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THE SINGULAR SEQUENCE PROBLEM*

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The so-called "singular sequence problem", to be described herein, was as far as I know formulated jointly by the present author and Konrad Jörgens in 1970 in Boulder. In its simplest form the question is: if a bounded self-adjoint operator B can be written as the difference $A_2 - A_1$ of two bounded self-adjoint operators A_2 and A_1 where A_2 and A_1 possess the same singular sequences, does it follow that B is compact? The condition of "same singular sequences", to be specified below, may for the moment be regarded as a certain type of strong condition relating the spectral measures of A_2 and A_1.

This problem is of interest for at least three reasons. First, it arises originally in differential equations and in particular in applications involving continuous spectra. Second, there are important connections to more general considerations and questions in Banach algebras. Third, although the degree of difficulty of the problem is not yet clear, it is nonetheless a question in the spectral theory for a single bounded self-adjoint operator, or, if you will, for pairs of bounded self-adjoint operators, which as of this writing has not been resolved. It has not been previously exposed in the open literature and I will do so (briefly) as follows. (1)

1. The original problem: Weyl's Theorems.
2. The generalized problem: Banach Algebras.
3. The specialized problem: Singular Sequences.

1. The original problem: Weyl's Theorems

Weyl [10] showed in 1909 that if a bounded self-adjoint A on a complex Hilbert space H is perturbed by a compact operator B, the essential spectrum is preserved:

$$\sigma_e(A+B) = \sigma_e(A).$$

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(1) The author became aware of the wider interest in this general area of problems at the Leipzig International Congress on Operator Algebras in September, 1977. Accordingly the author presented essentially this brief survey of these problems in the Applied Mathematics Seminar in Boulder on 6 October 1977, from which this paper is taken.

Weyl was interested in this result for application to ordinary differential operators, and this theorem and its extensions have since been useful in many other areas of differential equations and in other contexts. Von Neumann [4] later proved a converse of Weyl's Theorem, namely, if two bounded self-adjoint operators A_2 and A_1 on a separable complex Hilbert space have the same essential spectrum $\sigma_e(A_2) = \sigma_e(A_1)$ then there exists a unitary operator U such that the operator $A_2 - UA_1U^*$ is compact. That is, $B \equiv A_2 - A_1$ is "almost" compact.

Gustafson and Weidmann [2] later showed that under the stronger assumption that $\sigma_e(A+B) = \sigma_e(A)$ for all A in $B(H)$ one obtains the stronger conclusion that B is compact. This latter converse result applies to any B in $B(H)$, self-adjoint or not. The proof, interestingly, used only three operators A: B^*, $-B^*$, and 0.

Let us recall that the essential spectrum $\sigma_e(A)$ of a bounded self-adjoint operator A is its limit-point spectrum and is comprised of its continuous spectrum, cluster points of the spectrum, and eigenvalues of infinite multiplicity. Alternately, a real number λ is in $\sigma_e(A)$ if and only if the spectral measure of every neighborhood containing λ is infinite dimensional. A third characterization, and one that we will need in Section 3, is that of singular sequences: λ is in $\sigma_e(A)$ if and only if there exists a sequence of elements φ_n in the Hilbert space such that $\|\varphi_n\| = 1$, $\varphi_n \xrightarrow{w} 0$, $(A-\lambda)\varphi_n \to 0$. Such a sequence $\{\varphi_n\}$ will be called a *singular sequence* for A at λ.

We refer the reader to the survey paper Gustafson [3] and the references therein for further information on Weyl's Theorems and the essential spectrum. We note here (see [3]) that for non-self-adjoint operators A in a Hilbert space H or a Banach space X there are several (seven) in general different types of essential spectra $\sigma_e^i(A)$ in use, with the largest $\sigma_e^5(A)$ corresponding to the above-described limit-point spectrum. In [3] we treated, for the most part, the essential spectrum $\sigma_e^4(A)$, for reasons of simplicity of exposition and also because $\sigma_e^4(A)$ is the most natural in the context of the Fredholm perturbation theory. The latter is by virtue of the fact that $\sigma_e^4(A) = \bigcap \sigma(A+B) = \bigcap \sigma(A+F)$, where the first intersection is taken over all compact B and the second intersection may be taken over the smaller set of all bounded operators F of finite rank. However, another essential spectrum, and one that we will need in the discussion in Section 2, is the essential spectrum $\sigma_e^3(A)$. This essential spectrum has the property that for $A \in B(X)$ one has $\sigma_e^3(A) = \sigma(\hat{A})$, where $\hat{A}$ denotes the image of A in the Calkin algebra $C(X) = B(X)/K(X)$, $K(X)$ denoting the compact operators on the Banach space X.

