing  $A = [\eta_I, 1_I]$ .**

If  $\alpha: P_1 \rightarrow I^\#$  and  $\beta: P_1 \rightarrow I$  are the representing morphisms, we define morphisms

$$\alpha_n: P_n \rightarrow P_{n-1}^\# \quad \text{and} \quad \beta_n: P_n \rightarrow P_{n-1}$$



rise to a language



The aim of the rest of this section is to show that this language is a result of iteration of the language

$$[w_1, 1_Q + \delta]: Q \mapsto Q.$$

More precisely:

**3.4. PROPOSITION.** *If  $F$  is an algorithmic variety, then for any  $F$ -algebra  $FQ \xrightarrow{\circ} Q$  we have*

$$[w_1, 1_Q + \delta]^* = [1_{Q^\#}, r]: Q \mapsto Q.$$

*Proof.* Apply 3.1 to  $[a, \beta] = [w_1, 1_Q + \delta]$ . First, this language contains  $\Delta$  because for  $e_1 = k_0: Q \rightarrow Q + FQ$  we have  $k_0 \cdot w_1 = w = \eta_Q$  and  $k_0 \cdot (1_Q + \delta) = 1_Q$ . Thus, by 3.2,

$$[w_1, 1_Q + \delta]^* = [\bar{\alpha}, \bar{\beta}].$$

We shall prove that (a)  $\bar{\alpha} \cdot r = \bar{\beta}$  and (b)  $\bar{\alpha} \in \mathcal{E}$ . The latter implies  $[1_{Q^\#}, r] = [\bar{\alpha} \cdot 1_{Q^\#}, \bar{\alpha} \cdot r]$ ; therefore  $[1_{Q^\#}, r] = [\bar{\alpha}, \bar{\beta}]$ .

(a) We shall verify that

$$(*) \quad (\alpha_1^* \dots \alpha_n^*) \cdot r = (\beta_n \dots \beta_1)^\# \cdot r \quad \text{for } n = 1, 2, 3, \dots;$$

then for each  $n$

$$\begin{aligned}
 \bar{e}_n \cdot (\bar{\alpha} \cdot r) &= \alpha_n \cdot \alpha_{n-1}^* \cdot \dots \cdot \alpha_1^* \cdot r && \text{(definition of } \bar{\alpha}) \\
 &= \eta \cdot \alpha_1^* \cdot \alpha_{n-1}^* \cdot \dots \cdot \alpha_1^* \cdot r && (\alpha_n = \eta \cdot \alpha_n^*) \\
 &= \eta \cdot (\beta_n \dots \beta_1)^\# \cdot r && \text{(by } (*)) \\
 &= (\beta_n \dots \beta_1) \cdot \eta \cdot r && (\eta \cdot (-)^\# = (-) \cdot \eta) \\
 &= \beta_n \dots \beta_1 = \bar{e}_n \cdot \bar{\beta} && (\eta \cdot r = 1; \text{ definition of } \bar{\beta}).
 \end{aligned}$$

There follows  $\bar{\alpha} \cdot r = \bar{\beta}$ .

(A)  $n = 1$ . We have  $\alpha_1 = w_1 = w_0 + Fw_0 \cdot \varphi$  and, since  $r$  is a homomorphism extending  $1_Q$ , also  $\eta_Q \cdot r = 1_Q$  and  $\varphi \cdot r = F \cdot \delta$ . Thus

$$\alpha_1 \cdot r = (w_0 \cdot r) + (Fw_0 \cdot \varphi \cdot r) = 1_Q + (F(w_0 \cdot r) \cdot \delta) = 1_Q + \delta = \beta_1.$$

Hence,  $\alpha_1^* \cdot r$  and  $\beta_1^\# \cdot r$  are homomorphisms from  $((Q + FQ)^\#, \varphi)$  which agree on  $Q + FQ$  (for,  $\eta_{Q+FQ} \cdot \alpha_1^* \cdot r = \alpha_1 \cdot r$  while  $\eta_{Q+FQ} \cdot \beta_1^\# \cdot r = \beta_1 \cdot \eta_Q \cdot r = \beta_1$ ). Hence  $\alpha_1^* \cdot r = \beta_1^\# \cdot r$ .

