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ON SPECTRAL DISTRIBUTIONS OF DEFINITIZABLE OPERATORS IN KREIN SPACE

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Let X be a Krein space, i.e. a Hilbert space with respect to some scalar product $(\cdot\,,\,\cdot)$, equipped with an indefinite scalar product $[\cdot\,,\,\cdot]$ given by $[x, y] = (Jx, y)$, $x, y \in X$. Here J denotes the difference of two orthogonal projectors P_+, P_- with $P_+ + P_- = I$ (the identity on X): $J = P_+ - P_-$. Let $\varkappa_\pm(X) := \dim P_\pm X \in \{0, 1, \dots, \infty\}$. The quantities $\varkappa_\pm(X)$ are called the *rank of positivity* and the *rank of negativity*, respectively, of the Krein space X. A bounded operator A on X is said to be *J-self-adjoint* if $[Ax, y] = [x, Ay]$, $x, y \in X$. A J-self-adjoint bounded operator A on X is said to be *definitizable*, if there exists a real non-constant polynomial p with property $[p(A)x, x] \geqslant 0$, $x \in X$. The non-real spectrum of a definitizable operator can be proved ([6]) to consist of no more than a finite number of eigenvalues.

In what follows A denotes a bounded definitizable operator. We assume that the spectrum $\sigma(A)$ of A is real (for the following considerations this is, in fact, no restriction).

The spectral function $E(\cdot)$ of a definitizable operator was found by M. G. Krein and H. Langer ([5], [6]). It is a projector-valued interval function defined on all real intervals whose endpoints do not belong to the set $\{\mu_1, \dots, \mu_k\}$ of real zeros of the definitizing polynomial p.

The Riesz–Dunford functional calculus $f \mapsto f(A)$ can be extended (cf. [6], [4]) to an $L(X)$-valued distribution which on $\boldsymbol{R} \setminus \{\mu_1, \dots, \mu_k\}$ provides the measure corresponding to the interval function $E(\cdot)$. This distribution is also denoted by E. It is a spectral distribution in the sense of Foiaş (cf. [2]).

In the case of $\dim X < \infty$ (and for an arbitrary linear operator) the order of E in a neigbourhood of a point $\mu \in \sigma(A)$ is equal to the maximal length of Jordan chains in the root space of this point minus one.

In this note we are concerned with connections between the order of the distribution E on one-sided and deleted neighbourhoods of a point μ_i, $i = 1, \dots, k$, and the length of Jordan chains in certain subspaces of the root space to μ_i.

M. G. Krein and H. Langer ([5], [6]) proved connections of the type considered here in the case of a Pontrjagin space, i.e. $\min(\varkappa_+(X), \varkappa_-(X)) < \infty$. Here we do not assume that X is a Pontrjagin space. But in § 3 we make additional assumptions on the operator A. Under these assumptions, which are close to the case of a Pontrjagin space, we obtain the same relations as in [5], [6]. As a simple consequence we get conditions for the spectrality of definitizable operators. We mention that our results can be applied to the spectral theory of operator pencils having only real zeros (cf. e.g. [8]).

We shall confine ourselves to definitizable operators A with real spectrum such that

$$[A^n x, x] \geqslant 0, \quad x \in X,$$

where n is some non-negative integer. This is no restriction, as can easily be seen. The class of these operators is denoted by $D(0, n)$. Throughout the paper permanent use will be made of results (cf. [6] and also [7], [4]) on the spectral function of $A \in D(0, n)$.

I thank Professor H. Langer for many valuable suggestions.

1. Definitions and some auxiliary results

1.1. For any linear subspace Y of the Krein space X we put

$$Y^{[\perp]} := \{x \in X \colon [x, y] = 0 \text{ for all } y \in Y\}.$$

The subspace $Y \cap Y^{[\perp]}$ is called the *isotropic part of* Y. We set

$$\mathfrak{P}_{\pm} := \{x \in X \colon \pm [x, x] \geqslant 0\} \quad \text{and} \quad \mathfrak{P}_0 = \{x \in X \colon [x, x] = 0\}.$$