2. The generalized problem: Banach Algebras

The above described Weyl–von Neumann Converse Theorem in the form given by Gustafson–Weidmann [2] for operators on a Hilbert space, as described in Section 1 above, was extended to Banach space in Gustafson ([3], Theorem $2'_{00}$): If for B in $B(X)$ one has

$$\sigma_e(A+B) = \sigma_e(A)$$

for all A in $B(X)$, then B is inessential. Recall that an operator T in $B(X)$ is called *inessential* if its image $\hat{T}$ in the Calkin algebra $B(X)/K(X)$ is in the Jacobson radical there; recall also that the Jacobson radical of a Banach algebra with identity I may be characterized as the set of all elements B such that $I+CB$ is invertible for all invertible C, or, what is the same thing, as the set of all elements B such that $A+B$ is invertible for all invertible A.

The proofs in [2] and [3] were entirely different. The result in [2] for $B(H)$ was obtained by reducing the problem for a single operator B to the self-adjoint case, whereas the result in [3] for $B(X)$ necessitated demonstrating that the class of B under consideration formed an ideal of operators each possessing the property $\sigma_e(B) = \{0\}$. Although the result obtained in [3] was for the version σ_e^4 of the essential spectrum being treated in that paper, the proof given there applies in the same way to σ_e^3. One may just delete the "index zero" consideration and recall the characterization of σ_e^3 as a Fredholm spectrum.

Let us give here for completeness an alternate proof of the result of [3] for the version σ_e^3, which as mentioned just above, is an easier case. That is, we may regard σ_e^3 as a Calkin spectrum, rather than as a Fredholm spectrum as was done in [3], and we may make full use of the theory of the radical and in particular the characterization mentioned above, rather than using the properties of the inessential operators themselves as was done in [3]. Then from $\sigma_e^3(A+B) = \sigma_e^3(A)$ for all A in $B(X)$ we have $\sigma(\widehat{A+B}) = \sigma(\hat{A})$ for all $\hat{A}$ in the Calkin algebra $B(X)/K(X)$, from which the result follows.

Recently Zemánek [11], see also Zemánek and Pták [12] and the references therein, has generalized the result of [3] to an arbitrary closed two-sided ideal I in a Banach algebra $\mathscr{B}$: If $|\sigma(\widehat{A+B})| = |\sigma(\hat{A})|$ for all $\hat{A}$ in the quotient algebra $\mathscr{B}/I$, then B is in the Jacobson radical of $\mathscr{B}$. That $\sigma(\widehat{A+B}) = \sigma(\hat{A})$ for all $\hat{A}$ is sufficient would be expected from the known characterizations (e.g., those mentioned above) of the radical, but it should be pointed out that Zemánek's condition, interestingly, required a priori only the equality of the spectral radii rather than the previously supposed equality of spectra.

3. The specialized problem: Singular Sequences

The Weyl–von Neumann Converse Theorem described in Section 1 asserts that if two self-adjoint operators A_2 and A_1 have the same essential spectra

$$\sigma_e(A_2) = \sigma_e(A_1),$$

then A_2 minus a certain unitary equivalent of A_1 will be compact. Simple examples (e.g., take $A_i e_n = \lambda_n^i e_n$, $i = 1, 2$; $n = 1, 2, 3, \ldots$ in l_2 with the λ_n^i countable dense sets appropriately chosen in the unit interval) show that the difference $B = A_2 - A_1$ is not in general compact. That is, on the one hand one needs the unitary change of basis, and on the other hand it is somewhat remarkable that it then does the job so well.

In view of the result of [2], and as mentioned in the introduction, the speculation was that a condition somewhat stronger than $\sigma_e(A_2) = \sigma_e(A_1)$ yet of the same essential type would suffice to characterize the difference $B = A_2 - A_1$ as compact. The precise formulation was: if two bounded self-adjoint operators A_1 and A_2 on a complex Hilbert space H have the same singular sequences, i.e., written symbolically,

$$ss_2(\lambda) = ss_1(\lambda)$$

for all scalar λ, then they differ by a compact operator. If one prefers a statement for a single operator B: any bounded self-adjoint operator which can be written as the difference of two salf-adjoint operators possessing the same singular sequences is compact.

