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A new construction of a Kurepa tree with no Aronszajn subtree

by

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Abstract. In 1969, we asked whether $V = L$ implies the existence of a Kurepa tree having no Aronszajn subtrees. The affirmative answer to this question was supplied by Ronald Jensen in 1971, whose proof appeared in [2]. Jensen's proof was somewhat involved, and required some delicate argumentation. We present here a much simpler proof which has the same degree of complexity as the construction of any Kurepa tree in L.

Preliminaries. For terminology and notation covering trees we refer to either [1] or [2]. An in these references, for $\lambda \leqslant \omega_1$, by a *$\lambda$-tree* we mean a normal tree of height λ having countable levels. An *Aronszajn tree* is an ω_1-tree with no uncountable branch, a *Kurepa tree* is an ω_1-tree with at least $\aleph_2$ uncountable branches. Aronszajn trees can be constructed in ZFC. Kurepa trees can be constructed assuming $V = L$ (Solovay) or $\diamondsuit^+$ – which is true if $V = L$ (Jensen).

For background on constructibility we refer to [1]. We shall not require any fine structure theory.

The question as to whether $V = L$ implies the existence of a Kurepa tree with no Aronszajn subtrees was raised by me in 1969, and answered affirmatively by Jensen in 1971. Jensen's (rather involved) proof appeared in [2], together with an application of such a tree to solve a problem in partition calculus. At the time, it seemed as though, my application to combinatorics not withstanding, such trees were merely a curiosity. (Indeed, my original question was little more than a "coffee room" variety.) That this was not the case was demonstrated by Juhász and Weiss ([3]), who proved that the existence or such a tree is equivalent to the existence of an ω_1-metrizable, ω_1-compact space of cardinality at least ω_2, resolving an old question of Sikorski.

The new construction of such a tree (from $V = L$) does not involve any new methods, rather a refinement of the known tricks of the trade. That a rather simple modification to the standard construction of a Kurepa tree in L would give the required result occurred to me after a discussion with Bill Fleissner on some work of Ken Kunen and himself on the normal Moore space problem.

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The new construction.

THEOREM (Jensen). *Assume* $V = L$. *Then there is a Kurepa tree with no Aronszajn subtree.*

Proof (Devlin). For each $\alpha < \omega_1$, let

$$S_\alpha = \{\nu \in \omega_1 \mid L_\nu \models \mathrm{ZF}^- \text{ and } \alpha = \omega_1^{L_\nu}\}.$$

Define a function $f\colon \omega_1 \to \omega_1$ by setting

$$f(\alpha) = \begin{cases} \sup(S_\alpha), & \text{if } S_\alpha \text{ is non-empty and has no largest member;} \\ \text{the least } \gamma \text{ such that } \alpha \in L_\gamma \prec L_{\omega_1}, & \text{otherwise.} \end{cases}$$

We construct an ω_1-tree T by recursion on the levels, using countable ordinals as elements. For each $\alpha < \omega_1$, $T \upharpoonright \alpha$ will be an α-tree inside $L_{f(\alpha)}$.

To commence, set $T_0 = \{0\}$, and if $T \upharpoonright (\alpha+1)$ is defined, obtain $T_{\alpha+1}$ by using the first ω unused ordinals to give every member of T_α exactly two successors on $T_{\alpha+1}$ in some canonical fashion. Clearly, if $T \upharpoonright (\alpha+1)$ is an $(\alpha+1)$-tree, then $T \upharpoonright (\alpha+2)$ will be an $(\alpha+2)$-tree, and if $T \upharpoonright (\alpha+1) \in L_{f(\alpha+1)}$, then $T \upharpoonright (\alpha+2) \in L_{f(\alpha+2)}$.

Suppose now that $\lim(\alpha)$ and $T \upharpoonright \alpha$ is defined, an α-tree in $L_{f(\alpha)}$. To obtain T_α, use the first ω unused ordinals to give one-point extensions to each α-branch of $T \upharpoonright \alpha$ which lies in $L_{f(\alpha)}$. Now, in $L_{f(\alpha+1)}$, α is countable, so this extension procedure can be done canonically within $L_{f(\alpha+1)}$, thereby ensuring that $T \upharpoonright (\alpha+1) \in L_{f(\alpha+1)}$. The question is: is $T \upharpoonright (\alpha+1)$ an $(\alpha+1)$-tree? What we must show is that for every $x \in T \upharpoonright \alpha$ there is at least one α-branch of $T \upharpoonright \alpha$ which contains x and lies in $L_{f(\alpha)}$.