A linear space W equipped with an Hermitean sesquilinear form $[\cdot, \cdot]$ is called a *pseudo-Krein space* if W is the direct sum $W = W_1 + W_2$ of a space W_1, on which $[\cdot, \cdot]$ vanishes, and a Krein space W_2 (with respect to a suitable definite scalar product and $[\cdot, \cdot]$). A subspace Y of the Krein space X is a pseudo-Krein space if and only if it can be written as the direct sum of its isotropic part $Y \cap Y^{[\perp]}$ and a subspace of the form PX, where P is a J-self-adjoint projector (cf. e.g. [1]):

$$Y = Y \cap Y^{[\perp]} + PX. \tag{1}$$

Suppose that for a pseudo-Krein subspace Y we have two decompositions

$$Y = Y \cap Y^{[\perp]} + PX, \quad Y = Y \cap Y^{[\perp]} + QX$$

of type (1). Then it is easy to see that $\varkappa_+(PX) = \varkappa_+(QX)$ and $\varkappa_-(PX) = \varkappa_-(QX)$. We denote the quantities $\varkappa_+(PX)$ and $\varkappa_-(PX)$ by $\varkappa_+(Y)$ and $\varkappa_-(Y)$, respectively.

A pseudo-Krein subspace Y of the Krein space X is called *positive* (resp. *negative*) *pseudo-definite* if $Y \subset \mathfrak{P}_+$ (resp. $Y \subset \mathfrak{P}_-$). Note that every closed subspace Y of a Pontrjagin space X is a pseudo-Krein space with $\varkappa_{\pm}(Y) \leqslant \varkappa_{\pm}(X)$.

1.2. Let A be a definitizable operator belonging to $D(0, n)$ and let $E(\cdot)$ be its spectral function. The following subspaces are needed throughout the paper. Denoting open intervals by Δ we set

$$L_1' := \operatorname{span}\{E(\Delta)x \colon \overline{\Delta} \subset (-\infty, 0), x \in X\},$$
$$L_r' := \operatorname{span}\{E(\Delta)x \colon \overline{\Delta} \subset (0, \infty), x \in X\},$$
$$L_{(0)}' := \operatorname{span}\{E(\Delta)x \colon 0 \notin \overline{\Delta}, x \in X\},$$
$$L_1 := \overline{L_1'}, \quad L_r := \overline{L_r'}, \quad L_{(0)} := \overline{L_{(0)}'},$$
$$L_0 := \bigcap_{0 \in \Delta} E(\Delta)X = \{x \in X \colon 0 \notin \overline{\Delta} \Rightarrow E(\Delta)x = 0\},$$
$$L_{01} := L_0 \cap L_1, \quad L_{0r} := L_0 \cap L_r, \quad L_{00} := L_0 \cap L_{(0)}.$$

By well-known properties of E (see [6], [7]) we have

$$L_1 \subset \mathfrak{P}_+, \quad L_r \subset \mathfrak{P}_+ \quad \text{if and only if } n \text{ is even} \tag{2}$$

and

$$L_1 \subset \mathfrak{P}_-, \quad L_r \subset \mathfrak{P}_+ \quad \text{if and only if } n \text{ is odd.} \tag{3}$$

Furthermore, we have $L_{(0)} = \overline{L_1 + L_r}$ and

$$L_0^{[\perp]} = L_{(0)}, \quad L_0^{[\perp]} = L_0 \tag{4}$$

(see [5], [6]). Hence the subspace L_{00} is the isotropic part of L_0. For every non-negative integer i we set

$$N_i := \{x \in X \colon A^i x = 0\}.$$

In accordance with the usual notion of order in distribution theory we define the order of E at 0 to be the minimum of the integers m, such that on some interval $(-\varepsilon, \varepsilon)$, $\varepsilon > 0$, the spectral distribution E is the mth derivative of a bounded ($L(X)$-valued) measure. We denote by μ the order of E at 0. Owing to [4], Satz 4, we have

$$\mu \leqslant n. \tag{5}$$

LEMMA 1. $N_{\mu+1} = N_{\mu+2} = \ldots$

Proof. Let the spectrum $\sigma(A)$ be contained in the interval $[-M, M]$. The sequence $(\psi_{\mu+1,j})_{j=1}^{\infty}$ of real functions defined by

$$\psi_{\mu+1,j}(t) := \begin{cases} (t + j^{-1})^{\mu+1} & \text{if} \quad -M \leqslant t \leqslant -j^{-1}, \\ 0 & \text{if} \quad -j^{-1} \leqslant t \leqslant j^{-1}, \\ (t - j^{-1})^{\mu+1} & \text{if} \quad j^{-1} \leqslant t \leqslant M \end{cases}$$

converges in $C^{\mu}([-M, M])$ to the function $t \mapsto t^{\mu+1}$. Since the spectral distribution E is the μth derivative of a bounded measure, we have

$$\lim_{j \to \infty} \psi_{\mu+1,j}(A)x = A^{\mu+1}x, \quad x \in X. \tag{6}$$

If $A^{\mu+2}y = 0$ for some $y \in X$, then, obviously, $\psi_{\mu+1,j}(A)y = 0$ for all $j = 1, 2, \ldots$ and hence by (6) $A^{\mu+1}y = 0$.

