Stable shape concordance implies homeomorphic complements

by

Vo Thanh Liem (University) and **Gerard A. Venema** * (Grand Rapids)

Abstract. Let X and Y be compacta satisfying the inessential loops condition in the interior of a piecewise linear m-manifold M, $m \neq 3$ or 4, with $\mathrm{Fd}(X) = k \leqslant m-3$. If X and Y are shape concordant, and if pro-$\pi_i(X)$ is stable for $0 \leqslant i < r$ and Mittag-Leffler for $i = r$, where $r = 2k+2-m$, then $M-X$ is homeomorphic to $M-Y$.

1. Introduction. In this paper we prove a complement theorem for shape concordant compacta in a PL manifold. The main theorem (Theorem 5) is a generalization of recent results of Sher [S_1] and Liem [L]. Sher has shown [S_1] that if X_0 and X_1 are compact subsets of a PL n-manifold M^n, $n \geqslant 6$, such that both X_0 and X_1 satisfy the inessential loops condition (ILC), have the shape of a finite polyhedron K^k with $k \leqslant n-3$, and are shape concordant via an ILC compactum $Z \subset M \times I$, then $M-X_0 \cong M-X_1$. Liem subsequently improved Sher's theorem by showing [L] that the hypothesis that Z satisfy ILC in $M \times I$ could be dropped.

Our theorem generalizes that of Liem in two ways. First, the condition $n \geqslant 6$ is changed to $n \neq 4$. Second, the assumption of polyhedral shape is replaced by a considerably weaker condition on the homotopy pro-groups. We prove that if X_0 and X_1 are ILC compacta of fundamental dimension k in a PL n-manifold M^n, $k \leqslant n-3$, $n \neq 4$, if pro-$\pi_i(X_0)$ is stable for $i \leqslant 2k+1-n$ and Mittag-Leffler for $i = 2k+2-n$, and if X_0 and X_1 are shape concordant in M, then $M-X_0 \cong M-X_1$. We also prove that two shape concordant weak Z-set compacta in a Hilbert cube manifold have homeomorphic complements (Theorem 7).

In addition to being a direct generalization of the theorems of Liem and Sher, our theorem also indirectly implies most of the other known finite dimensional complement theorems. For example, the main result of [ISV] can be obtained as follows. Suppose X_0 and X_1 are two ILC compacta in E^n having fundamental dimension at most k where $k \leqslant n-3$ and $n \geqslant 5$. If $\mathrm{Sh}(X_0) = \mathrm{Sh}(X_1)$ and X_0 is

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$(2k-n+2)$-shape connected, then [HI], Corollary 2 gives a shape concordance from X_0 to X_1. So our theorem applies and we conclude that $E^n-X_0 \cong E^n-X_1$.

A compact subset X of a manifold M is said to satisfy the *incessential loops condition* (abbreviated ILC) if for every neighborhood U of X there is a neighborhood V of X such that each loop in $V-X$ which is null-homotopic in V is null-homotopic in $U-X$. Throughout this note, $I = [0, 1]$. Two compacta X_0, X_1 in the interior of a manifold M are said to be *shape concordant* if there is a compactum Z in $M\times I$ such that $X_\lambda\times\{\lambda\} = Z\cap(M\times\{\lambda\})\subset Z$ is a shape equivalence for each $\lambda = 0, 1$. Similarly, we can define the notion of shape concordance in Hilbert cube manifold theory.

We will work with pointed topological spaces; however, we will suppress the base points from our notations. We assume that the reader is familiar with the fundamentals of shape [B] and ANR-systems [MS]. For standard notions and notations in piecewise linear (abbreviated PL) topology and Hilbert cube manifold theory, we refer to [Hd$_1$] and [Ch] respectively if it is not specified otherwise.