(B) Let  $(*)$  be proved for  $1, 2, \dots, n-1$ . To prove it for  $n$ , notice that both sides of  $(*)$  are homomorphisms from  $(P_n^\#, \varphi)$ ; thus it suffices to show that they coincide on  $P_n$ . Indeed, since for each  $k$  we have  $\alpha_{k+1} \cdot \beta_k^\# = \beta_{k+1} \cdot \alpha_k$  (see 3.1) we get

$$\begin{aligned}
 \eta \cdot \alpha_n^* \cdot \dots \cdot \alpha_1^* \cdot r &= \eta \cdot \alpha_n^* \cdot (\beta_{n-1} \dots \beta_1)^\# \cdot r && \text{(by induction)} \\
 &= \alpha_n \cdot \beta_{n-1}^\# \cdot (\beta_{n-2} \dots \beta_1)^\# \cdot r && (\eta \cdot \alpha_n^* = \alpha_n) \\
 &= \beta_n \cdot \alpha_{n-1} \cdot (\beta_{n-2} \dots \beta_1)^\# \cdot r && (\alpha_n \cdot \beta_{n-1}^\# = \beta_n \cdot \alpha_{n-1}) \\
 &= \beta_n \cdot \eta \cdot \alpha_{n-1}^* \cdot (\beta_{n-2} \dots \beta_1)^\# \cdot r && (\eta \cdot \alpha_{n-1}^* = \alpha_{n-1}) \\
 &= \beta_n \cdot \eta \cdot \alpha_{n-1}^* \cdot \alpha_{n-2}^* \cdot \dots \cdot \alpha_1^* \cdot r && \text{(by induction)} \\
 &= \beta_n \cdot \eta \cdot (\beta_{n-1} \dots \beta_1)^\# \cdot r && \text{(by induction)} \\
 &= \eta \cdot (\beta_n \dots \beta_1)^\# \cdot r && (\eta \cdot \beta_n^\# = \beta_n \cdot \eta).
 \end{aligned}$$

(b) To prove  $\bar{\alpha} \in \mathcal{E}$  we shall exhibit, for every  $n = 0, 1, 2, \dots$ , a morphism  $\tau_n: W_n \rightarrow E$  with  $\tau_n \cdot \bar{\alpha} = w_n$ . Then, given an  $(\mathcal{E}, \mathcal{M})$ -factorization  $\bar{\alpha} = e \cdot m$ , we infer that  $\{(\tau_n \cdot e) \cdot m\}$  is a compatible family for the chain  $\{k_n\}$ ; hence so is  $\{\tau_n \cdot e\}$ ; therefore there exists a  $\lambda$  with  $\tau_n \cdot e = w_n \cdot \lambda$  and so

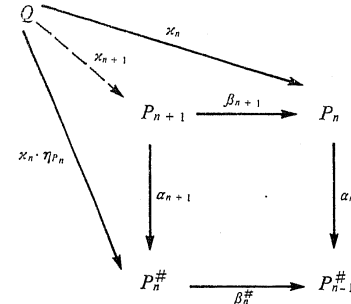
$$w_n = w_n \cdot (\lambda \cdot m) \quad \text{for each } n,$$

which implies  $\lambda \cdot m = 1$ . Thus  $m$  is an isomorphism and  $\bar{\alpha} = e \cdot m^{-1} \in \mathcal{E}$ .

(b1) Define a sequence  $\kappa_n: Q \rightarrow P_n$  with

$$(1) \quad \kappa_{n+1} \cdot \beta_{n+1} = \kappa_n \quad \text{and} \quad \kappa_{n+1} \cdot \alpha_{n+1} = \kappa_n \cdot \eta_{P_n}$$

by induction on  $n$ . First  $\kappa_0 = 1_Q: Q \rightarrow Q$  and  $\kappa_1$  is the injection  $\kappa_1: Q \rightarrow P_1 = Q + FQ$ . The inductive step uses the pullback of  $\alpha_n, \beta_n^\#$ :



Put  $\gamma_1 = 1_Q + \delta: Q + FQ \rightarrow Q$ ;  $\gamma_{n+1} = 1_Q + F\gamma_n: W_{n+1} \rightarrow W_n$ . Denote  $u_n = \eta_{P_n} + F\eta_{P_n} \cdot \varphi: P_n + FP_n \rightarrow P_n^\#$  and define a sequence  $\varrho_n: W_n \rightarrow P_n$  with

$$(2) \quad \varrho_n \cdot \beta_n = \gamma_n \cdot \varrho_{n-1} \quad \text{and} \quad \varrho_n \cdot \alpha_n = (\gamma_{n-1} + F\varrho_{n-1}) \cdot u_{n-1}.$$

