Applications of Luzinian separation principles (non-separable case)

by

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Abstract. Using the results of [F_3] and [F–$H_{1,2}$] we prove "the first separation principle", and use it to add some new properties of analytic and Luzin spaces. Further applications of the first, and the second Luzin separation principle respectively, extend the sooner results of Frolík [F_3], Luzin [L] and Purves [Pu] concerning images of measurable sets under measurable mappings.

This is a continuation of [F_3], and [F–$H_{1,2}$]. Here we introduce Baire sets, prove "the first separation principle" (1.3), and develop the properties of analytic and Luzin spaces, for which the first principle and Baire sets are relevant. The Baire sets in a space are usually defined as the smallest σ-algebra containing the zero-sets (equivalently: making measurable all uniformly continuous functions). Here the elements of this σ-algebra are called *ω-Baire sets*, and in general, *$\varkappa$-Baire sets* are elements of the smallest σ-algebra containing the zero-sets, which is, in addition, closed under taking the unions of discrete families of cardinal $\leqslant\varkappa$. Finally, a Baire set is a $\varkappa$-Baire set for some $\varkappa$. This terminology makes the wording of the first principle (§ 1), and also of some other theorems, almost identical with the corresponding results in separable theory. It may be more appropriate to speak about $\varkappa$-extended Baire sets or $\varkappa$-hyper-Baire sets (see Remarks 1.1), leaving the usual meaning to the term Baire sets.

In § 2 the Luzin spaces are characterized externally without using the d-Suslin operation.

§ 3 is devoted to point-analytic and point-Luzin spaces; note that point-ω-analytic spaces are called *Suslin spaces* by N. Bourbaki, and point-ω-Luzin spaces are called *Luzin spaces* by Bourbaki. These spaces are hereditarily paracompact, and Baire sets coincide with Borel sets (defined in 1.1). These spaces are characterized among all analytic or Luzin spaces by existence of a continuous (not necessarily uniformly continuous!) bijection onto a metric space.

In § 4 measurable maps of analytic spaces into point-Luzin spaces are studied. We know that the image of an analytic space under a usco-compact σ-dd-preserving correspondence is analytic. In § 4 we are trying to weaken "usco" to Suslin measurability.

In § 5 we prove an adaptation of two theorems of Luzin and Purves to the non-separable case (see Theorem 5.3).

§ 1. First separation principle. The main result is Theorem 1.3 below which generalizes the famous Luzin 1st Separation Principle, and also the version for ω-analytic spaces [F1] and metrizable analytic spaces [Ha2] to analytic spaces. Recall that by a space we mean a uniform space, and if we speak about topological spaces we have in mind the fine uniformity generated by all continuous pseudometrics.

1.1. Baire sets. If X is a space, and if $\varkappa$ is an infinite cardinal, we denote by $\mathbf{Ba}_\varkappa(X)$ the smallest collection of sets $\mathscr{M}$ which contains $Z(X)$ (= the collection of all zero-sets in X, i.e. the sets of the form $Z(f) = \{x \mid fx = 0\}$ where f is a uniformly continuous function), and which satisfies the following two conditions (i) and (ii):

(i) $\mathscr{M}$ is a σ-algebra;

(ii) $\mathscr{M}$ is closed under taking the unions of discrete families with the index set of cardinal $\leqslant\varkappa$.

The elements of $\mathbf{Ba}_\varkappa(X)$ are called *$\varkappa$-Baire sets in X*. The Baire sets in X are the elements of

$$\mathbf{Ba}(X) = \bigcup\{\mathbf{Ba}_\varkappa(X) \mid \varkappa\}.$$

Similarly we obtain the definition of $\varkappa$-Borel sets $(\mathbf{Bo}_\varkappa(X))$ and Borel sets $(\mathbf{Bo}(X))$; we just replace $Z(X)$ by the collection of all closed sets (denoted by $F(X)$).

Since every zero-set is closed, we have

$$\mathbf{Ba}_\varkappa(X) \subset \mathbf{Bo}_\varkappa(X),$$

and the two collections coincide if $Z(X) = F(X)$, e.g. if X is metrizable.