Recall (as stated in Section 1; for a proof see, for example, Riesz–Sz. Nagy [5]) that λ is in the essential spectrum of an operator A if and only if there exists a singular sequence $\{\varphi_n\}$ for A at λ. The condition in the speculation that $ss_2(\lambda) = ss_1(\lambda)$ is taken to mean that at each scalar λ the $\{\varphi_n\}$ for A_2 are the $\{\varphi_n\}$ for A_1, and vice-versa. The condition $ss_2(\lambda) = ss_1(\lambda)$ thus means that $\sigma_e(A_2) = \sigma_e(A_1)$ and in addition says a little more (than just their infinite dimensionality) about the relation between the spectral families $E_2(\lambda)$ and $E_1(\lambda)$ over the essential spectra.

Note that if B is compact then the assertion (which may if you wish be regarded as a theorem of Weyl type) $ss_{A+B}(\lambda) = ss_A(\lambda)$ is valid for any operator A, by the property that B maps weakly convergent sequences into strongly convergent sequences. That is, the speculation is an appropriate converse statement. Note also that if B is to be compact, virtually every singular sequence $\{\varphi_n\}$ must be a singular sequence for B at $\lambda = 0$.

Let us mention now some results for this singular sequence problem for bounded self-adjoint operators that have been obtained by students in Munich and Boulder. Jörgens gave the question to a student W. Tafel in Munich who obtained some partial results in his 1974 Diplomarbeit [6]. In particular it was shown in [6] that if B is a bounded self-adjoint operator on a separable Hilbert space such that $B\varphi_n \to 0$ for all orthonormal bases $\{\varphi_n\}$ in H, then B is compact. After being reminded of the question upon hearing of Tafel's work in [6] I mentioned it later to a student D. Barraza in Boulder who gave (joint with P. Bader; see Barraza [1], Chapter II, Section 3) an example of a bounded self-adjoint operator B on a separable Hilbert space which annihilates (i.e., $B\varphi_n \to 0$) an orthonormal basis $\{\varphi_n\}$ but which is not compact. Tafel [6] also showed that the singular sequence speculation is true in many cases: If A_1 and A_2 are bounded self-adjoint operators on a Hilbert space with the singular sequence property that $ss_{A_1}(\lambda) \subset ss_{A_2}(\lambda)$ and with the additional assumption that the number of limit points of the essential spectrum $\sigma_e(A_1)$ is finite, then the difference $B = A_2 - A_1$ is compact. This contains a result obtained later by Bader and Barraza that the speculation is true for projections.

Let us conclude by also mentioning that Weidmann has given, in a recent (1975) survey [9] of perturbation theory for self-adjoint partial differential equations, a result concerning perturbation of singular sequences. In [9], Appendix, Weidmann showed that for A a self-adjoint operator under perturbation by a symmetric operator B such that B is A-bounded, A^2-compact, and such that $A+B$ is self-adjoint, then not only is $\sigma_e(A+B) = \sigma_e(A)$ but also $ss_{A+B}(\lambda) \subset ss_A(\lambda)$. Let us recall and mention that for unbounded operators A and B the versions of Weyl's theorems that appear involve conditions such as that of B being A-compact, i.e., B is compact on the domain $D(A)$ equipped with the A operator norm; we refer the reader to [3] and [9] and the references therein for further information concerning the versions of these theorems for unbounded operators and for the applications to differential equations.

4. Additional remarks and observations(²)

As mentioned in Section 1 the result in [2] required not all A in $B(H)$ but only $A = B^*, -B^*, 0$. This suggests generalizations and formulations involving conditions involving only operators A in the vector space (or more generously the C^* algebra) generated by B.

In connection with the discussion in Section 2, Zemánek [11] has raised the question of whether for an arbitrary Banach algebra with identity the condition

$$\sigma(A+B) \cap \sigma(A) \neq 0$$

for all A in the algebra implies that B belongs to some proper two-sided ideal.

In a very recent paper [7] Tafel, Voigt, and Weidmann have extended the problem of singular sequences as discussed in Section 3 to unbounded self-adjoint operators: if two unbounded self-adjoint operators A_1 and A_2 have the same singular sequences and their difference $B = A_2 - A_1$ is A_1-bounded with relative bound zero, can it then be concluded that B is A_1-compact? Also in [7] an example is given for bounded self-adjoint operators in which $ss_{A_1}(\lambda) \subset ss_{A_2}(\lambda)$ but the number of limit points of the essential spectrum $\sigma_e(A_1)$ is not finite and the difference $B = A_2 - A_1$ is not compact.

