Continua whose local homeomorphisms are homeomorphisms

by

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Abstract. Let $f\colon X \to Y$ be a local homeomorphism between continua. Then an answer is given to the question: Under what conditions for X, is f a homeomorphism?

1. Introduction. Let X, Y be continua and $f\colon X \to Y$ a local homeomorphism of X onto Y. Then we give sufficient conditions for X that f is a homeomorphism. If X is a chainable continuum or more generally a tree-like continuum, then f is a homeomorphism ([5, p. 261], [3, p. 67], [2, p. 317], [4, p. 50]).

Our results are the following:

THEOREM 1. *Let X, Y be continua and $f\colon X \to Y$ a local homeomorphism of X onto Y. If X is the limit of an inverse sequence, with bonding maps onto, of simply connected Peano continua and X has the fixed point property for homeomorphisms, then f is a homeomorphism.*

THEOREM 2. *Let X, Y be continua and $f\colon X \to Y$ a local homeomorphism of X onto Y. If X is the intersection of a monotone decreasing sequence of simply connected Peano continua and X has the fixed point property for homeomorphisms, then f is a homeomorphism.*

COROLLARY. *Every local homeomorphism of a compact metric* AR *onto a space is a homeomorphism.*

2. Definitions and notation. A *Peano continuum* is a locally connected, connected, compact metrizable space. A space X is *simply connected* if it is arcwise connected and each closed path in X is homotopic to zero. A map means a continuous function. A *local homeomorphism* $f\colon X \to Y$ between topological spaces is a map having the following property: For each point x of X there exists an open neighborhood U of x such that $f(U)$ is open in Y, and f restricted to $U, f|U$, is a homeomorphism of U onto $f(U)$.

Let (M, d) be a metric space and δ a positive number. For points a, b of M, a *δ-chain* from a to b is a finite sequence $\alpha = \{a = x_1, x_2, \ldots, x_l = b\}$ of points such that $d(x_i, x_{i+1}) < \delta$ $(1 \leqslant i < l)$. If $a = b$, then α is a *δ-loop* based at a. Let $\alpha = \{a_1, \ldots, a_l\}$, $\beta = \{b_1, \ldots, b_m\}$ be δ-chains. If $\{a_1, \ldots, a_l, b_1, \ldots, b_m\}$ is also a δ-chain, then we denote it by $\alpha\beta$. Moreover $\{a_l, a_{l-1}, \ldots, a_1\}$ is denoted by α^{-1}. For $\delta > 0$, a finite set of points $\{x_{ij}\colon 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m\}$ is a *δ-net* provided that the diameter of $\{x_{ij}, x_{i,j+1}, x_{i+1,j}, x_{i+1,j+1}\}$ $(1 \leqslant i < l, 1 \leqslant j < m)$ is less than δ. Let $\alpha = \{a_1, a_2, \ldots, a_l = a_1\}$ be a δ-loop. If there exists a δ-net $\{x_{ij}\colon 1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m\}$ such that $X_{i1} = a_i$ and $x_{1j} = x_{im} = x_{lj} = a_1$ $(1 \leqslant i \leqslant l, 1 \leqslant j \leqslant m)$, then α is said to be *δ-homotopic to zero* in M, and we denote $\alpha \simeq 0(\delta)$.

Let X, Y be continua, and $f\colon X \to Y$ be a local homeomorphism of X onto Y. Hereafter we shall exclusively use the symbols $\mathscr{V}$, $\mathscr{U}$, ε, μ, ν, and λ:

$\mathscr{V}$ = an open covering $\mathscr{V}$ of Y as follows: For each $V \in \mathscr{V}$ there exists a finite collection $\{E_1, \ldots, E_k\}$, denoted by $\mathscr{C}(V)$, of mutually exclusive open sets of X, such that $f^{-1}(V) = \bigcup_{i=1}^{k} E_i$ and $f|E_i\colon E_i \to V$ $(1 \leqslant i \leqslant k)$ is a homeomorphism of E_i onto V.