We have shown that the root space of A to 0 is $N_{\mu+1} = N_{n+1}$. The following lemma asserts that the subspace L_0 defined above coincides with the root space of A to 0.

LEMMA 2 ([6]). $L_0 = N_{\mu+1}$.

Proof. 1. Let $x \in N_{\mu+1}$ and let Δ be a real interval with $0 \notin \bar{\Delta}$. Then we have

$$E(\Delta)x = A^{\mu+1}(A^{-(\mu+1)}E(\Delta))x = (A^{-(\mu+1)}E(\Delta))A^{\mu+1}x = 0.$$

Thus $x \in L_0$.

2. Let $x \in L_0$. We consider the sequence $(\psi_{\mu+1,j})_{j=1}^{\infty}$ from the proof of Lemma 1. Obviously, we have $\psi_{\mu+1,j}(\Delta)x = 0$ for all $j = 1, 2, \ldots$ and hence by (6) it follows that $x \in N_{\mu+1}$.

We shall need the following lemma, which can partly be found in [6].

LEMMA 3. *Let k be a non-negative integer. The following assertions are equivalent:*

I_k^r: *The integral $\int_{(0,\infty)} t^k dE(t)$ converges in the weak sense.*

II_k^r: *The integral $\int_{(0,\infty)} t^k dE(t)$ converges in the strong sense.*

III_k^r: *The interval function $A^k E(\cdot)$ on $(0, \infty)$ is bounded.*

IV_k^r: *The distribution E on $(0, \infty)$ is the k-th derivative of a bounded measure on $(0, \infty)$.*

Proof. It was shown in [4] that III_k^r is equivalent to IV_k^r. Evidently, I_k^r is equivalent to

I': *For every $\varepsilon > 0$ and every $x \in X$ there exists a $\delta > 0$ such that $[A^k E(\Delta)x, x] \leqslant \varepsilon$ for every interval Δ with $\bar{\Delta} \subset (0, \delta)$.*

Therefore, the assertions I_k^r and III_k^r are equivalent.

Now the lemma will be proved if we can show that I' and III_k^r imply II_k^r. Making use of the Schwartz inequality

$$\begin{aligned}\|A^k E(\Delta)x\|^2 &= [A^k E(\Delta)x, JA^k E(\Delta)x] \\ &\leqslant [A^k E(\Delta)x, x]^{1/2} [A^k E(\Delta)JA^k E(\Delta)x, JA^k E(\Delta)x]^{1/2},\end{aligned}$$

we infer from I' and III_k^r that: For every $\varepsilon > 0$ and every $x \in X$ there exists a $\delta > 0$ such that $\|A^k E(\Delta)x\| \leqslant \varepsilon$ for every interval Δ with $\bar{\Delta} \subset (0, \delta)$. Thus II_k^r is valid.

Obviously, the lemma remains true if we replace the interval $(0, \infty)$ by $(-\infty, 0)$. The assertions with $(-\infty, 0)$ instead of $(0, \infty)$ are denoted by I_k^l, II_k^l, III_k^l, IV_k^l, respectively. The conjunctions of I_k^r and I_k^l, ..., IV_k^r and IV_k^l are denoted by $I_k, \ldots, IV_k$.

The minimum of the non-negative integers k for which one of the assertions $I_k^r, \ldots, IV_k^r$ is valid is called the *right order* of E at 0 and is denoted by μ_r. Replacing $I_k^r, \ldots, IV_k^r$ by $I_k^l, \ldots, IV_k^l$ and $I_k, \ldots, IV_k$ we define the *left order* μ_l and the *reduced* order μ_0, respectively. We have

(7) $$\mu_r,\ \mu_l \leqslant \mu_0 \leqslant \mu.$$

Obviously, an operator $A \in D(0, n)$ is spectral if and only if $\mu_0 = 0$, and A is scalar if and only if $\mu = 0$.

