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Remarks on intrinsic isometries

by

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Abstract. A map $f\colon A \to A'$ of metric spaces is said to be an intrinsic isometry if it preserves the length of every arc. It is shown in this note that the Euclidean n-space E^n is intrinsically isometric to a subset A of E^{n+1} with arbitrarily small diameter $\delta(A)$. We also consider the intrinsic metric of a product of metric spaces.

1. Introduction. The notion of the intrinsic metric for metric spaces and related notions were introduced by K. Borsuk [1]. Let us say that a space A (with metric ϱ) is *geometrically acceptable* (notation: $A \in \mathrm{GA}$) if

(1.1) for every two points $x, y \in A$ there exists an arc $L \subset A$ with finite length such that $x, y \in L$

and

(1.2) for every point $x \in A$ and for every $\varepsilon > 0$ there is a neighborhood U of x in A such that for every point $y \in U$ there exists in A an arc L containing the two points x, y and such that the length $|L| < \varepsilon$.

Then setting

(1.3) $\varrho_A(x, y) =$ lower bound of the length of all arcs $L \subset A$ containing the two points x, y,

one gets a metric ϱ_A in A called the *intrinsic metric in A*. The topology in $A \in \mathrm{GA}$ induced by the metric ϱ_A is the same as the topology induced by the metric ϱ.

A function f mapping a GA-space A onto another GA-space A' is said to be an *intrinsic isometry* provided

$$\varrho_A(x, y) = \varrho_{A'}(f(x), f(y)) \quad \text{for every } x, y \in A. \tag{1.4}$$

A map f is an intrinsic isometry if and only if it preserves the length of every arc. Every intrinsic isometry is a homeomorphism.

K. Borsuk has proved [1] that for every $\varepsilon > 0$ there exists an intrinsic isometry mapping the Euclidean n-space E^n onto a subset $A \subset E^{2n}$ such that the diameter of A (by the usual metric in E^{2n}) is less than ε. We will prove the following

(1.5) THEOREM. *For every $\varepsilon > 0$ there exists a subset A of the $(n+1)$-space E^{n+1} with a diameter less than ε which is intrinsically isometric to the Euclidean n-space E^n.*

Let A and B be GA-spaces with metrics ϱ' and ϱ'', respectively. Then the product $A \times B$ with the metric ϱ given by

$$\varrho((a_1, b_1), (a_2, b_2)) = \sqrt{(\varrho'(a_1, a_2))^2 + (\varrho''(b_1, b_2))^2} \tag{1.6}$$

is a GA-space. The intrinsic $\varrho_{A\times B}$ in $A \times B$ is given by

$$\varrho_{A\times B}((a_1, b_1), (a_2, b_2)) = \sqrt{(\varrho'_A(a_1, a_2))^2 + (\varrho'_B(b_1, b_2))^2} \tag{1.7}$$

(see Theorem (3.7)). It follows that if $f\colon A \to A'$ and $g\colon B \to B'$ are intrinsic isometries of GA-spaces, then the map $f \times g\colon A \times B \to A' \times B'$ is an intrinsic isometry.

2. The Euclidean n-space E^n is intrinsically isometric to a subset of E^{n+1} with a small diameter. We denote by I the set of all integers. Let F be the union of segments A_i in E^n $(i \in I)$ such that the intersection $A_i \cap A_{i+1} = \dot{A}_i \cap \dot{A}_{i+1}$ is a point and segments A_i and A_j are disjoint if $|i-j| > 1$. We say that the broken line F is obtained by the reflection of a straight line K relative to a family $\{H_i\}_{i\in I}$ of hyperplanes ($(n-1)$-dimensional hyperplanes in E^n) if the following conditions are satisfied:

(2.1) $A_0 \subset K$,

(2.2) $A_{i-1} \cap A_i \subset H_i$,

(2.3) the hyperplane H_i is perpendicular to the bisectrix of the angle between the segments A_{i-1} and A_i,

(2.4) F is intrinsically isometric to K.