$\mathscr{U}$ = the covering $\bigcup_{V \in \mathscr{V}} \mathscr{C}(V)$ of X.

ε = the Lebesgue number of $\mathscr{V}$.

μ = the mesh of $\mathscr{U}$.

ν = the Lebesgue number of $\mathscr{U}$.

λ = a positive number such that if A is a subset of X with diameter $< \lambda$, then $f|A\colon A \to f(A)$ is a homeomorphism.

Let $\beta = \{b_1, \ldots, b_l\}$ be a chain in Y such that $\{b_i, b_{i+1}\}$ is contained in an element V_i of $\mathscr{V}$. If a chain $\alpha = \{a_1, \ldots, a_l\}$ in X satisfies the condition that $f(a_i) = b_i$ and $\{a_i, a_{i+1}\}$ is contained in an element of $\mathscr{C}(V_i)$, then we say that *α covers β*, or that α is a *lifting* of β. A homeomorphism g of X onto itself is said to be an *automorphism* of X with respect to f provided that $f\circ g = f$.

By the method of lifting similar to that in the theory of covering spaces, we have (2.1) and (2.2).

(2.1) *Let β be an ε-chain in Y, with initial point y, and let x be a point of X with $f(x) = y$. If $\mu < \lambda/2$, then there exists a unique chain in X with initial point x, covering β.*

(2.2) *Let β, β' be chains in Y from y to y', such that $\beta\beta'^{-1} \simeq 0$ (ε) in Y. Let x be a point of X with $f(x) = y$, and let α, α' be liftings of β, β' with initial point x, respectively. If $\mu < \lambda/2$, then α, α' have the same terminal point. Whence a lifting of a loop ε-homotopic to zero is also a loop.*

Obviously

(2.3) *If β, γ are η-chains in Y with $\gamma \subset \beta$, then $\beta\gamma^{-1} \simeq 0$ (η).*

3. Proof of Theorem 1. We first prove that if $a, b \in X$ with $f(a) = f(b)$, then there exists an automorphism g of X with respect to f such that $g(a) = b$. Next we show that if $a \neq b$, then g has no fixed point.

Let X be the limit of an inverse sequence, with bonding maps onto, of simply connected Peano continua X_i. Let π_i be the ith projection of X onto X_i, and d_i a metric on X_i bounded by number 1. Then a metric d' on X is given by

(1) $d'(x, x') = \sum_{i=1}^{\infty} 2^{-i} d_i(\pi_i(x), \pi_i(x'))$.

We may assume that

(2) $\mu < \lambda/2$.

Let τ_0 be a positive number such that

(3) If A is a subset of X with $\operatorname{diam}(A) < \tau_0$, then $\operatorname{diam}(f(A)) < \varepsilon$.

By (1) there exist a positive integer n and $\gamma > 0$ such that

(4) If K is a subset of X_n with $\operatorname{diam}(K) < \gamma$, then $\operatorname{diam}(\pi_n^{-1}(K)) < \tau_0$.

Choose $\delta > 0$ such that

(5) If $z, z' \in X_n$ and $d_n(z, z') < \delta$, then z, z' can be joined by an arc in X_n with diameter $< \gamma$.

Moreover choose $\tau > 0$ such that

(6) $\tau \leqslant \min\{\tau_0, \delta/2^n\}$

and

(7) If α is a τ-chain in X, then every lifting of $f(\alpha)$ is a $\delta/2^n$-chain.

(a) The definition of $g\colon X \to X$: Let x be a point of X and join a to x by a τ-chain α. Then by (6) and (3), $f(\alpha)$ is an ε-chain in Y. Since ε is the Lebesgue number of $\mathscr{V}$, by (2) and (2.1) we can lift $f(\alpha)$ to a unique chain α' in X with initial point b. The terminal point of α' is our desired $g(x)$. Obviously $f\circ g(x) = f(x)$.