In what follows we need notations for the maximal lengths of those parts of the Jordan chains of A to the eigenvalue 0 which lie in the subspaces of L_0 defined above:

$$\nu := \min\{i\colon A^i L_0 = \{0\}, i = 0, 1, \ldots\},$$
$$\nu_q := \min\{i\colon A^i L_{0q} = \{0\}, i = 0, 1, \ldots\},$$

where q stands for one of the symbols r, l, 0. The integer ν is called the *index* of 0 and ν_r, ν_l, ν_0 are called the *right index* of 0, the *left index* of 0, and the *reduced index* of 0, respectively.

Obviously,

$$\nu_r, \nu_l \leqslant \nu_0 \leqslant \nu.$$

2. Connections between order and index quantities; general case

In the following we are concerned with the connections between the order quantities μ_r, μ_l, μ_0 and the index quantities ν_r, ν_l, ν_0.

THEOREM 1. *Let $A \in D(0, n)$. Then the following holds.*

(a) $\nu_r \leqslant [\frac{1}{2}(\mu_r+1)]$, $\nu_l \leqslant [\frac{1}{2}(\mu_l+1)]$.

(b) *If n is even* (cf. (2)), *then*

$$\nu_0 \leqslant [\tfrac{1}{2}(\mu_0+1)].$$

(c) *If n is odd* (cf. (3)), *then*

$$\nu_0 \leqslant \min\{[\tfrac{1}{2}\mu_0+1], \mu_0\}.$$

Here the square brackets [ ] *stand for "integral part".*

Proof. First, let μ_r be an even integer and let $x \in L_{0r}$. There exist a sequence (x_i) of elements of L_r' and a sequence (Δ_i) of open intervals with $\bar{\Delta}_i \subset (0, \infty)$ such that $x_i = E(\Delta_i)x_i$ and $x_i \to x$, $i \to \infty$. Then for every $y \in X$ we have

$$|[A^{\mu_r/2}x_i, y]| = |[x_i, A^{\mu_r/2}E(\Delta_i)y]| \leqslant [x_i, x_i]^{1/2}\,[A^{\mu_r}E(\Delta_i)y, y]^{1/2}.$$

Since $[x_i, x_i] \to [x, x] = 0$, $i \to \infty$, and the interval function $A^{\mu_r}E(\cdot)$ is bounded on $(0, \infty)$, we obtain $A^{\mu_r/2}x = 0$, and hence $\nu_r \leqslant \frac{1}{2}\mu_r$ or, equivalently, $\nu_r \leqslant [\frac{1}{2}(\mu_r + 1)]$.

For odd μ_r we get $\nu_r \leqslant \frac{1}{2}(\mu_r+1)$ in the same way. This proves the first inequality of (a).

The proof of the remaining relations is similar. In the case (c) one has to use the form $[A\cdot, \cdot]$ instead of $[\cdot, \cdot]$.

Lemmas 1 and 2 imply the inequality $\nu \leqslant \mu+1$. Under an additional assumption we can easily obtain another estimate for the index ν: Making use of Theorem 1 and of [1], Theorem IX.4.9, we verify the following

COROLLARY. *Let $A \in D(0, n)$ and let L_0 be a pseudo-Krein space with $\varkappa(L_0) = \min(\varkappa_+(L_0), \varkappa_-(L_0)) < \infty$. Then we have*

(8) $$\nu \leqslant [\tfrac{1}{2}(\mu_0+1)] + 2\varkappa(L_0) + 1 \quad \textit{for even } n$$

and

(9) $$\nu \leqslant \min\{[\tfrac{1}{2}\mu_0+1], \mu_0\}+2\varkappa(L_0)+1 \quad \textit{for odd } n.$$

Now we assume that the root space L_0 is a pseudo-Krein space. If L_0 is finite-dimensional or if X is a Pontrjagin space, this condition is fulfilled. Under this assumption we can estimate the reduced index ν_0 from below:

THEOREM 2. *Let $A \in D(0, n)$ and let L_0 be a pseudo-Krein space. Then we have*
$$\mu_0 \leqslant 2\nu_0 \quad \textit{for even } n \text{ (cf. (2))}$$
and
$$\mu_0 \leqslant 2\nu_0+1 \quad \textit{for odd } n \text{ (cf. (3))}.$$