Remarks. (1) In the case when X is metrizable the Borel sets were introduced by Hansell in his theses under the name hyper-Borel sets; in his subsequent papers R. Hansell is using the term extended Borel sets. The first author of this note introduced and studied Baire sets in uniform spaces under the name hyper-Baire sets, using the term Baire sets for what we call here ω-Baire sets (i.e. following the usual way from topological spaces).

(2) The $\varkappa$-Baire sets in a topological space are defined to be the $\varkappa$-Baire sets w.r.t. the fine uniformity of X, i.e. w.r.t. the uniformity defined by all continuous pseudometrics on X. Similarly we define $\mathbf{Ba}(X)$, $\mathbf{Bo}_\varkappa(X)$ and $\mathbf{Bo}(X)$ if X is a topological space.

It is very important for our purposes, that $\mathbf{Ba}(X)$ can be defined without the assumption of being closed under taking of complements. In fact, the following result is true.

PROPOSITION. *Let X be a space. Then $\mathbf{Ba}_\varkappa(X)$ is the smallest collection $\mathscr{M}$ of sets such that $\mathscr{M} \supset Z(X)$, condition* (ii) *is fulfilled, and also the following condition is fulfilled*:

(iii) $\mathscr{M}_\sigma = \mathscr{M}_\delta = \mathscr{M}$.

Of course, $\mathscr{M}_\sigma$ stands for the collection of the countable unions of elements of $\mathscr{M}$, and $\mathscr{M}_\delta$ stands for the collection of the countable intersections of elements of $\mathscr{M}$.

The proof is quite easy. To make it more easier we formulate a more general result (have in mind that the complements of the zero-sets are in $Z_\sigma(X)$).

LEMMA. *Assume that $\mathscr{N} \supset Z(X)$ is a collection of subsets of X. Let $\mathscr{M}$ be the smallest collection of subsets of X such that $\mathscr{M} \supset \mathscr{N}$, and $\mathscr{M}$ satisfies* (ii) *and* (iii). *If the complement of each set in $\mathscr{N}$ is in $\mathscr{M}$, then $\mathscr{M}$ is closed under the taking of complements, and hence $\mathscr{M}$ satisfies* (i).

Proof. Consider

$$\mathscr{C} = \{Y \mid X \in \mathscr{M},\ X \setminus Y \in \mathscr{M}\}.$$

By our assumption $\mathscr{C} \supset \mathscr{N}$. It is enough to show that $\mathscr{C}$ satisfies (ii) and (iii) (with $\mathscr{M}$ replaced by $\mathscr{C}$).

The statement (iii) is quite easy. Let $\{C_a\}$ be a discrete collection of $\varkappa$ sets from $\mathscr{C}$. There is a uniformly continuous pseudometric α on X such that $\{C_a\}$ is discrete in $\langle X, \alpha\rangle$, and thus there are zero sets Z_a in $\langle X, \alpha\rangle$ with $C_a \subset Z_a$ and $\{Z_a\}$ discrete in $\langle X, \alpha\rangle$. The sets $(X \setminus C_a) \cap Z_a = Z_a \setminus C_a$ are elements of $\mathscr{C} \subset \mathscr{M}$, and the cozero set $X \setminus \bigcup\{Z_a\}$ is also in $\mathscr{C} \subset \mathscr{M}$. Thus both $\bigcup\{C_a\}$ and its complement belong to $\mathscr{M}$. For a partial result see [F5].

The collection $S(X)(\supset F(X))$ of all Suslin sets in X is closed under the discrete unions, countable intersections and countable unions, and hence we get from the proposition:

COROLLARY. *$S(X) \supset \mathbf{Ba}(X)$, and hence*

$$\text{bi-}S(X) \supset \mathbf{Ba}(X).$$

Of course, bi-$S(X)$ stands for the collection of all bi-Suslin sets, i.e. the sets M such that both M and $X \setminus M$ are Suslin.

Worning. The collection $S(Z(X))$ of all Suslin sets derived from the zero sets does not need to contain $\mathrm{Ba}(X)$. The point is that the union of a discrete family of zero sets does not need to be a zero set. The usual example is the space $\varkappa^\lambda$ where both $\varkappa$ and λ are uncountable cardinals; the argument is that the zero sets, and hence the Suslin sets derived from the zero sets depend on a countable number of coordinates, while the union of a discrete family of zero sets does not need to depend on a countable number of coordinates. On the other hand, if in a space X the union of a discrete family of zero sets is a zero set (e.g., if X is topologically fine, or more generally, locally fine), then $S(X)$ may be replaced by $S(Z(X))$ in the corollary.