(b) The map g is well defined, i.e., $g(x)$ is a unique point independent of τ-chains from a to x. For let $\alpha = \{a = x_1, x_2, \ldots, x_l = x\}$, $\alpha' = \{a = x_{l+m}, x_{l+m-1}, \ldots, x_{l+1}, x_l = x\}$ be τ-chains in X from a to x. Put $z_k = \pi_n(x_k)$ $(1 \leqslant k \leqslant l+m)$. Then $\pi_n(\alpha\alpha'^{-1}) = \{z_1, \ldots, z_l, \ldots, z_{l+m}\}$ is a δ-loop in X_n based at $\pi_n(a)$ (cf. (1), (6)). Let α_k be an arc in X_n, from z_k to z_{k+1}, whose diameter $< \gamma$ (cf. (5)), and let $\psi\colon I = [0, 1] \to X_n$ be a parametrization of the closed curve $\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_{l+m-1}$.

Since X_n is simply connected, there exists a map $F\colon I \times I \to X_n$ such that

$$\begin{aligned} &F(s, 0) = \psi(s), \\ &F(s, 1) = F(0, t) = F(1, t) = \pi_n(a) \end{aligned} \qquad (0 \leqslant s, t \leqslant 1).$$

Then we can find numbers $0 = s_1 < s_2 < \ldots < s_p = 1$ a.d $0 = t_1 < t_2 < \ldots < t_q = 1$ such that

(8) $\operatorname{diam}\{z_{ij}, z_{i,j+1}, z_{i+1,j}, z_{i+1,j+1}\} < \gamma$,

where $z_{ij} = F(s_i, t_j)$, and such that $\{s_i\}$ has a subsequence, $0 = s_{i(1)} < \ldots < s_{i(k)} < \ldots < s_{i(l+m)} = 1$, with $z_{i(k),1} = z_k$. Choose a point x_{ij} of $\pi_n^{-1}(z_{ij})$ so that $x_{iq} = x_{1j} = x_{pj} = a$ $(1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q)$ and $x_{i(k),1} = x_k$ $(1 \leqslant k \leqslant l+m)$. Then by (8) and (4) $\{x_{ij}\}$ is a τ_0-net in X, and by (3) $\{f(x_{ij})\}$ is an ε-net in Y. Therefore if $\beta = \{f(a) = f(x_{11}), f(x_{21}), \ldots, f(x_{i(l),1}) = f(x)\}$ and $\beta' = \{f(a) = f(x_{p1}), f(x_{p-1,1}), \ldots, f(x_{i(l),1}) = f(x)\}$, then $\beta\beta'^{-1} \simeq 0$ (ε). Hence by (2) and (2.2) the liftings of β, β' with initial point b have the same terminal point.

On the other hand, since α, α' are τ-chains, by (6) and (3) $f(\alpha), f(\alpha')$ are ε-chains with $f(\alpha) \subset \beta, f(\alpha') \subset \beta'$. Since β, β' are ε-chains, by (2.3) we have $f(\alpha)\beta^{-1} \simeq 0$ (ε), $f(\alpha')\beta'^{-1} \simeq 0(\varepsilon)$. Thus the liftings of $f(\alpha)$ and $f(\alpha')$ with initial point b have the same terminal point, $g(x)$ (cf. (2.2)).

(c) The map g is a local homeomorphism. For let U be any neighborhood of $g(x)$. Then there exist $V \in \mathscr{V}$, $\{E_1, E_2\} \subset \mathscr{C}(V)$ such that $f(x) \in V$, $x \in E_1$ and $g(x) \in E_2$. Let V' be an open set such that $f(x) \in V' \subset V$, $\operatorname{diam} V' < \varepsilon$, $\operatorname{diam}(E_1') < \tau$ and $E_2' \subset U$, where $E_1' = (f|E_1)^{-1}(V')$, $E_2' = (f|E_2)^{-1}(V')$. If u is any point of E_1' and α is a τ-chain from a to x, then $\alpha \cup \{u\}$ is a τ-chain from a to u and $f(\alpha) \cup \{f(u)\}$ is an ε-chain in Y. Lifting the ε-chain to a chain with initial point b, we see that $g(u) \in E_2' \subset U$ and hence $g(E_1') \subset U$. Thus g is continuous. Clearly $g(E_1') = E_2'$, and g is a local homeomorphism.