In connection with the related result of Weidmann ([9], Appendix) mentioned in Section 3 above, Weidmann raised the following conjecture: for self-adjoint operators A_1, A_2 and an operator B, such that $D(A_1) = D(A_2) \subset D(B)$, does it follow that B is A_1^2-compact if and only if B is A_2^2-compact? As T. Kato observed (Kato also went a bit further there) in Weidmann ([9], Appendix), the answer is yes in the case that $B = A_2 - A_1$. In a recent preprint [8] Voigt has answered that conjecture negatively with a (rather complicated) counterexample. We should perhaps observe here that in the case that the operator B is closable and $D(A_1^2) = D(A_2^2)$, the answer is affirmative, as is easily seen by use of the closed graph theorem; and (as we have suggested to Voigt in a private communication) this sufficient condition is easily generalized to $D(A_1^p) = D(A_2^p)$ for some $p > 1$ by

(²) Added later.

following the argument in [2], Section 3, Remark (5). However, a more generally formulated theorem would perhaps be desirable — which gave a necessary and sufficient condition appropriately involving domains.

Finally we would like to mention that although the singular sequence problem may remain of interest for normal and some normal-like operators also, it is not of interest very much beyond them. To show this we would like to give a short construction which might be of some use elsewhere in these considerations. Let B be any bounded noncompact operator such that $\sigma_e(B) = 0$ and $B+I$ is invertible. Let A be B^2. It follows immediately (from $A+B = (B+I)B$) that $\sigma_e(A) = \sigma_e(A+B) = 0$ and that $A+B$ and A have the same singular sequences. But B was not compact. One may use for an explicit example the $B = \left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$ of [2]. In that case one even has A $(= 0)$ self-adjoint, $ss_{A+B}(\lambda) = ss_A(\lambda)$, and a negative conclusion for the singular sequence question.

Added in proof. For W^* algebra the question of Zemánek [11] may be answered affirmatively, and negatively for general C^* algebras. For further references see K. Gustafson and M. Seddighin, *Nonperturbing algebras*, to appear, and the survey by the present author to appear in the proceedings of the Spectral Theory Semester (2[nd]), 1982.

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ON THE SPECTRAL PROPERTIES OF TENSOR PRODUCTS OF LINEAR OPERATORS IN BANACH SPACES*

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1. Introduction

Let X and Y be complex Banach spaces and let α be a uniform reasonable norm on $X\otimes Y$. The completion of $X\otimes Y$ with respect to α is denoted by $X\hat{\otimes}_\alpha Y$. Let $A\colon D[A] \subset X \to X$ and $B\colon D[B] \subset Y \to Y$ be densely defined closed linear operators with nonempty resolvent sets. Associated with each polynomial of degrees m in ξ and n in η

$$P(\xi, \eta) = \sum_{jk} c_{jk}\xi^j\eta^k \tag{1.1}$$

is a polynomial operator

$$P\{A\otimes I, I\otimes B\} \equiv \sum_{jk} c_{jk}A^j\otimes B^k \tag{1.2}$$

in $X\hat{\otimes}_\alpha Y$ with domain $D[A^m]\otimes D[B^n]$. In particular, to $\xi+\eta$ and $\xi\eta$ correspond respectively $A\otimes I+I\otimes B$ and $A\otimes B$. The identity operators in both X and Y are denoted by the same I. Assume that (1.2) *is closable in* $X\hat{\otimes}_\alpha Y$ *with closure* $\boldsymbol{P}$. This is the case, for instance, if α is faithful on $X\otimes Y$, i.e. if the natural continuous linear mapping $j_\varepsilon^\alpha\colon X\hat{\otimes}_\alpha Y \to X\hat{\otimes}_\varepsilon Y$ is one-to-one.

We are interested in the problem of what spectral contributions $\boldsymbol{P}$ gets from A and B.

The aim of this note is to make a brief survey of our results ([9], [10], [11], [12]) on the exact representations of the spectrum, essential spectra, approximate point spectrum and approximate deficiency spectrum of $\boldsymbol{P}$ by the parts of the spectra of A and B. By the essential spectra are meant those in the sense of F. E. Browder [3], F. Wolf [22], M. Schechter [18], Gustafson–Weidmann [7] and T. Kato [14]. Further we refer to the formulae expressing the nullity, deficiency and index of $\boldsymbol{P}$ in terms of the quantities concerning A and B.

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