In many considerations the following simple remark is essential.

PROPOSITION. *Let Y be a subspace of X. Then*:

$$\mathbf{Ba}_\varkappa(Y) = \{B \cap Y \mid B \in \mathbf{Ba}_\varkappa(X)\},$$
$$\mathbf{Bo}_\varkappa(Y) = \{B \cap Y \mid B \in \mathbf{Bo}_\varkappa(X)\}.$$

Moreover,

(a) *if $\{Y_a\}$ is a discrete family of $\varkappa$-Baire sets in Y, then there exists a discrete family $\{X_a\}$ of $\varkappa$-Baire sets in X such that $Y_a = Y \cap X_a$ for each a,*

(b) *if* $\{Y_a\}$ *is a* σ*-dd family of Baire sets in* Y, *then there exists a* σ*-dd family* $\{X_a\}$ *of Baire sets in* X *such that* $Y_a = Y \cap X_a$ *for each* a; *if* $\{Y_a\}$ *is disjoint then* $\{X_a\}$ *may be chosen disjoint.*

The proof may be left to the reader; we just remark that it is essential for the proof that a discrete family on Y is discrete in X, and $\boldsymbol{Z}(Y) = \{Z \cap Y|\ Z \in \boldsymbol{Z}(X)\}$. It should be noted that a similar result for topological spaces fails; indeed if a topological space Y is a subspace of a topological space X then the fine uniformity of Y may be quite finer than the relativization of the fine uniformity of X to Y, and it may be so bad that $\boldsymbol{Z}(Y)$ is much smaller than $\{B \cap Y|\ B \in \mathbf{Bo}(X)\}$, discrete families in X are finite while Y contains arbitrarily large discrete families. For example take for Y a discrete space, and the one-point compactification of Y for X.

1.2. Say, that sets A and S are separated by a set B if

$$A \subset B \subset X \setminus S .$$

SEPARATION LEMMA. *Let* $\{X_a|\ a \in A\}$ *be a* σ*-dd family of subsets of a space* X, $|A| \leqslant \varkappa$. *Let* $\{Y_n\}$ *be a countable sequence of subsets of* X, *and let every pair* X_a, Y_n *be separated by a* $\varkappa$*-Baire set* B_{an}. *Then, the unions*

$$\bigcup \{X_a|\ a \in A\} \quad \text{and} \quad \bigcup \{Y_n|\ n \in \omega\}$$

are separated by a $\varkappa$*-Baire set.*

Proof. Let $\{X_{ak}|\ a \in A, k \in \omega\}$ be a σ-discrete decomposition of $\{X_a\}$ and $\{C_{ak}\}$ be a discrete family of zero-sets such that $C_{ak} \supset X_{ak}$ for each k. Put

$$B = \bigcap_{n \in \omega} \bigcup_{k \in \omega} \bigcup_{a \in A} (B_{an} \cap C_{ak}) .$$

The set B is $\varkappa$-Baire, and separates the two sets.

1.3. 1ST SEPARATION PRINCIPLE. *Let* X *be a space,* A *be a* $\varkappa$*-analytic subset of* X, *and* S *a Suslin subset of* X *with* $A \cap S = \emptyset$. *Then,* A *and* S *are separated by a* $\varkappa$*-Baire set* B.

Proof. Suppose that A and S are not separated by a $\varkappa$-Baire set, and that $f: \varkappa^\omega \to A$ is an analytic parametrization of A [F-H$_2$, 3.2]. Let

$$S = \bigcup \{\bigcap \{S_{\sigma|n+1}|\ n \in \omega\}|\ \sigma \in \omega^\omega\}$$

be a Suslin stratification of S, i.e.

$$\bigcap \{\bar{S}_{\sigma|n+1}|\ n \in \omega\} = \bigcap \{S_{\sigma|n+1}|\ n \in \omega\}$$

for each $\sigma \in \omega^\omega$, and

$$S_{i_0,\ldots,i_n} = \bigcup \{S_{i_0,\ldots,i_n,k}|\ k \in \omega\}$$

for each finite sequence $\{i_0, \ldots, i_n\} \in \omega^{n+1}$. According to Separation Lemma we find inductively $d = (d_0, d_1, \ldots) \in \varkappa^\omega$, and a σ in ω^ω such that:

($*$) the sets $f\,[\varkappa^\omega(d|n+1)]$ and $S_{\sigma|n+1}$ are not separated for $n \in \omega$. (Here $\varkappa^\omega(d|n+1) = \{e \in \varkappa^\omega|\ e|n+1 = d|n+1\}$.) For $n = n_0$ large enough the compact set

$$\bigcap \{f\,[\varkappa^\omega(d|n+1)]|\ n \in \omega\} = K,$$

and the closed set $\bar{S}_{\sigma|n_0+1}$ are disjoint because $\bigcap \bar{S}_{\sigma|n+1} = \bigcap S_{\sigma|n+1}$. Since f is upper semi-continuous there is an integer $n_1 \geqslant n_0$ such that $f\,[\varkappa^\omega(d|n_1+1)]$ is contained in the open set $X \setminus \bar{S}_{\sigma|n_0+1}$, and thus it is contained in $X \setminus \bar{S}_{\sigma|n_1+1}$ because the stratification is "monotone". This contradicts ($*$).

1.4. Here we are going to state several simple consequences of the 1st Separation Principle.

COROLLARY 1. (a) *If* X *is analytic, and if* A *and* S *are two disjoint Suslin sets in* X, *then* A *and* S *are separated by a Baire set. In particular,*

$$\mathbf{bi}\text{-}S(X) = \mathbf{Ba}(X)$$

if X *is analytic.*

(b) *If* A *and* S *are two disjoint analytic subspaces of* X, *then* A *and* S *are separated by a Baire set in* X, *in particular if* Y *and* $X \setminus Y$ *are analytic then* Y *is a Baire set (however then* X *is analytic).*

Proof. One only needs to know that a Suslin subset of an analytic space is analytic in the subspace uniformity, and if the relativization of the uniformity of X to a subset is analytic, then the subset is Suslin in X [F-H$_2$, §§ 3, 4].

A mapping F: $X \to Y$ is called $(\mathscr{A} \leftarrow \mathscr{B})$*-measurable* if $f^{-1}[B] \in \mathscr{A}$ for each B in $\mathscr{B}$.

COROLLARY 2. *Assume that* $\mathscr{M} \subset \exp Y$ *is closed under taking the operation of complementation (i.e. if* $M \in \mathscr{M}$ *then* $Y \setminus M \in \mathscr{M}$*). If* f: $X \to Y$ *is* $(\boldsymbol{S}(X) \leftarrow \mathscr{M})$*-measurable, and if* X *is analytic then* f *is* $(\mathbf{Ba}(X) \leftarrow \mathscr{M})$*-measurable.*

Proof. Apply Corollary 1.

COROLLARY 3. *Let* $\{X_a|\ a \in A\}$ *be a* σ*-dd disjoint family of* $\varkappa$*-analytic subsets of* X. *Then there exists a disjoint family* $\{B_{an}|\ a \in A, n \in \omega\}$ *of* $\varkappa$*-Baire sets in* X *such that* $\{B_{an}|\ a \in A\}$ *is discrete for each* n, *and* $X_a \subset \bigcup \{B_{an}|\ n \in \omega\}$ *for each* a *in* A.

Proof. Put $Y = \bigcup \{X_a|\ a \in A\}$. To apply Proposition 1.1(a) we need to check that each X_a is a $\varkappa$-Baire set in Y, and this follows from the 1st Principle because X_a is analytic in Y, and $Y \setminus X_a$ is Suslin in Y as a σ-dd union of Suslin sets.

Remark. Note that Y in Corollary 3 does not need to be analytic; of course Y is analytic in the fine uniformity.

COROLLARY 4. *Let* $\{\mathscr{A}_n\}$ *be a Luzin sequence of covers. Then the elements of* $\mathscr{A} = \bigcup \{\mathscr{A}_n\}$ *are Baire sets.*

Proof. The elements of $\mathscr{A}$ are Luzin by [F-H$_2$, 4.3(b)]. Corollary 3 applies. One can also use the 1st principle directly on a given $A \in \mathscr{A}$ and the complement of A.