A map $f\colon X\to Y$ between ANR's is *r-connected* if $f_*\colon \pi_i(X)\to\pi_i(Y)$ is an isomorphism when $0\leqslant i\leqslant r-1$ and an epimorphism when $i = r$. A shape morphism $f\colon X\to Y$ between pointed 1-movable continua is *shape r-connected* if $f_*\colon$ pro-$\pi_i(X)\to$ pro-$\pi_i(Y)$ is an isomorphism of pro-groups for $0\leqslant i\leqslant r-1$ and an epimorphism for $i = r$. Recall that a pro-group $\underline{G} = \{G_\alpha, g_{\alpha\beta}, A\}$ is *stable* if $\underline{G}$ is isomorphic in the category of pro-groups to a group, and that $\underline{G}$ satisfies the *Mittag-Leffler condition* if for each $\alpha\in A$ there is a $\beta\geqslant\alpha$ such that for all $\gamma\geqslant\beta$, $g_{\alpha\gamma}(G_\gamma) = g_{\alpha\beta}(G_\beta)$.

Let X be a compactum in Hilbert cube (or PL) manifold M. By a *defining sequence* for X in M, we mean a sequence $\{U_n|\ n = 1, 2, \ldots\}$ of compact Hilbert cube (or PL) manifold neighborhoods of X such that $U_{n+1}\subset \mathrm{Int}_M U_n$ for $n = 1, 2, \ldots$, and that $X = \bigcap\{U_n|\ n = 1, 2, \ldots\}$. A defining sequence is said to be *r-connected* if $U_{n+1}\subset U_n$ is r-connected for each $n = 1, 2, \ldots$ The *fundamental dimension of* X is defined by $\mathrm{Fd}(X) = \min\{\dim Y|\ Y$ is a compactum and $\mathrm{Sh}(X) = \mathrm{Sh}(Y)\}$. We use the notation $X\cong Y$ to mean that X and Y are homeomorphic spaces and the notation $f\simeq g$ to mean that f and g are homotopic maps.

2. Constructing polyhedral concordances. In this section we show how to use the shape concordance Z to construct a polyhedral concordance from a polyhedron approximating X_0 to a polyhedron approximating X_1. The proof of the first lemma is similar to that of [V$_2$, Lemma 3.6].

Remark. We assume that pro-$\pi_1(X_0)$ satisfies the Mittag-Leffler condition in order to overcome the base point problem which arises. This is the standard assumption which is used to handle base point problems and implies that X_0 is pointed 1-movable. (See [DS], Lemma 7.1.2, and [DS], Theorem 7.1.3, for example.) In the proof of our main theorem the compacta either have fundamental dimension in the trivial range (in which case the problem does not arise) or else have pro-π_1 which satisfies the Mittag-Leffler condition.

Lemma 1. *Suppose X_0 and X_1 are continua in the* PL *manifold M^n,* pro-$\pi_1(X_0)$ *satisfies the Mittag-Leffler condition, and X_0 and X_1 are shape concordant via a compactum $Z\subset M\times I$. Then for every neighborhood U of Z in $M\times I$, there exists a neighborhood V of Z in U such that the inclusion-induced homomorphism*

$$\pi_i(V, V\cap(M\times\{1\}))\to\pi_i(U, U\cap(M\times\{1\}))$$

is trivial for every $i\geqslant 0$.

Proof. The case $i = 0$ is trivial, so we begin with the case $i = 1$.

Pick base points $z\in Z$ and $x\in X_1\times\{1\}$ for pro-$\pi_1(Z)$ and pro-$\pi_1(X_1\times\{1\})$ respectively. Let U be given. Suppose $f = \{f_j\}$ is a fundamental sequence from Z to $X_1\times\{1\}$ which is a shape-inverse of the inclusion-induced sequence $\underline{i}\colon X_1\times\{1\}\to Z$. We may assume that each f_j is a map of the form $f_j\colon U\to M\times\{1\}$.

Use the fact that pro-$\pi_1(Z)$ is Mittag-Leffler to choose a neighborhood V_1 of Z in U having the following property: If V is any neighborhood of Z in $M\times I$ and l is a loop in V_1 based at z, then l is homotopic in U (rel z) to a loop in V. Next, choose a neighborhood V of Z and an integer j such that $f_j|V\simeq \mathrm{id}_V$ in V_1 and $f_j|V\cap(M\times\{1\})$ is homotopic to $\mathrm{id}_{V\cap(M\times\{1\})}$ in $V_1\cap(M\times\{1\})$. We may assume that $f_j(z) = x$.