If $f(\alpha)$ is the least γ such that $\alpha \in L_\gamma \prec L_{\omega_1}$ there is no problem, since α is countable in $L_{f(\alpha)}$ in this case, and the construction of α-branches within $L_{f(\alpha)}$ is straightforward. So suppose $f(\alpha) = \sup(S_\alpha)$. Then $\alpha = \omega_1^{L_{f(\alpha)}}$ and $L_{f(\alpha)}$ thinks that $T \upharpoonright \alpha$ is an ω_1-tree. (Since we use ω intervals of ordinals for the levels of T, $L_{f(\alpha)}$ recognises that each level of $T \upharpoonright \alpha$ is countable.) For some $\lambda \in S_\alpha$, $T \upharpoonright \alpha \in L_\lambda$. (In fact it is easy to see that $T \upharpoonright \alpha \in L_{\min(S_\alpha)}$, but we do not need this fact.) Given $x \in T \upharpoonright \alpha$, we construct, within L_λ, an α-branch, b_x, of $T \upharpoonright \alpha$ containing x. Let us use $b_x(\gamma)$ to denote the member of b_x in T_γ. So the definition of $b_x(\gamma)$ for $\gamma \leqslant \gamma_0 = ht(x)$ is determined by the requirement $x \in b_x$. For $\gamma > \gamma_0$, if $b_x(\gamma)$ is defined, let $b_x(\gamma+1)$ be the least (as an ordinal) extension of $b_x(\gamma)$ on $T_{\gamma+1}$. And if $\gamma > \gamma_0$ is a limit ordinal and $b_x(\delta)$ is defined for all $\delta < \gamma$, let $b_x(\gamma)$ be the unique extension of all $b_x(\delta)$, $\delta < \gamma$, on T_γ. By induction we see that $\langle b_x(\delta) \mid \delta < \gamma \rangle \in L_{f(\gamma)}$, so such a $b_x(\gamma)$ exists. This defines b_x within L_λ, as required.

We now know that $T = \bigcup_{\alpha < \omega_1} T \upharpoonright \alpha$ is well-defined and is an $\aleph_1$-tree. We prove that T is Kurepa. This is practically identical to Solovay's proof.

Notice first that the function f is definable within L_{ω_2} (by the definition given). Hence T is definable within L_{ω_2} (again by the given definition). Suppose T were not Kurepa. Then T will have exactly $\aleph_1$ many ω_1-branches, which can be enumerated as $\langle B_\nu \mid \nu < \omega_1 \rangle$. (A simple variant of our construction of the limit branches b_x given above shows that T certainly has $\aleph_1$ many ω_1-branches.) Let this enumeration be the $<_L$-least such. Then it too is definable in L_{ω_2}.

By recursion, define an increasing chain

$$N_0 \prec N_1 \prec \ldots \prec N_\nu \prec \ldots \prec L_{\omega_2} \quad (\nu < \omega_1)$$

of elementary submodels of L_{ω_2} thus:

$$N_0 = \text{the smallest } N \prec L_{\omega_2};$$
$$N_{\nu+1} = \text{the smallest } N \prec L_{\omega_2} \text{ such that } N_\nu \cup \{N_\nu \cap \omega_1\} \subseteq N;$$
$$N_\lambda = \bigcup_{\nu<\lambda} N_\nu, \text{ if } \lim(\lambda).$$

For each $\nu < \omega_1$, $N_\nu \cap \omega_1$ is transitive, so let $\alpha_\nu = N_\nu \cap \omega_1$. Then $\langle \alpha_\nu \mid \nu < \omega_1 \rangle$ is strictly increasing, continuous, and cofinal in ω_1. Let $\pi_\nu\colon N_\nu \cong L_{\beta(\nu)}$. Then $\pi_\nu \upharpoonright \alpha_\nu = \mathrm{id} \upharpoonright \alpha_\nu$, $\pi_\nu(\omega_1) = \alpha_\nu$, $\pi_\nu(T) = T \upharpoonright \alpha$, $\pi_\nu(\langle B_\gamma \mid \gamma < \omega_1 \rangle) = \langle B_\gamma \cap T \upharpoonright \alpha_\gamma \mid \gamma < \alpha_\gamma \rangle$, for each $\nu < \omega_1$.