First,  $\varrho_1 = 1_{W_1}: W_1 \rightarrow P_1 = W_1$ . Given  $\varrho_n$ , let us verify that

$$(3) \quad (\gamma_{n+1} \cdot \varrho_n) \cdot \alpha_n = (\gamma_n + F\varrho_n) \cdot u_n \cdot \beta_n^\#$$

and then define  $\varrho_{n+1}$ , using the same pullback as above:

$$\begin{array}{ccccc}
 W_{n+1} & & & & \\
 \swarrow \gamma_{n+1} \cdot \rho_n & & & & \\
 & P_{n+1} & \xrightarrow{\beta_{n+1}} & P_n & \\
 \swarrow \rho_{n+1} & \downarrow \alpha_{n+1} & & \downarrow \alpha_n & \\
 & P_n^\# & \xrightarrow{\beta_n^\#} & P_{n-1}^\# & \\
 \swarrow (\gamma_n + F\rho_n) \cdot u_n & & & & 
 \end{array}$$

Indeed, since

$$(4) \quad u_n \cdot \beta_n^\# = (\beta_n + F\beta_n) \cdot u_{n-1},$$

we have

$$\begin{aligned}
 (\gamma_n + F\varrho_n) \cdot u_n \cdot \beta_n^\# &= (\gamma_n \cdot \beta_n + F(\varrho_n \cdot \beta_n)) \cdot u_{n-1} && \text{(by (4))} \\
 &= (\gamma_{n-1} + F(\gamma_n \cdot \varrho_{n-1})) \cdot u_{n-1} && \text{(by (1) and (2))} \\
 &= (1_Q + F\gamma_n) \cdot (\gamma_{n-1} + F\varrho_{n-1}) \cdot u_{n-1} \\
 &= (1_Q + F\gamma_n) \cdot \varrho_n \cdot \alpha_n && \text{(by induction and (2))} \\
 &= (\gamma_{n+1} \cdot \varrho_n) \cdot \alpha_n && (\gamma_{n+1} = 1_Q + F\gamma_n).
 \end{aligned}$$

(b2) Define  $\tau_n = \varrho_n \cdot \bar{e}_n: W_n \rightarrow E$  (where  $\bar{e}_n: P_n \rightarrow E$  are the colimit injections of  $E = \text{colim}\{\varrho_n\}$ ). Then we prove

$$\tau_n \cdot \bar{\alpha} = w_n$$

by induction. First,  $\tau_1 \cdot \bar{\alpha} = \varrho_1 \cdot \alpha_1 = 1 \cdot \alpha_1 = \alpha_1 = w_1$ . For the induction first notice that

$$(5) \quad u_n \cdot \alpha_n^* = \alpha_n + F\alpha_n \cdot \varphi \quad \text{for each } n$$

(because in the construction of  $P_n^\#$  as a colimit  $w^{(P_n)}: W_k^{(P_n)} \rightarrow P_n^\#$  we have  $u_n = w_1^{(P_n)}$  and  $\alpha_n^*$  is a free extension of  $\alpha_n$ , see 2.1). Further

$$(6) \quad \gamma_n \cdot \eta_{P_n} \cdot \alpha_n^* \dots \cdot \alpha_1^* = \gamma_Q \quad \text{for each } n$$

because of (1), which yields  $\gamma_n \cdot \eta \cdot \alpha_n^* = \gamma_n \cdot \alpha_n = \gamma_{n-1} \cdot \eta$ . Finally, since  $\alpha_n^* \dots \cdot \alpha_1^*: (P^\#, \varphi) \rightarrow (Q^\#, \varphi)$  is a homomorphism, we have

$$(7) \quad \varphi \cdot \alpha_n^* \dots \cdot \alpha_1^* = F(\alpha_n^* \dots \cdot \alpha_1^*) \cdot \varphi.$$

