Chainable continua and homeomorphisms of the plane onto itself

by

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Abstract. It is shown that for each chainable plane continuum S there exists a chainable plane continuum S' such that no homeomorphism of the plane onto itself carries S' into S.

Let S be a continuum. An ε-chain with links $U_1, U_2, \ldots, U_n$ is the family of open subsets U_i of S such that the diameter of U_i is less than ε for $i = 1, 2, \ldots, n$ and for $i, j = 1, 2, \ldots, n$, $U_i \cap U_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$. S is said to be *chainable* if, for each $\varepsilon > 0$, S is covered by an ε-chain. Chainable continua are sometimes called *snake-like continua*. A continuum X is said to be *triodic* if it contains three continua such that the common part of all three is both a nonempty proper subcontinuum of each of them and the common part of each two of them. A continuum is atriodic if it is not triodic. Every chainable continuum is atriodic [1]. Two continua in the plane M and N are *equivalent* if there exists a homeomorphism of the plane onto itself which carries M onto N.

It is well known that every chainable continuum can be embedded in the plane [1]. In 1965 Shori constructed a universal chainable continuum [3]. Bing asked if there exists a chainable plane continuum with some universality property [3]. Namely, if there exists a chainable plane continuum S such that for each chainable plane continuum S' there is a homeomorphism of the plane onto itself which carries S' into S. Here we show that such a continuum does not exist.

THEOREM. *For each chainable plane continuum S there exists a chainable plane continuum S' which is not equivalent to any subcontinuum of S.*

Proof. We construct a family S_a, $a \in A$ (where A is an uncountable set of indexes), of chainable plane continua with the following property: if h_a, $a \in A$, are homeomorphisms of the plane onto itself, then for some $a, b \in A$ the set $h_a(S_a) \cup h_b(S_b)$ is a triodic continuum. Hence, if S is a chainable plane continuum then, for some $a \in A$, S_a is not equivalent to any subcontinuum of S.

In our construction we use a chainable plane continuum M_1 from Example 4 [1]. It is contained in the set $\{(x, y) |\ 0 \leqslant x \leqslant 1\}$ and is a compactification of a topological ray with origin $(1,0)$ by interval $\{(x,y)|\ x=0, |y|\leqslant 3\}$. M_1 has following property:

$(*)$ there exists no uncountable family of mutually exclusive continua each of which is equivalent to M_1.

W. Dębski in [4] shows that there are continuum topological types among indecomposable chainable continua which can be obtained as inverse limits of sequences $I \xleftarrow{p_1} I \xleftarrow{p_2^3} I \ldots$ where I is the unit interval and p_n^{n+1} are open continuous maps. Every proper subcontinuum of such continua is an arc. Let C_a, $a \in A$, denote a family of topologically different continua such as in Dębski's work, contained in the interior of the unit disc K with centre $(2,0)$. Let L_a be a topological ray with origin $(1,0)$ contained in $K \setminus C_a$ and such that L_a is compactified by C_a. Finally we define $S_a = M_1 \cup L_a \cup C_a \cup M'_1 \cup L'_a \cup C'_a$ where M'_1, L'_a, C'_a are symmetric to M_1, L_a, C_a with respect to the point $(0,0)$. It is easy to verify that S_a is a chainable continuum. Let h_a, $a \in A$, be homeomorphisms of the plane onto itself. By property $(*)$ of M_1 there are $a \neq b$ such that $h_a(M_1) \cap h_b(M_1) \neq \emptyset$. Then we have three continua $A_1 = h_a(M_1 \cup L_a \cup C_a)$, $A_2 = h_a(M_1 \cup M'_1 \cup L'_a \cup C'_a)$, $A_3 = h_b(M_1 \cup L_b \cup C_b)$, such that $A_1 \cap A_2 \cap$ $\cap A_3 \neq \emptyset$. Moreover, A_1 is not contained in $A_2 \cup A_3$, because only an indecomposable subcontinuum of A_1 is $h_a(C_a)$, $h_a(C_a) \cap A_2 = \emptyset$ and only an indecomposable subcontinuum of A_3 is homeomorphic to C_b; analogously, A_2 is not contained in $A_1 \cup A_3$, and A_3 is not contained in $A_1 \cup A_2$, because only indecomposable subcontinuum of A_3 is homeomorphic to C_b and only indecomposable subcontinua of $A_1 \cup A_2$ are homeomorphic to C_a. Hence the set $h_a(S_a) \cup h_b(S_b)$ is a triodic continuum (Theorem 1.8 [5]).

References

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