Now suppose that $g\colon (\Delta^1, \partial\Delta^1)\to(V, V\cap(M\times\{1\}))$ where $\Delta^1 = [0, 1]$. Adjust g if necessary so that $g(0) = x$. First, observe that $g|\partial\Delta^1$ can be extended to a map $g_1\colon \Delta^1\to V_1\cap(M\times\{1\})$. The reason is that $g|\partial\Delta^1\simeq f_jg|\partial\Delta^1$ and $f_jg|\partial\Delta^1$ extends to a map of Δ^1 into $V_1\cap(M\times\{1\})$ (namely f_jg). Let a be the path traced out by z during the homotopy from id_V to $f_j|V$ and let b be the loop which consists of $g(\Delta^1)$ plus $g_1(\Delta^1)$ with reverse orientation. Then, aba^{-1} is a loop based at z and contained in V_1. Thus aba^{-1} is homotopic in U (rel z) to a loop c in V (by the choice of V_1). Furthermore, the choice of a shows that $c\simeq a[f_j(c)]a^{-1}$ (rel z). Therefore

$$b\simeq a^{-1}aba^{-1}a\simeq a^{-1}ca\simeq a^{-1}a[f_j(c)]a^{-1}a\simeq f_j(c),$$

all homotopies being in U and rel x. Thus g is homotopic in U (rel endpoints) to the path consisting of the loop $f_j(c)$ followed by $g_1(\Delta^1)$. This completes the proof of the case $i = 1$.

Now suppose $i\geqslant 0$ and that the neighborhood U has been given. Use the first part of the proof to choose a neighborhood V_1 of Z such that the inclusion induced map $\pi_1(V_1, V_1\cap(M\times\{1\}))\to\pi_1(U, U\cap(M\times\{1\}))$ is trivial. Next choose a neighborhood V_2 of Z and an integer j such that $f_j|V_2\simeq\mathrm{id}_{V_2}$ in V_1. Finally, choose a neighborhood V and an integer k such that $f_k|V\cap(M\times\{1\})\simeq\mathrm{id}_{V\cap(M\times\{1\})}$ in $V_2\cap(M\times\{1\})$.

Let $g\colon (\Delta^i, \partial\Delta^i)\to(V, V\cap(M\times\{1\}))$ be given, where Δ^i is an i-simplex. Exactly as in the proof of the case $i = 1$, the choice of V guarantees that $g|\partial\Delta^i$ extends to a map $g_1\colon \Delta^i\to V_2\cap(M\times\{1\})$. Let $\hat{g}\colon S^i\to V_2$ be the map which agrees with g on the northern hemisphere of S^i and with g_1 on the southern hemisphere. Then $\hat{g}\simeq f_j\hat{g}$ in V_1 and $f_j\hat{g}(S^i)\subset V_1\cap(M\times\{1\})$. Let p be the path followed by the base point x during that homotopy. Use the choice of V_1 to homotope p (rel endpoints) to a path $\hat{p}$ in $U \cap (M\times\{1\})$. Now we see that the original map g is homotopic in U (rel $\partial\Delta^i$) to g_1 plus the singular i-sphere which is $f_j\hat{g}(S^i)$ acted on by $\hat{p}$. ■

LEMMA 2. *Suppose X_0 and X_1 are continua in the* PL *manifold M^n,* pro-$\pi_1(X_0)$ *satisfies the Mittag-Leffler condition, and X_0 and X_1 are shape concordant via a compactum $Z \subset M\times I$. Then for every neighborhood U of Z in $M\times I$ and every integer p, there exists a neighborhood V of Z in U such that if P is any compact polyhedron in V with* $\dim P \leqslant p$, *then there is a homotopy $f_t\colon P \to U$ such that f_0 = inclusion, $f_1(P) \subset U \cap (M\times\{1\})$, and $f_t|P \cap (M\times\{1\}) = \mathrm{id}$ for every t.*

Proof. Define V_{p+1} to be U and use Lemma 1 to choose neighborhoods $V_p \supset \ldots \supset V_0$ such that the inclusion induced map

$$\pi_i(V_i, V_i \cap (M\times\{1\})) \to \pi_i(V_{i+1}, V_{i+1} \cap (M\times\{1\}))$$

is trivial for $i = 0, 1, \ldots, p$. Let $V = V_0$.