Now, assuming  $\tau_n \cdot \bar{\alpha} = w_n$ , we get

$$\begin{aligned}
 \tau_{n+1} \cdot \bar{\alpha} &= \varrho_{n+1} \cdot (\bar{e}_{n+1} \cdot \bar{\alpha}) && \text{(definition of } \tau_{n+1}) \\
 &= \varrho_{n+1} \cdot \alpha_{n+1} \cdot \alpha_n^* \dots \cdot \alpha_1^* && \text{(definition of } \bar{\alpha}) \\
 &= (\gamma_{n+1} + F\varrho_n) \cdot u_n \cdot \alpha_n^* \dots \cdot \alpha_1^* && \text{(by (2))} \\
 &= (\gamma_n + F\varrho_n) \cdot (\alpha_n + F\alpha_n \cdot \varphi) \cdot \alpha_{n-1}^* \dots \cdot \alpha_1^* && \text{(by (5))} \\
 &= (\gamma_n \cdot \alpha_n \cdot \alpha_{n-1}^* \dots \cdot \alpha_1^*) + (F\varrho_n \cdot F\alpha_n \cdot \varphi \cdot \alpha_{n-1}^* \dots \cdot \alpha_1^*) \\
 &= \gamma_Q + F(\varrho_n \cdot \alpha_n \cdot \alpha_{n-1}^* \dots \cdot \alpha_1^*) \cdot \varphi && \text{(by (6) and (7))} \\
 &= \gamma_Q + F(\varrho_n \cdot \bar{e}_n \cdot \bar{\alpha}) \cdot \varphi && \text{(definition of } \bar{\alpha}) \\
 &= \gamma_Q + Fw_n \cdot \varphi && \text{(induction)} \\
 &= w_{n+1} && \text{(see 2.1).}
 \end{aligned}$$

This concludes the proof.

#### 4. The proof of the theorem on analyses

Extend  $i$  and  $1_Q$  freely to homomorphisms

$$i^*: (I^\#, \varphi) \rightarrow (Q, \delta) \quad \text{and} \quad r = 1_Q^*: (Q^\#, \varphi) \rightarrow (Q, \delta).$$

Then for  $i^\#: (I^\#, \varphi) \rightarrow (Q^\#, \varphi)$  (see 1.6) we have

$$i^* = i^\# \cdot r: (I^\#, \varphi) \rightarrow (Q, \delta)$$

because both  $i^*$  and  $r$  are homomorphisms extending  $i: I \rightarrow Q$ . Consider pullback of  $r$  and  $t$  (square I) and the pullback of  $i^\#$  and  $\nu$  (square II):

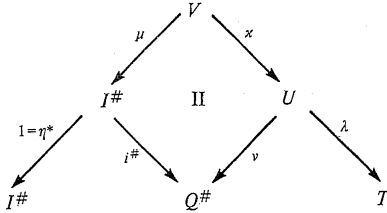
$$\begin{array}{ccccc}
 V & \xrightarrow{\kappa} & U & \xrightarrow{\lambda} & T \\
 \mu \downarrow & & \downarrow \nu & \text{I} & \downarrow t \\
 I^\# & \xrightarrow{i^\#} & Q^\# & \xrightarrow{r} & Q
 \end{array}$$

Then the outer square is a pullback of  $t$  and  $i^* = i^\# \cdot r$ ; hence

$$L(A) = [\mu, \kappa \cdot \lambda]$$

by Definition 1.5. Further,

$$[\mu, \kappa \cdot \lambda] = [\eta_I, i] \odot [\nu, \lambda] = L_A \odot [\nu, \lambda].$$



and so, to prove the theorem, it suffices to verify

$$[\nu, \lambda] = (A \cup M_A)^* \odot R_A.$$

Now, in the construction of  $Q^\#$  (2.1) we have  $\eta_Q = w_0$  and  $Fw_0 \cdot \varphi = \varphi_0 \cdot w_1$ ; therefore  $w_1 = \eta_Q + F\eta_Q \cdot \varphi$ . Thus

$$(A \cup M_A) = [\eta_Q, 1_Q] \cup [F\eta_Q \cdot \varphi, \delta] = [\eta_Q + F\eta_Q \cdot \varphi, 1_Q + \delta] = [w_1, 1_Q + \delta].$$

By 3.3,

$$(A \cup M_A)^* = [1_{Q^\#}, r].$$

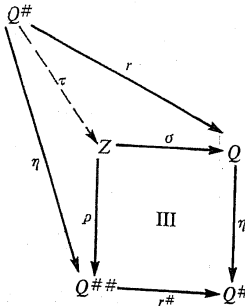
Hence

$$\begin{aligned} (A \cup M_A)^* \odot R_A &= [1_{Q^\#}, r] \odot [t \cdot \eta_Q, 1_T] = [1_{Q^\#}^*, r^\#] \odot [t \cdot \eta_Q, 1_T] \\ &= [1_{Q^\#}^*, r^\#] \odot [\eta_Q, 1_Q] \odot [t, 1_T]. \end{aligned}$$