We try to define an ω_1-branch, b, of T by recursion. Let $b(0) = 0$. The idea now is to define $b(\alpha_\nu)$, $\nu < \omega_1$, by recursion on ν. Noting that $T_1 = \{1, 2\}$, by definition, we let (if this is possible) $b(\alpha_0)$ be the extension on T_{α_0} of the $<_L$-least α_0-branch of $T \upharpoonright \alpha_0$ containing 1. Then in general, if $b(\alpha_\nu)$ is defined, let x_ν be that element of $T_{\alpha_\nu+1}$ not in B_ν and let $b(\alpha_{\nu+1})$ be the extension on $T_{\alpha_{\nu+1}}$ of the $<_L$-least $\alpha_{\nu+1}$-branch of $T \upharpoonright \alpha_{\nu+1}$ containing x_ν (if possible). Finally, if $\lim(\lambda)$ and we have defined $b(\alpha_\nu)$, $\nu < \lambda$, we let $b(\alpha_\lambda)$ be the unique point of T_{α_λ} extending all $b(\alpha_\nu)$, $\nu < \lambda$ (again, if possible).

Now, providing the above definition goes through, b will be an ω_1-branch of T distinct from each B_ν, $\nu < \omega_1$, (since $x_\nu \in b - B_\nu$ for all $\nu < \omega_1$), which will give us our desired contradiction. We prove by induction on ν that $b(\alpha_\nu)$ is well-defined for all $\nu < \omega_1$.

Well, $b(0)$ is well-defined, and since the $<_L$-least branch of $T \upharpoonright \alpha_0$ containing 1 is clearly an element of $L_{f(\alpha_0)}$, $b(\alpha_0)$ is well-defined. Moreover, if $b(\alpha_\nu)$ is defined, the $<_L$-least branch of $T \upharpoonright \alpha_{\nu+1}$ containing x_ν is an element of $L_{f(\alpha_{\nu+1})}$, so $b(\alpha_{\nu+1})$ is well-defined. (Note that whatever x_ν is, it is one of jus two ordinals, both available within $L_{f(\alpha_{\nu+1})}$.) So there remains the case of $b(\alpha_\lambda)$, where $\lim(\lambda)$, and $b(\alpha_\nu)$ is well-defined for all $\nu < \lambda$. We must show that the α_λ-branch of $T \upharpoonright \alpha_\lambda$ determined by $\{b(\alpha_\nu) \mid \nu < \lambda\}$ is an element of $L_{f(\alpha_\lambda)}$. Well, this branch is clearly definable from $\langle T \upharpoonright \alpha_\lambda < \alpha_\nu \mid \nu < \lambda \rangle$, $\langle B_\nu \cap T \upharpoonright \alpha_\lambda \mid \nu < \lambda \rangle$ in $\mathrm{ZF}^- + V = L$. (The above definition made no use of the power set axiom). So it suffices to show that each of these sets is a member of $L_{f(\alpha_\lambda)}$. (In the case where $f(\alpha_\lambda)$

$= \sup(S_{\alpha_\lambda})$, we can then drop down to some L_μ which contains these sets and is a model of ZF^- to define the required branch within L_μ; in the other case $L_{f(\alpha_\lambda)}$ is already a model of $ZF^- + V = L$.)

Now, $\alpha_\lambda = \omega_1^{L_{\beta(\lambda)}}$ (since $\alpha_\lambda = \pi_\lambda(\omega_1)$) and $L_{\beta(\lambda)} \models ZF^-$. Thus if $f(\alpha_\lambda) = \sup(S_{\alpha_\lambda})$, we have $\beta(\lambda) \in S_{\alpha_\lambda}$, so $\beta(\lambda) < f(\alpha_\lambda)$. And in the other case where $\alpha_\lambda \in L_{f(\alpha_\lambda)} \prec L_{\omega_1}$, we have

$$L_{f(\alpha_\lambda)} \models \text{“}\alpha_\lambda \text{ is countable”},$$

so again $\beta(\lambda) < f(\alpha_\lambda)$.