(d) The map g is a homeomorphism. For suppose that there exist distinct points x, x' with $g(x) = g(x')$. Let α, α' be τ-chains from a to x, x' respectively, and let β, β' be the liftings of $f(\alpha), f(\alpha')$ with initial point b. Then β, β' are $\delta/2^n$-chains from b to $g(x)$ (cf. (7)). As in Paragraph (b), we see that α, α' have the same terminal poinp, which contradicts to $x \neq x'$. Therefore g is one-to-one. Obviously g is onto.

(e) If $a \neq b$, then g has no fixed point. For otherwise there would exist $x \in X$ with $g(x) = x$. Let α be a τ-chain from a to x and let α' be the chain with initial point b, covering $f(\alpha)$. Then α^{-1}, α'^{-1} cover $f(\alpha)^{-1}$ and have the common initial point x. By (2) and (2.1), we have $a = b$, contrary to $a \neq b$.

(f) Suppose that there exist distinct points $a, b \in X$ with $f(a) = f(b)$. Then by (a) $\sim$ (e) there exists an automorphism g of X without fixed point, which contradicts to our assumption that X has the fixed point property for homeomorphisms. Thus f is a homeomorphism.

4. Proof of Theorem 2. Let a, b be points of X with $f(a) = f(b)$. We first show the existence of an automorphism g of X with respect to f such that $g(a) = b$.

We may assume that $\mu < \lambda/2$. Let τ_0 be a positive number such that if A is a subset of X with $\operatorname{diam}(A) < \tau_0$, then $\operatorname{diam}(f(A)) < \varepsilon$. Then we can find a positive integer n such that X_n is contained in a $\tau_0/4$-neighborhood of X. Choose $\delta > 0$ such that if $z, z' \in X_n$ and $d(z, z') < \delta$, then z, z' can be joined by an arc in X_n with diameter $< \tau_0/2$. There exists $\tau > 0$ such that $\tau \leqslant \min\{\tau_0, \delta\}$ and such that if α is a τ-chain in X, then each lifting of $f(\alpha)$ is a δ-chain.

(a) The definition of $g\colon X \to X$ is the same as (a) in the preceding section.

(b) The map g is well defined. For let $\alpha = \{a = x_1, x_2, \ldots, x_l = x\}$, $\alpha' = \{a = x_{l+m}, x_{l+m-1}, \ldots, x_l = x\}$ be τ-chains in X, and let α_k be an arc in X_n, from x_k to x_{k+1}, whose diameter $< \tau_0/2$. If $\psi\colon I \to X_n$ is a parametrization of the loop $\alpha_1 \cup \alpha_2 \cup \ldots \cup \alpha_{l+m-1}$, then there exists a map $F\colon I \times I \to X_n$ such that

$$\begin{aligned} F(s, 0) &= \psi(s), \\ F(s, 1) &= F(0, t) = F(1, t) = a \end{aligned} \qquad (0 \leqslant s, t \leqslant 1).$$

We can find numbers $0 = s_1 < s_2 \ldots < s_p = 1$ and $0 = t_1 < t_2 < \ldots < t_q = 1$ such that