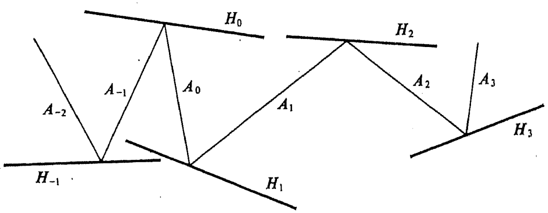

Fig. 1

For positive real numbers a and b we define

$$p_i = \left(\frac{i}{|i|+1}a, (-1)^i b\right) \quad \text{for every } i \in I.$$

Let $A_i(a, b)$ be the segment which joins the points p_i and p_{i+1}. Let $L_i(a, b)$ be a straight line which contains the point p_i and which is perpendicular to the bisectrix of the angle between the segments $A_{i-1}(a, b)$ and $A_i(a, b)$; i.e. the broken line $F(a, b) = \bigcup_{i\in I} A_i(a, b)$ is obtained by the reflection of the straight line K which contains the segment $A_0(a, b)$ relative to the family $\{L_i(a, b)\}_{i\in I}$ (see Fig. 2).

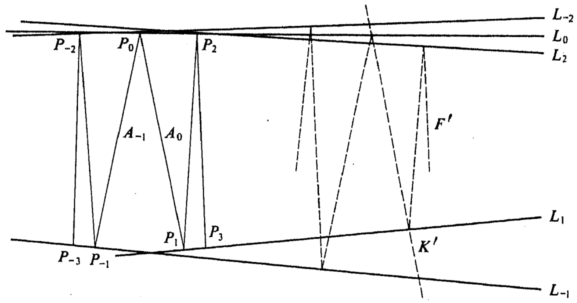

Fig. 2

One can prove that for any positive real numbers a, d and ε there exists a positive number b (sufficiently small) such that for any straight line K' parallel to the segment $A_0(a, b)$ where the distance of this segment from K' is less than d the broken line F' which is obtained by the reflection of K' relative to the family $\{L_i(a, b)\}_{i\in I}$ has a diameter $\delta(F')$ less than $2a+\varepsilon$.

Thus we can formulate the following

(2.5) LEMMA. *For any positive real numbers ε and d there exist positive real numbers a and b such that for any straight line K' parallel to the segment $A_0(a, b)$ where the distance of this segment from K' is less than d the broken line F' which is obtained by the reflection of K' relative to the family $\{L_i(a, b)\}_{i\in I}$ has a diameter $\delta(F')$ less than ε. The sets F' and $f(F')$ are disjoint for any nontrivial translation f of the direction of the second axis.*

Now we will prove the following

(2.6) LEMMA. *Let X be a subset of the Euclidean n-space E^n which does not intersect the set $f(X)$ for any nontrivial translation f in the direction of some vector α. Then for any $\varepsilon > 0$ there exists an embedding*

$$g\colon X \times E \to E^{n+1}$$

which is an intrinsic isometry such that

$$\delta(g(X \times E)) \leqslant \delta(X) + 2\varepsilon, \tag{2.7}$$

(2.8) *the sets* $g(X\times E)$ *and* $f'(g(X\times E))$ *are disjoint for any nontrivial translation* f' *in the direction of some vector* α'.

Proof. We will consider E^n as a subset of E^{n+1}. Let P be a plane in E^{n+1} parallel to the $(n+1)$-axis and to the vector α such that the intersection $P\cap X$ is nonempty (any such plane intersects the set X in at most one point). For ε and $d=\delta(X)$ we find positive real numbers a and b which satisfy the conditions of Lemma (2.5). Let $F(a,b)=\bigcup_{i\in I}A_i(a,b)$ be a broken line in P defined as above in such a way that the straight line which contains the segment $A_0(a,b)$ intersects the set X and is parallel to the $(n+1)$-axis of E^{n+1}. Let $L_i(a,b)$, $i\in I$, be the straigth line in the plane P defined as above. By $H_i(a,b)$, $i\in I$, we denote the n-dimensional hyperplane in E^{n+1} with the image $L_i(a,b)$ under the orthogonal projection onto the plane P.

The straight line parallel to the $(n+1)$-axis which contains a point $x\in X$ we denote by $K(x)$. Let $F(x)$ be the broken line which is obtained by the reflection of the straight line $K(x)$ relative to the family $\{H_i(a,b)\}_{i\in I}$. Observe that $F(x)$ is contained in the plane $P(x)\subset E^{n+1}$ which is parallel to P and which contains the line $K(x)$. We know that $P(x)\cap X=\{x\}$; thus $F(x)$ and $F(x')$ are disjoint if x and x' are different points of X. By Lemma (2.5) the diameter $\delta(F(x))$ is less than ε for every point $x\in X$. Thus the diameter of the set $Y=\bigcup_{x\in X}F(x)$ is not greater than $\delta(X)+2\varepsilon$.