But $T \restriction \alpha_\lambda = \pi_\lambda(T) \in L_{\beta(\lambda)}$ and $\langle B_\nu \cap T \restriction \alpha_\lambda \mid \nu < \alpha_\lambda \rangle = \pi_\lambda(\langle B_\nu \mid \nu < \omega_1 \rangle) \in L_{\beta(\lambda)}$, so it remains only to show that $\langle \alpha_\nu \mid \nu < \lambda \rangle \in L_{f(\alpha_\lambda)}$. But it is easily seen that $\langle \alpha_\nu \mid \nu < \lambda \rangle$ is definable from $L_{\beta(\lambda)}$ in exactly the same way that $\langle \alpha_\nu \mid \nu < \omega_1 \rangle$ was defined from L_{ω_2} (see [1] for details), so in fact $\langle \alpha_\nu \mid \nu < \lambda \rangle \in L_{f(\alpha_\lambda)}$ also, and we are done.

We turn now to the proof that T has no Aronszajn subtree. Suppose, on the contrary, that there were such a subtree, and let A be one such. Let γ be the least ordinal such that $T, A \in L_\gamma$, and for each $n < \omega$ let $\varkappa(n)$ be the $(n+1)$-th ordinal greater than γ such that $L_{\varkappa(n)} \models ZF^-$. Define a chain

$$N_0^{(n)} \prec N_1^{(n)} \prec \ldots \prec N_\nu^{(n)} \prec \ldots \prec L_{\varkappa(n)} \qquad (\nu < \omega_1)$$

as we defined $N_\nu \prec L_{\omega_2}$ earlier, except that we demand that $T, A \in N_0^{(n)}$, and let $\alpha_\nu^n = N_\nu^{(n)} \cap \omega_1$. Set

$$C_n = \{\alpha_\nu^n \mid \alpha_\nu^n = \nu < \omega_1\},$$

a club subset of ω_1. Let α be the least element of $\bigcap_{n<\omega} C_n$. For all $n < \omega$, $\alpha_\alpha^n = \alpha$ and $\alpha = \pi_n(\omega_1)$, where $\pi_n\colon N_\alpha^{(n)} \cong L_{\nu(n)}$. Moreover $\pi_n \restriction \alpha = \mathrm{id} \restriction \alpha$, $\pi_n(T) = T \restriction \alpha$, $\pi_n(A) = A \cap T \restriction \alpha$, and (hence) $\pi_n(\gamma) = \bar{\gamma}$ for some $\bar{\gamma}$ independent of n.

For each n,

$$L_{\varkappa(n)} \models \text{“}A \text{ has no } \omega_1\text{-branches”}$$

so

$$L_{\varkappa(n)} \models \text{“}A \cap T \restriction \alpha \text{ has no } \alpha\text{-branches”}.$$

Thus if we can show that $f(\alpha) = \sup_{n<\omega} \nu(n)$ we shall be done, for it will then follow that no α-branch of $T \restriction \alpha$ lying within $L_{f(\alpha)}$ is contained in A, so that no element of T_α will determine an α-branch through A, whence $A \cap T_\alpha = \emptyset$, a contradiction.

Certainly, $\nu(n) \in S_\alpha$ for all $n < \omega$. So we must show that if $\mu \geq \sup_{n<\omega} \nu(n)$, then $\mu \notin S_\alpha$. Let $\nu = \sup_{n<\omega} \nu(n)$. For each $n < \omega$, $\nu(n)$ is the $(n+1)$-th ordinal greater than $\bar{\gamma}$ such that $L_{\nu(n)} \models ZF^-$ (this is easily seen), so the sequence $\langle \nu(n) \mid n < \omega \rangle$ is definable from $\bar{\gamma}$ over L_ν. Thus L_ν not $\models ZF^-$, and we know that $\nu \notin S_\alpha$. Now suppose $\mu > \nu$. If L_μ not $\models ZF^-$, then we know that $\mu \notin S_\alpha$. Suppose $L_\mu \models ZF^-$. If α is countable in L_μ, then again $\mu \notin S_\alpha$. Otherwise $\alpha = \omega_1$. Now it is easily seen that, working inside L_μ we can construct the sequence $\langle C_n \cap \alpha \mid n < \omega \rangle$ from $\bar{\gamma}$ in exactly the same way that the sequence $\langle C_n \mid n < \omega \rangle$ was constructed from γ. (In particular, we know that $\langle \nu(n) \mid n < \omega \rangle \in L_\mu$.) Inside L_μ, each set $C_n \cap \alpha$ is club in α, so $\bigcap_{n<\omega} (C_n \cap \alpha)$ is club in α. But α is the least member of $\bigcap_{n<\omega} C_n$, so $\bigcap_{n<\omega} (C_n \cap \alpha) = \emptyset$, so this is absurd. The proof is complete, since we have now shown that $\alpha \neq \omega_1^{L_\mu}$.

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