$$\operatorname{diam}\{z_{ij}, z_{i,j+1}, z_{i+1,j}, z_{i+1,j+1}\} < \tau_0/2,$$

where $z_{ij} = F(s_i, t_j)$, and such that there exists a subsequence of $\{s_i\}$, $0 = s_{i(1)} < \ldots < s_{i(k)} < \ldots < s_{i(l+m)} = 1$, with $z_{i(k),1} = x_k$. Choose a point x_{ij} of X so that $d(x_{ij}, z_{ij}) < \tau_0/4$, $x_{iq} = x_{1j} = x_{pj} = a$ $(1 \leqslant i \leqslant p,\ 1 \leqslant j \leqslant q)$ and $x_{i(k),1} = x_k (1 \leqslant k \leqslant l+m)$. Then $\{x_{ij}\}$ is a τ_0-net in X and $\{f(x_{ij})\}$ is an ε-net in Y. Therefore if we put $\beta = \{f(a) = f(x_{11}), f(x_{21}), \ldots, f(x_{i(l),1}) = f(x)\}$ and $\beta' = \{f(a) = f(x_{p1}), f(x_{p-1,1}), \ldots, f(x_{i(l),1}) = f(x)\}$, then $\beta\beta'^{-1} \simeq 0$ (ε). Hence the liftings of β, β' with initial point b have the same terminal point (cf. (2.2)). On the other hand, since $f(\alpha) \subset \beta, f(\alpha') \subset \beta'$, by (2.3) and (2.2) the liftings of $f(\alpha), f(\alpha')$ with initial point b have the same terminal point, $g(x)$.

By the same procedure as (c) $\sim$ (f) in Section 3, we can complete the proof.

Addendum. The following Propositions 1 and 2 correspond to Theorems 1 and 2, respectively.

Proposition 1. *Let X, Y be continua, and $f\colon X \to Y$ a local homeomorphism of X onto Y. If Y is the limit of an inverse sequence, with bonding maps onto, of simply connected Peano continua, then f is a homeomorphism.*

Proposition 2. *Let X, Y be continua, and $f\colon X \to Y$ a local homeomorphism of X onto Y. If Y is the intersection of a monotone decreasing sequence of simply connected Peano continua, then f is a homeomorphism.*

I am much indebted to Professor Y. Kodama who indicated to me that Propositions 1 and 2 above are consequences of the Fox's overlay theorem [1, (5.2), p. 60]. Also after submitting the manuscript, I have known Lau's theorem (Bull. Acad. Polon. Sci. Sér. Sci. Math. 27 (1979), p. 382) deeply related to this paper.

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A sum theorem for A-weakly infinite-dimensional spaces

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Abstract. In this note we shall establish a hereditarily closure-preserving sum theorem for A-weakly infinite-dimensional spaces. The applications of this theorem to the closed mappings defined on A-weakly infinite-dimensional spaces are given in [5].

Our terminology and notation follow [2]. Let us recall that a normal space X is said to be *A-weakly infinite-dimensional* (abbrev. A-w.i.d.) if for every sequence (A_1, B_1), (A_2, B_2), ... of pairs of disjoint closed subsets of X there exists a sequence $L_1, L_2, \ldots$ of closed subsets of X such that, for each positive integer i, the set L_i is a partition between A_i and B_i in X (meaning that there exist disjoint open subsets U_i, V_i of X such that $A_i \subset U_i$, $B_i \subset V_i$ and $X \setminus L_i = U_i \cup V_i$), and $\bigcap_{i=1}^{\infty} L_i = \emptyset$. It is manifest that every closed subspace of an A-w.i.d. space is A-w.i.d.

We begin with the following obvious lemma (cf. the proof of Lemma 1.2.9 in [2]).

LEMMA 1. *Let F be a closed subspace of a hereditarily normal space X and A, B a pair of disjoint closed subsets of X. For every partition L between $A \cap F$ and $B \cap F$ in F with $F \setminus L = G \cup H$, where disjoint open subsets G, H of F are such that $A \cap F \subset G$ and $B \cap F \subset H$, there exists a partition L' between A and B in X with $X \setminus L' = M \cup N$, where disjoint open subsets M, N of X are such that $A \subset M$, $B \subset N$, $M \cap F = G$ and $N \cap F = H$.*

The next lemma deals with countable families of partitions.

LEMMA 2. *Let F be a closed subspace of a hereditarily normal A-w.i.d. space X and $(A_1, B_1), (A_2, B_2), \ldots$ a sequence of pairs of disjoint closed subsets of X. For every sequence $L_1, L_2, \ldots$, where L_i is a partition between $A_i \cap F$ and $B_i \cap F$ in F for $i = 1, 2, \ldots$, such that $\bigcap_{i=1}^{\infty} L_i = \emptyset$, there exists a sequence*