Let g_x be the intrinsic isometry of $K(x)$ onto $F(x)$ which is the identity on the segment of $K(x)$ between the hyperplanes $H_0(a,b)$ and $H_1(a,b)$. The embedding

$$g\colon X\times E\to Y\subset E^{n+1}$$

defined by

$$g(z)=g_x(z)\quad \text{if}\quad z\in K(x)\text{ and } x\in X$$

is an intrinsic isometry since g restricted to any arc in $X\times E$ is a composition of the reflections relative to some hyperplanes $H_i(a,b)$.

Observe that for any nontrivial translation f' in the direction of the bisectrix of the angle between the segments $A_{-1}(a,b)$ and $A_0(a,b)$ the sets Y and $f'(Y)$ are disjoint.

From Lemma (2.6) one can obtain (by induction) Theorem (1.5).

It is easy to see [1] that there exists a smooth embedding $g\colon E^n\to Y\subset E^{2n}$ which is an intrinsic isometry such that the diameter $\delta(Y)$ of Y is small. There is no smooth intrinsic isometry $g\colon E^n\to E^{n+1}$ with the diameter $\delta(Y)$ finite, $n>1$. This follows by the Hartman–Nirenberg theorem of [2] (see Theorem (5.3) Chapter VI, [3]). Let us formulate the following question:

(2.9) Is it true that there is no smooth intrinsic isometry $g\colon E^n\to Y\subset E^m$ with the diameter $\delta(Y)$ finite if m is less than $2n$?

3. The intrinsic metric of the product of metric spaces. By R^+ we denote the set of all nonnegative real numbers. We will consider functions $f\colon R^+\times R^+\to R^+$ which satisfy the following conditions:

$$f(s_1,t_1)+f(s_2,t_2)\geqslant f(s_1+s_2,t_1+t_2), \tag{3.1}$$

$$f(s_1,t_1)\leqslant f(s_2,t_2)\quad\text{if}\quad s_1\leqslant s_2 \text{ and } t_1\leqslant t_2, \tag{3.2}$$

$$f(as,at)=a\cdot f(s,t), \tag{3.3}$$

$$f(s,t)=0\quad\text{if and only if}\quad s=0=t. \tag{3.4}$$

Let X and Y be metric spaces with metrics ϱ' and ϱ'' respectively. If $f\colon R^+\times R^+\to R^+$ is a function which satisfies conditions (3.1)–(3.4), then setting

$$\varrho((x_1,y_1),(x_2,y_2))=f(\varrho'(x_1,x_2),\varrho''(y_1,y_2)) \tag{3.5}$$

one gets a metric ϱ on the product $X\times Y$ which induced the product topology. Observe that the function $f\colon R^+\times R^+\to R^+$ defined by

$$f(s,t)=\sqrt{s^2+t^2} \tag{3.6}$$

satisfies conditions (3.1)–(3.4).

Now we will prove the following

(3.7) THEOREM. *Let* ϱ_X *and* ϱ_Y *be the intrinsic metrics in* GA-*spaces* (X,ϱ') *and* (Y,ϱ'') *respectively. Let* ϱ *be the metric in the product* $X\times Y$ *given by* (3.5), *where* $f\colon R^+\times R^+\to R^+$ *is a function satisfying conditions* (3.1)–(3.4). *Then the intrinsic metric in the metric space* $(X\times Y,\varrho)$ *is given by*

$$\varrho_{X\times Y}((x_1,y_1),(x_2,y_2))=f(\varrho_X(x_1,x_2),\varrho_Y(y_1,y_2)). \tag{3.8}$$

Proof. Let $z_1=(x_1,y_1)$ and $z_2=(x_2,y_2)$ be points in $X\times Y$. Let L be an arc lying in $X\times Y$ and joining the points z_1 and z_2 with a parametric representation given by a homeomorphism

$$h\colon \langle 0,1\rangle\to L.$$

Let $h_X=p_X\circ h$ and $h_Y=p_Y\circ h$, where p_X and p_Y are the natural projections of $X\times Y$ onto X and Y respectively. For every $\varepsilon>0$ there exists a sequence $0=t_0<t_1<\ldots<t_k=1$ such that

$$\sum_{i=1}^{k}d_i'>\varrho_X(x_1,x_2)-\varepsilon\quad\text{and}\quad\sum_{i=1}^{k}d_i''>\varrho_Y(y_1,y_2)-\varepsilon$$

where

$$d_i'=\varrho'(h_X(t_i),h_X(t_{i-1}))\quad\text{and}\quad d_i''=\varrho''(h_Y(t_i),h_Y(t_{i-1})).$$