Suppose P is a compact subpolyhedron of V such that $\dim P \leqslant p$. Triangulate P so that $P \cap (M\times\{1\})$ is a subcomplex. Each vertex of P in $P-(M\times\{1\})$ can be homotoped through V_1 into $V_1 \cap (M\times\{1\})$ by the choice of V_0. Use the homotopy extension property to extend that homotopy to a homotopy of all of P, being careful that the extension is the identity on $P \cap (M\times\{1\})$. Now all the 1-simplices of P have their vertices in $V_1 \cap (M\times\{1\})$ and the choice of V_1 allows us to homotope them through V_2 into $V_2 \cap (M\times\{1\})$, keeping the boundaries fixed. We again extend that homotopy to all of P using the homotopy extension property. The construction is continued inductively. We next homotope the 2-simplices into $M\times\{1\}$, then the 3-simplices, etc., until all of P has been homotoped into $M\times\{1\}$. ■

LEMMA 3. *Suppose X_0 and X_1 are continua in a* PL *n-manifold M^n which are shape concordant via a compactum $Z \subset M\times I$. Suppose further that* $\mathrm{Fd}(X_0) = k \leqslant n-3$, $n \geqslant 5$, *X_0 satisfies* ILC, *and either $2k+2 \leqslant n$ or* pro-$\pi_1(X_0)$ *satisfies the Mittag-Leffler condition. Then for every neighborhood U of Z in $M\times I$, there exist a k-dimensional polyhedron $K \subset M^n$, a regular neighborhood N_0 of K in M^n and a $(k+1)$-dimensional polyhedron $L \subset U$ such that*

(3.1) $X_0 \subset N_0$,
(3.2) $N_0 \times \{0\} \subset U \cap (M\times\{0\})$,
(3.3) $L \cap (M\times\{0\}) = K\times\{0\}$, *and*
(3.4) $L \searrow L \cap (M\times\{1\})$.

Proof. Let $r = 2k+2-n$. Fix a fundamental sequence $\{f_j\}$ from Z to $X_1\times\{1\}$ as in the proof of Lemma 1.

Consider the case $r \leqslant 0$. Choose a neighborhood V of Z and an integer j such that $f_j|V \simeq \mathrm{id}_V$ in U. By [V_1, Theorem 4.1], there exist a compact k-dimensional polyhedron K in M and a regular neighborhood N_0 of K such that $X_0\times\{0\} \subset N_0 \times \{0\} \subset V \cap (M\times\{0\})$. The choice of V implies that there exists a map $g\colon K\times [0,1] \to U$ such that $g(x, 0) = (x, 0) \in M\times\{0\}$ and $g(x, 1) = f_j(x, 0) \in M\times\{1\}$. We may assume that g is a PL map in general position and that $g(K\times\{\lambda\}) = g(K\times I) \cap (M\times\{\lambda\})$ for $\lambda = 0, 1$. By general position, the dimension of the singular set of g is no more than $2(k+1)-(n+1) = 2k+1-n = r-1 < 0$, and so g is an embedding. We simply take L to be $g(K\times I)$ in this case.

Next, consider the case $r > 0$. In that case, pro-$\pi_1(X_0)$ is Mittag-Leffler and so Lemma 2 applies. Let $V_0 = U$. Choose neighborhoods $V_1, V_2, \ldots, V_r$ of Z such that $V_0 \supset V_1 \supset \ldots \supset V_r$ and such that each inclusion satisfies the conclusion of Lemma 2 with $p = r$, Finally, choose a neighborhood V of Z and an integer j such that $f_j|V \simeq \mathrm{id}_V$ in V_r. By [V_1, Theorem 4.1] again, there exist a compact, connected, k-dimensional polyhedron $K \subset M$ and a regular neighborhood N_0 of K such that $X_0\times\{0\} \subset N_0\times\{0\} \subset V \cap (M\times\{0\})$. The choice of V implies that there exists a map $g_1\colon K\times I \to V_r$ such that g_1 is a PL map in general position, $g_1(K\times\{0\}) = K\times\{0\}$, and $g_1(K\times\{\lambda\}) = g_1(K\times I) \cap (M\times\{\lambda\})$ for $\lambda = 0, 1$.