Proof. One easily verifies that X is the orthogonal sum

(10) $$X = \overline{L_0+L_{(0)}} \oplus JL_{00}$$

with respect to the definite scalar product. Let
$$\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$$
be the operator matrix of A with respect to (10). The relation $A^{\nu_0}L_{00} = \{0\}$ and the J-self-adjointness of A imply $A_{22}^{\nu_0} = 0$. Thus we have

(11) $$A^{\nu_0}X \subset \overline{L_0+L_{(0)}}.$$

Let $y \in \overline{L_0+L_{(0)}}$. Since L_0 is a pseudo-Krein space, by (4) the element y can be written as a sum $y = u+v$, where $u \in L_0$ and $v \in L_{(0)}$. Let Δ be an interval with $0 \notin \overline{\Delta}$. If n is even, we have
$$[E(\Delta)y, y] = [E(\Delta)v, v] \leqslant [E(\Delta)v, v]^{1/2}[v, v]^{1/2}.$$
Thus
$$[E(\Delta)y, y] \leqslant [v, v].$$
Then from (11) we infer that the set $\{[A^{2\nu_0}E(\Delta)x, x]\colon 0 \notin \overline{\Delta}\}$ is bounded for any $x \in X$. If n is odd, we have
$$[AE(\Delta)y, y] = [AE(\Delta)v, v] \leqslant [AE(\Delta)v, v]^{1/2}\,[Av, v]^{1/2}.$$
Thus
$$[AE(\Delta)y, y] \leqslant [Av, v].$$
Then from (11) we see that the set $\{[A^{2\nu_0+1}E(\Delta)x, x]\colon 0 \notin \overline{\Delta}\}$ is bounded for any $x \in X$.

Theorems 1 and 2 yield:

THEOREM 3. *Let $A \in D(0, n)$ and let L_0 be a pseudo-Krein space. Then we have*
(a)

(12) $$\nu_r \leqslant [\tfrac{1}{2}(\mu_r+1)], \quad \nu_l \leqslant [\tfrac{1}{2}(\mu_l+1)].$$

(b) *If n is even* (cf. (2)), *then*
$$\nu_0 = [\tfrac{1}{2}(\mu_0+1)].$$

(c) *If n is odd* (cf. (3)), *then*

(13) $$\nu_0 \leqslant [\tfrac{1}{2}\mu_0+1] \leqslant \nu_0+1 \quad \textit{and} \quad \nu_0 \leqslant \mu_0.$$

Remark. It can be seen that the estimates (12) and (13) are sharp. Indeed, for an operator of the special class considered in § 3 the equality sign holds in (12). Further, for an operator of this class and odd μ_0 we have $\nu_0 = [\frac{1}{2}\mu_0+1]$. It is easy to construct operators of $D(0, 1)$ with $\mu_0 = 1$ and $\nu = \nu_0 = 0$, in other words: J-positive operators which are invertible and not spectral. See, for example, [3]. For these operators the equality sign holds in $[\frac{1}{2}\mu_0+1] \leqslant \nu_0+1$.

Theorem 3 implies a criterion for spectrality: For even n an operator $A \in D(0, n)$ whose root space is a pseudo-Krein space is spectral if and only if it has no eigenvectors in the isotropic part of the root space L_0.

3. Connections between order and index quantities; a special class of operators

3.1. Now, imposing stronger conditions on our operator A, we shall prove some closer connections between the order and the index quantities. We suppose that the subspaces L_r and L_l are pseudo-definite. If X is a Pontrjagin space, this condition is fulfilled for every $A \in D(0, n)$.

THEOREM 4. *Let $A \in D(0, n)$ and let the subspace L_r (resp. L_l) be pseudo-definite. Then we have*

(14) $$\nu_r = [\tfrac{1}{2}(\mu_r+1)] \quad (\textit{resp. } \nu_l = [\tfrac{1}{2}(\mu_l+1)]).$$

Remark. We can write the relations (14) in the form (cf. Lemma 3)
$$\nu_r = \min\{k\colon \mathrm{III}_{2k}^r \text{ is valid}\},$$
$$\nu_l = \min\{k\colon \mathrm{III}_{2k}^l \text{ is valid}\}.$$

Proof. The proof is given for ν_r and μ_r. In the other case a similar reasoning applies. In view of Theorem 1(a) it remains to prove the inequality