By conditions (3.1) and (3.2) we obtain

$$|L| \geqslant \sum_{i=1}^{k} \varrho(h(t_i), h(t_{i-1})) = \sum_{i=1}^{k} f(d_i', d_i'')$$

$$\geqslant f\Big(\sum_{i=1}^{k} d_i', \sum_{i=1}^{k} d_i''\Big) \geqslant f\big(\varrho_X(x_1, x_2)-\varepsilon, \varrho_Y(y_1, y_2)-\varepsilon\big).$$

Since f is continuous, we obtain

$$|L| \geqslant f\big(\varrho_X(x_1, x_2), \varrho_Y(y_1, y_2)\big).$$

Thus

(3.9) $$\varrho_{X\times Y}(z_1, z_2) \geqslant f\big(\varrho_X(x_1, x_2), \varrho_Y(y_1, y_2)\big).$$

For every $\varepsilon > 0$ there exist arcs L' in X joining x_1 and x_2 and L'' in Y joining y_1 and y_2 such that

(3.10) $$\varrho_X(x_1, x_2)+\varepsilon > |L'| \quad \text{and} \quad \varrho_Y(y_1, y_2)+\varepsilon > |L''|.$$

Let the parametric representations

$$h'\colon \langle 0, u\rangle \to L' \quad \text{and} \quad h''\colon \langle 0, v\rangle \to L''$$

be intrinsic isometries. Let

$$\varphi\colon \langle 0, 1\rangle \to \langle 0, u\rangle \times \langle 0, v\rangle$$

be a map given by $\varphi(t) = (t\cdot u, t\cdot v)$. Let $\tilde{\varphi} = (h'\times h'')\circ\varphi$, i.e. $\tilde{\varphi}(t) = \big(h'(t\cdot u), h''(t\cdot v)\big)$. For every $\delta > 0$ there exists a sequence $0 = t_0 < t_1 < \ldots \ldots < t_k = 1$ such that

$$|\tilde{L}| < \sum_{i=1}^{k} \varrho\big(\tilde{\varphi}(t_i), \tilde{\varphi}(t_{i-1})\big)+\delta$$

where $\tilde{L}$ is the arc with the parametric representation $\tilde{\varphi}$. For any GA-spaces A and for any points $a, b\in A$ we have $\varrho_A(a, b) \geqslant \varrho(a, b)$. Thus, by conditions (3.2) and (3.3), we obtain

$$\sum_{i=1}^{k} \varrho\big(\tilde{\varphi}(t_i), \tilde{\varphi}(t_{i-1})\big) = \sum_{i=1}^{k} f\Big(\varrho'\big(h'(t_i\cdot u), h'(t_{i-1}\cdot u)\big), \varrho''\big(h''(t_i\cdot v), h''(t_{i-1}\cdot v)\big)\Big)$$

$$\leqslant \sum_{i=1}^{k} f\big(|t_i\cdot u - t_{i-1}\cdot u|, |t_i v - t_{i-1} v|\big)$$

$$= \sum_{i=1}^{k} |t_i - t_{i-1}|\, f(u, v) = f(u, v) = f(|L'|, |L''|).$$

Thus

$$|\tilde{L}| < f(|L'|, |L''|)+\delta$$

for every $\delta > 0$. By conditions (3.2) and (3.10), it follows that

$$|\tilde{L}| \leqslant f(|L'|, |L''|) \leqslant f\big(\varrho_X(x_1, x_2)+\varepsilon, \varrho_Y(y_1, y_2)+\varepsilon\big).$$

Since $\varrho_{X\times Y}(z_1, z_2) \leqslant |\tilde{L}|$, we obtain

$$\varrho_{X\times Y}(z_1, z_2) \leqslant f\big(\varrho_X(x_1, x_2)+\varepsilon, \varrho_Y(y_1, y_2)+\varepsilon\big)$$

for every $\varepsilon > 0$. Since f is continuous, we obtain

(3.11) $$\varrho_{X\times Y}(z_1, z_2) \leqslant f\big(\varrho_X(x_1, x_2), \varrho_Y(y_1, y_2)\big).$$

From Theorem (3.7) follows

(3.12) COROLLARY. *If $f'\colon X' \to Y'$ and $f''\colon X'' \to Y''$ are intrinsic isometries of* GA-*spaces, then the map $f'\times f''$ is an intrinsic isometry.*

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