Let S_1 denote the singular set of g_1 and let $L_1 = g_1(K\times I)$. By general position we have $\dim S_1 \leqslant 2(k+1)-(n+1) = r-1$. Since $K\times I \searrow K\times\{1\}$, Lemma 7.3 of [$\mathrm{Hd}_1$] implies that there is a subpolyhedron Σ_1 of $K\times I$ such that $S_1 \subset \Sigma_1$, $\dim\Sigma_1 \leqslant r$, and $K\times I \searrow \Sigma_1 \cup (K\times\{1\})$. (We call Σ_1 the *shadow* of S_1.) We now define L_1^* to be the polyhedron formed by taking $L_1 \cup [g_1(\Sigma_1)\times I]$ and identifying each point $x \in g_1(\Sigma_1) \subset L_1$ with the point $(x, 0) \in g_1(\Sigma_1)\times I$. Notice that $L_1^* \searrow g_1(K\times\{1\}) \cup (g_1(\Sigma_1 \cap K\times\{1\})\times I) \cup (g_1(\Sigma_1)\times\{1\})$. By the choice of V_r, the inclusion map $L_1 \subset V_r$ can be extended to a map $g_2\colon L_1^* \to V_{r-1}$ such that $g_2(L_1^*) \cap (M\times\{0\}) = g_1(K\times\{0\})$ and $g_2(L_1^*) \cap (M\times\{1\}) = g_2(g_1(K\times\{1\}) \cup (g_1(\Sigma_1 \cap K\times\{1\})\times I) \cup (g_1(\Sigma_1)\times\{1\}))$. We may assume that g_2 is a PL, general position map.

We now repeat the entire construction in the previous paragraph. Let $L_2 = g_2(L_1^*)$ and let S_2 denote the singular set of g_2. Then $\dim S_2 \leqslant (k+1)+(r+1)-(n+1) \leqslant r-2$. By [$\mathrm{Hd}_1$], Lemma 7.3 again, there is a shadow Σ_2 of S_2 in L_1^*. We next form L_2^* by attaching $g_2(\Sigma_2)\times I$ to L_2. The choice of V_{r-1} gives a map $g_3\colon L_2^* \to V_{r-2}$ such that g_3 extends the inclusion $L_2 \subset V_{r-2}$ and such that $g_3(L_2^*) \cap (M\times\{1\}) = g_3([g_2(\Sigma_2) \cap (M\times\{1\})]\times I \cup [g_2(\Sigma_2)\times\{1\}] \cup [L_2 \cap (M\times\{1\})])$. Put g_3 in general position and let $L_3 = g_3(L_2^*)$.

The construction is continued inductively and produces a sequence $L_1, L_2, \ldots, L_{r+1}$ of $(k+1)$-dimensional polyhedra such that $L_i \subset V_{r-i+1}$, $L_i \cap (M\times\{0\}) = K\times\{0\}$, and $L_i \searrow L_i \cap (M\times\{1\}) \cup \Sigma_i$ where Σ_i is a subpolyhedron of dimension $\leqslant r+1-i$.

Take L to be L_{r+1}. Notice that $L \subset V_0 = U$, $L \cap (M\times\{0\}) = K\times\{0\}$ and $L \searrow L \cap (M\times\{1\}) \cup \Sigma_{r+1}$ where $\dim\Sigma_{r+1} \leqslant 0$. On the other hand, L can only collapse to a connected set. So we must have that $L \searrow L \cap (M\times\{1\})$.

Remark. The construction in the second part of the proof above is reminiscent of the construction in the proof of an engulfing theorem. Notice, however, that nothing like a piping argument is needed because of the fact that K has codimension 4 in $M\times I$.