The proof will be concluded when we show

$$(*) \quad [1_{Q^\#}^*, r^\#] \odot [\eta_Q, 1_Q] = [1_{Q^\#}, r].$$

Consider the pullback of  $r^\#$  and  $\eta_Q$  (square III):



By definition of  $r^\#$  we have  $r \cdot \eta = \eta \cdot r^\#$ ; hence there exists a  $\tau: Q^\# \rightarrow Z$  with  $\tau \cdot \varrho = \eta$ . It follows that

$$\tau \cdot (\varrho \cdot 1_{Q^\#}^*) = \eta \cdot 1_{Q^\#}^* = 1_{Q^\#}; \quad \text{thus} \quad 1_{Q^\#}^* \in \mathcal{E}.$$

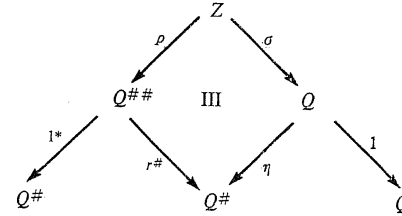
This implies

$$[1_{Q^\#}, r] = [\varrho \cdot 1_{Q^\#}^*, \varrho \cdot 1_{Q^\#}^* \cdot r].$$

Further, both  $r^\# \cdot r$  and  $1_{Q^\#}^* \cdot r$  are homomorphisms from  $(Q^{\#\#}, \varphi)$  to  $(Q, \delta)$  which equal  $r$  on  $Q$ ; hence  $r^\# \cdot r = 1_{Q^\#}^* \cdot r$ . Therefore  $\varrho \cdot 1_{Q^\#}^* \cdot r = \varrho \cdot r^\# \cdot r = \sigma \cdot \eta \cdot r = \sigma$  (for  $\eta \cdot r = 1$ ). We get

$$[1_{Q^\#}, r] = [\varrho \cdot 1_{Q^\#}^*, \sigma].$$

This proves (\*), because  $[\varrho \cdot 1_{Q^\#}^*, \sigma] = [1_{Q^\#}^*, r^\#] \odot [\eta_Q, 1_Q]$ .



This proves the theorem.

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## WEAK AUTOMORPHISMS OF 1-UNARY ALGEBRAS

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Groups of weak automorphisms of 1-unary algebras have been described only in the case of free algebras (see [5] and [1]). Proposition 1 of this note generalizes these results and gives a description for the general case. Proposition 2 tells which groups are of the form “group of all weak automorphisms modulo group of all automorphisms” for 1-unary algebras, Proposition 3 shows that the class of all groups of weak automorphisms of 1-unary algebras is at least as rich as the class of all groups of automorphisms of these, and Proposition 4 ensures that this class does not contain all groups.

I would like to express my thanks for inspiration to my students R. Ptáček and Z. Svoboda, the authors of [4]. In fact, they have studied groups of weak automorphisms for some 1-unary algebras but without the help of groups of automorphisms of those algebras.

Note that groups of automorphisms of 1-unary algebras were characterized in [2].

For a 1-unary algebra  $\mathfrak{A} = (A, f)$  let  $\mathcal{A}\mathfrak{A} = (A\mathfrak{A}, \cdot)$  and  $\mathcal{W}\mathfrak{A} = (W\mathfrak{A}, \cdot)$  denote the group of all automorphisms and the group of all weak automorphisms of it, respectively. Let  $f^n$  stand for the  $n$ th iteration of  $f$  (i.e.  $f^0 = \text{id}_A$ ,  $f^{n+1}a = f(f^n a)$  for all  $a \in A$ ,  $n = 1, 2, \dots$ ). By  $N$  we denote the set of all positive natural numbers and let  $\mathcal{Z}_d = (Z_d = \{0, 1, \dots, d-1\}, \cdot)$  be a semigroup, where the operation  $\cdot$  is the usual multiplication modulo  $d$ ,  $d \in N$ .

PROPOSITION 1. *Let  $\mathfrak{A} = (A, f)$  be a 1-unary algebra.*

(1) *If for no  $n \in N$   $f^{n+1} = f$ , then any weak automorphism of  $\mathfrak{A}$  is an automorphism of  $\mathfrak{A}$ .*

(2) *Let  $d$  be the smallest  $n \in N$  such that  $\mathfrak{A}$  satisfies  $f^{n+1} = f$ . Then the set*