(15) $$\mu_r \leqslant 2\nu_r.$$

Let P be a J-self-adjoint projector such that (1) holds with $Y = L_r$. Let Δ be an interval with $\overline{\Delta} \subset (0, \infty)$ and let $x, y \in X$. Since $A^{\nu_r}L_{0r} = \{0\}$ and $(I-P)L_r \subset L_{0r}$, we obtain

(16) $$[x, PA^{\nu_r}E(\Delta)y] = [A^{\nu_r}E(\Delta)Px, y] = [A^{\nu_r}\{P+(I-P)\}E(\Delta)Px, y] = [A^{\nu_r}PE(\Delta)Px, y].$$

It is easy to see that the operators $PE(\Delta)P$, $\overline{\Delta} \subset (0, \infty)$, are orthogonal projectors in the Hilbert space $(PX, [\cdot, \cdot])$ and, therefore, uniformly bounded. Then from (16) it follows that the operators $PA^{\nu_r}E(\Delta)$, $\overline{\Delta} \subset (0, \infty)$, are uniformly bounded.

For every interval Δ with $\overline{\Delta} \subset (0, \infty)$ we have

(17) $$[PA^{\nu_r}E(\Delta)x, PA^{\nu_r}E(\Delta)x] = [A^{2\nu_r}E(\Delta)x, x].$$

Making use of this equation we see that the operators $A^{2\nu_r}_{u}E(\Delta)$, $\overline{\Delta} \subset (0, \infty)$, are uniformly bounded. This proves (15).

Now under our stronger assumptions we are going to show a closer relation between the reduced order μ_0 and the reduced index ν_0 of $A \in D(0, n)$ for odd n. Such a relation results immediately from the following lemma.

LEMMA 4. *Let $A \in D(0, n)$. Let the subspaces L_r and L_l be pseudo-definite, and let P_r and P_l be some corresponding* (cf. (1)) *J-self-adjoint projectors.*

Then $L_{(0)}$ and L_0 are pseudo-Krein spaces and $L_{(0)}$ can be written as the direct sum

(18) $$L_{(0)} = L_{00}+P_rX+P_lX,$$

where the summands are pairwise J-orthogonal. Further, we have

(19) $$L_{00} = \overline{L_{0r}+L_{0l}}.$$

Hence

(20) $$\nu_0 = \max\{\nu_r, \nu_l\}.$$

Proof. Let $x \in L_{(0)}$. There exist sequences $y_i \in L_r$ and $z_i \in L_l$, $i = 1, 2, \ldots$, such that $y_i+z_i \to x$ for $i \to \infty$. Then $(P_r y_i)$ and $(P_l z_i)$ converge to $P_r x$ and $P_l x$, respectively. Thus

$$(y_i+z_i-P_r y_i-P_l z_i) = \big((I-P_r)y_i+(I-P_l)z_i\big)$$

also converges and we have

$$x = \lim_{i\to\infty}[(I-P_r)y_i+(I-P_l)z_i]+P_r x+P_l x.$$

Consequently, $L_{(0)}$ has the decomposition (18) and $L_{(0)}$ is a pseudo-Krein space. Setting above $x \in L_{00}$, we obtain $L_{00} \subset \overline{L_{0r}+L_{0l}}$ and hence (19).

By (4) it easily follows that L_0 is a pseudo-Krein space.

Making use of Theorem 4, Lemma 4, and of the obvious relation $\mu_0 = \max\{\mu_r, \mu_l\}$, we get the following theorem, which is a slight generalization of a result of M. G. Krein and H. Langer ([5], [6]).

THEOREM 5 (M. G. Krein, H. Langer). *Let $A \in D(0, n)$. Suppose that the subspaces L_r and L_l are pseudo-definite. Then we have*

$$\nu_0 = [\tfrac{1}{2}(\mu_0+1)].$$

Remark. We can write the preceding equation in the form

$$\nu_0 = \min\{k\colon \mathrm{III}_{2k} \text{ is valid}\}$$

(cf. Lemma 3).

COROLLARY 1. *Let $A \in D(0, n)$ and let the subspaces L_r and L_l be pseudo-definite. If $\varkappa(L_0) = \min\{\varkappa_+(L_0), \varkappa_-(L_0)\} < \infty$, then we have*

$$\nu \leqslant [\tfrac{1}{2}(\mu_0+1)]+2\varkappa(L_0)+1$$

(cf. (8), (9)).