3. Main theorem. In this section we state and prove our main theorem.

LEMMA 4. *Suppose X_0 and X_1 are continua in the interior of a* PL *manifold M^n which are shape concordant via a compactum $Z \subset M \times I$. Suppose further that* $\operatorname{Fd}(X_0) = k \leqslant n-3$, $n \geqslant 5$, *X_0 satisfies* ILC *and either $2k+2-n \leqslant 0$ or pro-$\pi_1(X_0)$ satisfies the Mittag-Leffler condition. Then for every neighborhood M_1 of X_1 there exists a* PL *isotopy h_t of M^n such that $h_0 = \mathrm{id}$, $h_1(X_0) \subset M_1$ and X_1 and $h_1(X_0)$ have the same relative shape in M_1.*

Proof. Let $\{f_j\}$ be a fundamental sequence from Z to $X_1 \times \{1\}$ as in the proof of Lemma 1. Choose a neighborhood U of Z and an integer j_0 such that

$$f_j|U \simeq f_{j+1}|U \quad \text{and} \quad f_j|U \cap M \times \{1\} \simeq \mathrm{id}_{U \cap (M \times \{1\})} \text{ in } M_1,$$

for every $j \geqslant j_0$. Apply Lemma 3 to this neighborhood U; let K^k, N_0, and L^{k+1} be as in the conclusion of Lemma 3. Choose a regular neighborhood N of L in U such that N meets the boundary of $M \times I$ regularly and $N \cap (M \times \{0\}) = N_0 \times \{0\}$.

Let N_1 be the PL submanifold of M such that $N \cap (M \times \{1\}) = N_1 \times \{1\}$. We claim that N is homeomorphic with $N_1 \times I$ via a PL homeomorphism $g: N_1 \times I \to N$ such that $g|N_1 \times \{1\} =$ inclusion. To see this, notice that we may assume that $L \cap (M \times [0, \varepsilon]) = K \times [0, \varepsilon]$ for some small positive number ε. Then $N - M \times [0, \varepsilon)$ is a regular neighborhood of $L \cap (M \times \{1\})$ which meets the boundary of $M \times I$ regularly and hence $N - (M \times [0, \varepsilon)) \cong N_1 \times [0, 1]$ by [Hd$_1$, Theorem 2.16, p. 65]. Adding back the collar $N \cap (M \times [0, \varepsilon])$ does not change the PL homeomorphism type and so the claim is correct.

Now $\partial N = g(N_1 \times \{0\}) \cup g(\partial N_1 \times I) \cup g(N_1 \times \{1\})$. We can adjust g near $\partial(N_1 \times I)$ so that the collar $g(\partial N_1 \times I)$ has very short fibres but $g|N_1 \times \{1\}$ is not changed. It will then be the case that $K \times \{0\} \subset g(N_1 \times \{0\})$. So we can define an embedding $F_1: K \to N_1$ by $(x, 0) = g(F_1(x), 0)$ for every $x \in K$ and an embedding $F: K \times I \to M \times I$ by $F(x, t) = g(F_1(x), t)$. Then $F|K \times \{0\} =$ inclusion and $F(K \times \{1\}) \subset N_1 \times \{1\}$.

We now apply Hudson's Concordance Implies Isotopy Theorem ([Hd$_2$], Theorem 1.1) to F. There is a PL isotopy H_t of $M \times I$ such that $H_0 = \mathrm{id}$, $H_t|M \times \{1\} = \mathrm{id}$ for every t, and $H_1F(x, t) = (F_1(x), t)$ for all $(x, t) \in K \times I$. By first pushing N_0 radially along the product structure of $N_0 - K$ if necessary, we can adjust H_t so that it has further property that $H_1(N_0 \times \{0\}) \subset M_1 \times \{0\}$.

Let h_t be the isotopy of M defined by $(h_t(x), 0) = H_t(x, 0)$. Notice that $h_0 = \mathrm{id}$ and the adjustment made just above implies that $h_1(N_0) \subset M_1$ and *a fortiori* $h_1(X_0) \subset M_1$. To see that $h_1(X_0)$ and X_1 have the same relative shape in M_1, we construct a relative fundamental sequence $\{f'_j\}$ from $h_1(X_0)$ to X_1. For $j \geqslant j_0$ define $f'_j: h_1(N_0) \to M$ by $(f'_j(x), 1) = f_j(h_1^{-1}(x), 0)$, where $\{f_j\}$ is the fundamental sequence mentioned at the beginning of the proof. It is obvious that $\{f'_j\}$ is a fundamental sequence. By the choice of j_0 it follows that $f'_j|h_1(N_0) \simeq \mathrm{id}_{h_1(N_0)}$ for $j \geqslant j_0$, and so $\{f'_j\}$ is a relative fundamental sequence. ∎