§ 2. **Luzin spaces.** Recall [F-H$_2$, 3.1] that by a $\varkappa$*-Luzin space* we mean a space X for which there exist a complete metric space M of weight $\varkappa \geqslant \omega$, and a disjoint usco-compact σ-dd-preserving correspondence from M onto X. Here we use the 1st separation principle to prove a nice external behavior of Luzin subspaces. The first result implies that Luzin subspaces of any space X are Borel subsets of X. The second result characterizes Luzin subspaces of a Luzin space in terms of closed and Baire sets. The two results generalize to Luzin spaces the results of the first author for ω-Luzin spaces, called Borelian by him, and descriptive Borel by C. A. Rogers [Ro]. Certainly the proof of the first result follows the classical pattern used for proving that a 1-1 continuous image of a Borel set in a complete separable metric space is Borel in any metric space.

2.1. THEOREM. *Let L be a $\varkappa$-Luzin subspace of X. Then L is a countable intersection of countable disjoint unions of sets of the form $F \cap B$, where F is closed and B is $\varkappa$-Baire in X. In particular, L is a $\varkappa$-Borel set in X.*

It should be noted that the conclusion can be written in symbols as follows:

$$L \in ([F(X)] \cap [\mathbf{Ba}_\varkappa(X)])_{\sigma_d \delta}$$

where $\mathscr{M}_{\sigma_d}$ means the collection of all countable disjoint unions of sets in $\mathscr{M}$, and

$$[\mathscr{M}] \cap [\mathscr{N}] = \{M \cap N |\ M \in \mathscr{M}, N \in \mathscr{N}\}.$$

Proof. It seems the most convenient way is to use the Luzin sequence of covers of L. We shall work directly with the defining parametrization from a Baire space. Let f be a σ-dd-preserving disjoint usco-compact correspondence from $\varkappa^\omega$ onto L. For each a in $\varkappa^n$ let L_a be the image under f of the Baire interval

$$\{d |\ d|n = a\}.$$

Obviously the sets L_a are Luzin, and of course each $\{L_a |\ a \in \varkappa^n\}$ is a disjoint σ-dd family. By 1.4(c) we can find $\varkappa$-Baire sets B_a^k in X such that

$$\{B_a^k |\ a \in \varkappa^n, k \in \omega\} \text{ is disjoint for each } n,$$
$$\{B_a^k |\ a \in \varkappa^n\} \text{ is discrete for each } n \text{ and } k,$$

and

$$L_a \subset \bigcup \{B_a^k |\ k \in \omega\} \quad \text{for each } a.$$

We may, and shall, assume that

$$\{B_a^k |\ k \in \omega, a \in \varkappa^{n+1}\} \quad \text{refines} \quad \{B_b^k |\ k \in \omega, b \in \varkappa^n\}$$

for $n = 1, 2, \ldots$, and this means, in particular, that if

$$B_a^k \cap B_b^l \neq \emptyset, \quad a \in \varkappa^n, \quad b \in \varkappa^{n+1}, \quad \text{then} \quad a = b|n.$$

Now it is to check that

$$L = \bigcap \{\bigcup \{\overline{L_a \cap B_a^k} \cap B_a^k |\ a \in \varkappa^{n+1}, k \in \omega\} |\ n \in \omega\}.$$

Indeed, clearly the inclusion $\subset$ holds, and to check the converse inclusion, if x belongs to the right-hand side, then

$$x \in \bigcap \{\overline{L_{a_n} \cap B_{a_n}^{k_n}} \cap B_{a_n}^{k_n} |\ n \in \omega\}$$

and by our choice of B_a^k, necessarily the family $\{a_n\}$ defines a d in $\varkappa^\omega$ such that

$$a_n = d|n+1 \quad \text{for each } n.$$

Hence

$$x \in \bigcap \{\overline{L}_{d|n+1} |\ n \in \omega\} = f[d] \subset L.$$

It remains to check that for each $n \in \omega$ and each $k \in \omega$ the set

$$\bigcup \{\overline{L_a \cap B_a^k} \cap B_a^k |\ a \in \varkappa^{n+1}\}$$

is of the form $F \cap B$ with F a closed set and B a Baire set. The family is discrete, and hence we can take a discrete family $\{Z_a\}$ of zero sets such that $\overline{L_a \cap B_a^k} \subset Z_a$. Put

$$B = \bigcup \{B_a^k \cap Z_a |\ a\},$$
$$F = \bigcup \{\overline{L_a \cap B_a^k} |\ a\}.$$

2.2. THEOREM. *The collection* $\mathbf{Luz}(X)$ *of Luzin subsets of a Luzin space X coincides with the sets of the form described in Theorem* 2.1.