COROLLARY 2. *Let $A \in D(0, n)$ and let the subspaces L_r and L_l be pseudo-definite. Then A is spectral if and only if it has no eigenvector in the isotropic part of the root space.*

3.2. We shall see from the following example that, generally, Theorem 4, (19) and (20) do not hold without our pseudo-definiteness condition.

EXAMPLE. We construct a compact operator $A_1 \in D(0, 1)$ such that

(i) $\mu_0 = \mu_r = \mu_l = 1$,

(ii) $\nu_0 = 1$, $\nu_r = \nu_l = 0$,

(iii) $L_0 = L_{00}$, $\dim L_0 = 1$.

Let Y be a complex Hilbert space. Consider the Hilbert space $Z := Y\oplus Y\oplus \boldsymbol{C}\oplus\boldsymbol{C}$. We introduce an indefinite scalar product on Z by setting

$$J := \begin{pmatrix} I & 0 & 0 & 0 \\ 0 & -I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & -I \end{pmatrix}$$

and $[z_1, z_2] := (Jz_1, z_2)$, $z_1, z_2 \in Z$.

First, we are going to define some subspaces L_+ and L_- of Z. Let $(e_i)_1^\infty$ be a complete orthonormal system of Y and let g be an element of $\boldsymbol{C}\oplus\boldsymbol{C}$ with $[g, g]_{\boldsymbol{C}\oplus\boldsymbol{C}} = 0$ ($[\cdot, \cdot]_{\boldsymbol{C}\oplus\boldsymbol{C}}$: induced indefinite scalar product) and $\|g\| = 1$. Let $(\gamma_i)_1^\infty$ and $(\sigma_i)_1^\infty$ be sequences of positive numbers with the properties $\gamma_i \in (\frac{1}{2}, 1)$, $\gamma_i \to 1$ for $i \to \infty$, $\sum_i \sigma_i^2 < \infty$ and

(21) $$(1-\gamma_i)\sigma_i^{-1} \to 0, \quad i \to \infty.$$

Define a self-adjoint operator $B \in L(Y)$ by $Be_i = \gamma_i e_i$, $i = 1, 2, \ldots$, and put $s = \sum_i \sigma_i e_i \in Y$. We write the elements of $Z = Y\oplus Y\oplus \boldsymbol{C}^2$ as triples and define the following two closed linear subspaces of Z:

$$L_+ := \left\{\begin{pmatrix} y \\ By \\ (y+By, s)g \end{pmatrix}\colon y \in Y\right\}, \quad L_- := \left\{\begin{pmatrix} By \\ y \\ -(y+By, s)g \end{pmatrix}\colon y \in Y\right\}.$$

Obviously, $[L_+, L_-] = \{0\}$ and $L_\pm \subset (\mathfrak{P}_\pm \setminus \mathfrak{P}_0)\cup\{0\}$, and hence

(22) $$L_\pm \cap L_\pm^{[\perp]} = \{0\}.$$

We put

$$\psi_{+,i} := \begin{pmatrix} e_i \\ \gamma_i e_i \\ (e_i+\gamma_i e_i, s)g \end{pmatrix} \in L_+, \quad \psi_{-,i} := \begin{pmatrix} \gamma_i e_i \\ e_i \\ -(e_i+\gamma_i e_i, s)g \end{pmatrix} \in L_-, \quad i = 1, 2, \ldots$$

Note that $[\psi_{+,j}, \psi_{+,k}] = [\psi_{-,j}, \psi_{-,k}] = 0$ for $j \neq k$. Making use of (21), one easily verifies the convergence

$$\tfrac{1}{2}(\sigma_i(1+\gamma_i))^{-1}(\psi_{+,i}-\psi_{-,i}) \to \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}, \quad i \to \infty.$$

It follows that

$$\overline{L_+ + L_-} = Y \oplus Y \oplus \langle g \rangle, \tag{23}$$

where $\langle a \rangle$ denotes the linear span of an element a.