THEOREM 5. *Suppose X_0 and X_1 are continua in the interior of the* PL *n-manifold M^n, $n \neq 4$, which are shape concordant via a compactum $Z \subset M \times I$. In case $n = 3$, assume that M contains no fake 3-cells. If X_0 and X_1 satisfy* ILC *in M^n,* $\operatorname{Fd}(X_0) = k \leqslant n-3$, *and pro-$\pi_i(X_0)$ is stable for $0 \leqslant i \leqslant r-1$ and Mittag-Leffler for $i = r$, where $r = 2k+2-n$, then $M - X_0 \cong M - X_1$.*

Proof. Consider first the case $n = 3$. Then X_0 and X_1 are cell-like subsets of M^3 which satisfy McMillan's cellularity criterion, and so $M^3 - X_0 \cong M^3 - X_1 \cong M^3 -$ point.

Now suppose $n \geqslant 5$. By the observation following the proof of Theorem 1 in [IS], X_1 has a PL manifold neighborhood M_1 such that the inclusion $X_1 \subset M_1$ is shape r-connected. By Lemma 4, there exists a PL isotopy h_t of M^n such that $h_1(X_0) \subset \operatorname{Int} M_1$ and X_1 and $h_1(X_0)$ have the same relative shape in M_1. We can, therefore, apply [IS], Theorem 3 to obtain a homeomorphism from $M_1 - h_1(X_0)$ to $M_1 - X_1$ which is the identity on ∂M_1. Extend that homeomorphism via the identity to a homeomorphism of $M^n - h_1(X_0)$ to $M^n - X_1$. ∎

Remark. A version of Theorem 5 could be proved for compacta instead of continua. In case $r \leqslant 0$, the proof goes through unchanged for (nonconnected) compacta. If $r \geqslant 1$, the fact that pro-$\pi_0(X_0)$ is stable means that X_0 has only a finite number of components. If each of them satisfied the hypotheses of Theorem 5, we could do the construction of Lemma 4 for each component and still conclude that $M - X_0 \cong M - X_1$.

4. Concordance implies homeomorphic complements in Hilbert-cube manifolds. Following [S$_2$], we say that a compact subset X of a Hilbert cube manifold M is a *weak Z-set* if for each closed neighborhood U of X and closed set $A \subset U$ there is a homotopy $H: A \times I \to U$ such that $H(A \times \{1\}) \cap X = \emptyset$ and $H(x, t) = x$ if either $t = 0$ and $x \in A$ or $x \in \operatorname{Fr}_M U$. The following is a different form of the Z-set unknotting theorem in Hilbert cube manifold theory that we will use in the proof of Theorem 7 below.

LEMMA 6. *Let X_0 and X_1 be two Z-set copies of a compactum X in a Hilbert cube manifold M. If X_0 and X_1 are shape concordant in M, say by a compactum $Z \subset M \times I$, then $M - X_0 \cong M - X_1$.*

Proof. For each $\lambda = 0, 1$, let $\{U_{\lambda,n} |\ n = 1, 2, \ldots\}$ and $\{W_n |\ n = 0, 1, \ldots\}$ denote defining sequences (connectedness is not necessary) for X_λ and Z; and $\underline{f}^\lambda = \{f_s^\lambda: W_s \to U_{\lambda,s}\}$ a shape inverse of $\underline{i}^\lambda$: $X_\lambda \subset Z$ where $\underline{i}^\lambda = \{i_s^\lambda\}$ and each i_s^λ: $U_{\lambda,s} \to W_{s-1}$. Observe that $\underline{f}^1 \underline{i}^0$ and $\underline{f}^0 \underline{i}^1$ are inverses each other.

The proof similar to that of Theorem 25.2 [Ch] is divided into three steps:

(i) Given a small neighborhood V_1 of X_1, there is a homeomorphism h_1 of M such that $h_1(X_0) \subset V_1$ and $h_1|X_0 \simeq f_s^1 i_{s+1}^0|X_0$ (in V_1) for all large s.