We are now coming to the definition of the announced operator A_1. Denoting by $P_{\pm,i}$ the J-self-adjoint projector on $\langle \psi_{\pm,i} \rangle$, we choose a sequence (ν_i) of positive numbers with the properties

$$\sum_{i=1}^{\infty} \nu_i \|P_{+,i}\| < \infty \quad \text{and} \quad \sum_{i=1}^{\infty} \nu_i \|P_{-,i}\| < \infty.$$

Obviously, the operator

$$A_1 := \sum_{i=1}^{\infty} \nu_i P_{+,i} - \sum_{i=1}^{\infty} \nu_i P_{-,i}$$

is compact and belongs to $D(0, 1)$. Let $E(\cdot)$ be the spectral function of A_1. We have $E(\Delta) = \sum_{\nu_i \in \Delta} P_{+,i}$, $\overline{\Delta} \subset (0, \infty)$. In the case $\overline{\Delta} \subset (-\infty, 0)$ a similar relation holds. We easily obtain $\mu_0 = \mu_r = \mu_l = 1$. By the definition of A_1 and by (23) we have

$$L_r = L_+, \quad L_l = L_-, \quad L_{(0)} = Y \oplus Y \oplus \langle g \rangle.$$

Hence, on account of (22),

$$L_{0r} = \{0\}, \quad L_{0l} = \{0\}, \quad L_{00} = \{0\} \oplus \{0\} \oplus \langle g \rangle = L_0.$$

Thus $\nu_r = \nu_l = 0$, $\nu_0 = 1$.

3.3. Finally, assuming the pseudo-definiteness of the spaces L_r and L_l, we give assertions that are equivalent to III^r_{2k}, III^l_{2k} and III_{2k}, respectively. Hence by the remarks on Theorems 4 and 5 we get another formulation of these theorems.

LEMMA 5. *Let $A \in D(0, n)$ and let the subspaces L_r and L_l be pseudo-definite. Then* III^r_{2k} (resp. III^l_{2k}) *is equivalent to: For every* $z \in L_{0r}$ (resp. $z \in L_{0l}$) *the interval function $A^{2k}E(\cdot)Jz$ is bounded on* $(0, \infty)$ (*resp. on* $(-\infty, 0)$).

Proof. Suppose that the interval functions $A^{2k}E(\cdot)Jz$, $z \in L_{0r}$, are bounded on $(0, \infty)$. We have to show that $A^{2k}E(\cdot)x$ is bounded on $(0, \infty)$ for every $x \in X$.

Let P_r and P_l be as in Lemma 4. According to (17) the lemma will be proved if we can show that $P_r A^k E(\cdot)x$ is bounded on $(0, \infty)$ for every $x \in X$. Owing to (10) and to (18), we can write an element $x \in X$ in the form

$$x = P_r x + P_l x + u + Jw + Jz,$$

where $u \in L_0$, $z \in L_{0r}$, $w \in L_{00} \ominus L_{0r}$.

Obviously the interval functions $P_r A^k E(\cdot) P_l x$ and $P_r A^k E(\cdot) u$ vanish on $(0, \infty)$. We have $P_r A^k E(\cdot) P_r x = P_r A^k P_r E(\cdot) P_r x$ on $(0, \infty)$. The operators $P_r E(\Delta) P_r$, $\overline{\Delta} \subset (0, \infty)$, being orthogonal projectors in the Hilbert space $(P_r X, [\cdot, \cdot])$, are uniformly bounded. Therefore, the interval function $P_r A^k E(\cdot) P_r x$ is bounded on $(0, \infty)$.

Further, we have

$$\begin{aligned}[P_r A^k E(\Delta) Jw, y] &= [Jw, E(\Delta) A^k P_r y] \\ &= (w, (I-P_r)E(\Delta)A^k P_r y) + [Jw, P_r E(\Delta) P_r A^k P_r y] \\ &= [Jw, P_r E(\Delta) P_r A^k P_r y]\end{aligned}$$

for $\overline{\Delta} \subset (0, \infty)$ and $y \in X$. Hence by the same argument the interval function $P_r A^k E(\cdot) Jw$ is bounded on $(0, \infty)$.

Summing up, we conclude that $A^{2k}E(\cdot)x$ is bounded on $(0, \infty)$.

The assertion for III^l_{2k} can be shown in the same way.

THEOREM 6. *Let $A \in D(0, n)$ and let the subspaces L_r and L_l be pseudo-definite. Then ν_0 (resp. ν_r, ν_l) is equal to the least integer k such that the interval function $A^{2k}E(\cdot)Jz$ is bounded on $(-\infty, 0) \cup (0, \infty)$ (resp. $(0, \infty)$, $(-\infty, 0)$) for every $z \in L_{00}$ (resp. L_{0r}, L_{0l}).*

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