(ii) If U_1 is a small neighborhood of X_0 such that $h_1(U_1) \subset V_1$, then there is a homeomorphism h_2 of M such that $X_1 \subset h_2 h_1(U_1)$ and $h_2 = \mathrm{id}$ on $M - V_1$.

(iii) If V_2 is small neighborhood of X_1 such that $V_2 \subset h_2 h_1(U_1)$, then there is a homeomorphism h_3 of M such that $h_3 h_2 h_1(X_0) \subset V_2$ and $h_3 = \mathrm{id}$ on $M - h_2 h_1(U_1)$.

The proofs of (ii) and (iii) are exactly the same as those of steps II and III on pp. 41–42 [Ch], by use of the fact that $f^1 i^0$ and $f^0 i^1$ are inverses of each other. The following is a proof of (i).

Let s be an integer such that $U_{1,s-1} \subset V_1$. Then we can show that the map $f^1_{s-1} i^0_s | X_0 \colon X_0 \to U_{1,s-1}$ is homotopic in W_{s-2} to the inclusion $X_0 \subset W_{s-2}$. Consequently, $\mathrm{proj}_M f^1_{s-1} i^0_s \colon X_0 \to U_{1,s-1} \simeq X_0 \subset M$. Let $\alpha_1 \colon X_0 \to \mathrm{int}\, U_{1,s-1}$ be a Z-embedding homotopic in $U_{1,s-1}$ to $\mathrm{proj}_M f^1_{s-1} i^0_s$. Then, we can extend α_1 to the homeomorphism h_1 we want of M by using Theorem 11.1 [Ch]. Hence, the lemma follows. ■

THEOREM 7. *Let X_0 and X_1 be weak Z-sets in a Hilbert cube manifold M. If X_0 and X_1 are shape concordant, say by $Z \subset M \times I$, then $M - X_0 \cong M - X_1$.*

Proof. For each $\lambda = 0, 1$, there is from [S_2] an open neighborhood V_λ of X_λ in M such that V_λ is homeomorphic to an open subset of the Hilbert cube. Let $U_\lambda \subset V_\lambda$ be a compact neighborhood of X_λ, and $\tilde{\imath}_\lambda \colon X_\lambda \to \mathrm{Int}_M U_\lambda$ a Z-set embedding approximating $i_\lambda \colon X_\lambda \subset \mathrm{Int}_M U_\lambda$ such that $\tilde{\imath}_\lambda \simeq i_\lambda$ in $\mathrm{Int}_M U_\lambda$. Now, from the proof of Theorem 3.1 [S_2], there is a homeomorphism $h \colon V_\lambda - X_\lambda \cong V_\lambda - \tilde{\imath}_\lambda(X_\lambda)$ such that $h|(V_\lambda - U_\lambda) = \mathrm{id}$. Then, by extending h via the identity, we obtain $M - X_\lambda \cong M - \tilde{\imath}_\lambda(X_\lambda)$. Therefore, combining with Lemma 6, we have $M - X_0 \cong M - X_1$. The proof is now complete. ■

Remarks 1. Following the proof of Theorem 25.1 [Ch], we can prove that if X, Y are weak Z-sets in a compact Hilbert-cube manifold M such that $M - X \cong M - Y$, then X and Y have the same shape.

2. Let X and Y be weak Z-sets in the Hilbert cube Q such that $Q - X \cong Q - Y$, then X and Y are shape concordant in Q.

3. From the proof of Theorem 5 and Theorem 7 above, there actually is an I-level-preserving homeomorphism $H \colon (M \times I) - (X_0 \times \{1\}) \to (M \times I) - (X_1 \times \{1\})$ such that $H(x, 0) = (x, 0)$ and $H(x, 1) = (h(x), 1)$, where h is a homeomorphism obtained in Theorem 5 or Theorem 7 correspondingly. Conversely, for given compacta $X, Y \subset M$, if such a level-preserving homeomorphism H exists, it is easy to prove that X and Y are shape concordant.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ALABAMA
University, AL 35486

DEPARTMENT OF MATHEMATICS
CALVIN COLLEGE
Grand Rapids, MI 49506

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