Proof. By Theorem 2.1 each Luzin subspace can be written in that way, and to prove the converse, it is enough to verify the following:

LEMMA (a). *Every Baire set in a Luzin space is Luzin.*

Lemma (a) follows immediately from the following description of Baire sets.

LEMMA (b). *For any space X, $\mathbf{Ba}_\varkappa(X)$ is the smallest collection $\mathscr{M}$ of sets such that $\mathscr{M} \supset Z(X)$, $\mathscr{M}_\delta = \mathscr{M}_{\sigma_d} = \mathscr{M}$, and $\mathscr{M}$ is closed under the operation of taking the unions of discrete families of cardinal $\leqslant \varkappa$.*

The proof follows from the following useful observation. Write $\mathscr{M}_{\sigma_d}$ for the collection consisting of the unions of disjoint sequences in $\mathscr{M}$.

LEMMA (c). *Let $\mathscr{M}$ be a collection of subsets of a set X such that $\mathscr{M}_\delta = \mathscr{M}_{\sigma_d} = \mathscr{M}$. Then $\mathscr{C}_\delta = \mathscr{C}_\sigma = \mathscr{C}$ where*

$$\mathscr{C} = \{Y |\ Y \in \mathscr{M}, X \setminus Y \in \mathscr{M}\}.$$

If, in addition, X is a space, $\mathscr{C} \supset Z(X)$, then if $\mathscr{M}$ is closed under taking discrete unions of $\varkappa$-elements, then so is $\mathscr{C}$.

Proof of Lemma (c). (1) First check that if $Y_1, Y_2 \in \mathscr{C}$ then $Y_1 \setminus Y_2 \in \mathscr{C}$. One should show that $Y_1 \setminus Y_2$ and $X \setminus (Y_1 \setminus Y_2)$ belong to $\mathscr{M}$, and this follows from:

$$Y_1 \setminus Y_2 = Y_1 \cap (X \setminus Y_2), \quad X \setminus (Y_1 \setminus Y_2) = (X \setminus Y_1) \cup (Y_1 \cap Y_2)$$

(the union is disjoint).

(2) Show that $\mathscr{C}$ is closed under finite unions, and then under countable unions. Then, of course, $\mathscr{C}$ is closed under countable intersections because it is complemented.

(3) The invariance under discrete unions is shown like in the proof of Lemma 1.1.

Proof of Lemma (b): Let $\mathscr{M}$ be the collection from Lemma (b). Clearly $\mathscr{M} \subset \mathbf{Ba}_\varkappa(X)$. Let $\mathscr{C}$ be the collection from Lemma (c). If we shaw that each cozero set is in $\mathscr{C}$, then $\mathbf{Ba}_\varkappa(X)\subset\mathscr{C}$, hence $\mathscr{M} = \mathbf{Ba}_\varkappa(X)$. To do that check that each cozero set is in

$$(Z(X))_{\sigma_d\delta\sigma_d}.$$

It should be noted that this fact has been observed by J. Jayne. It is enough to check it for the open unit interval J on the real line. The interval J is a disjoint union of one F_{σ_d} set, and one G_δ nowhere dense set $H = \bigcap G_n$, where G_n's are open. It suffices to find F_{σ_d} sets F_n such that $H\subset F_n\subset G_n$. Notice that this follows from the simple case where G_n's are open subintervals of J.

§ 3. Point-analytic spaces. A space X is called *point-$\varkappa$-analytic* if there exists a single-valued $\varkappa$-analytic parametrization of X; this is equivalent to saying thas there exists a σ-dd-preserving continuous mapping of a complete metric space of weight $\leqslant\varkappa$ onto X. Similarly, X is *point-$\varkappa$-Luzin* if there exists a single-valued $\varkappa$-Luzin parametrization of X; this is equivalent to saying that there exists a σ-dd-preserving 1-1 continuous mapping of a complete metric space of weight $\leqslant\varkappa$ onto X.

It should be noted that point-ω-analytic spaces are just the completely regular "Suslin" spaces in the terminology of N. Bourbaki, and point-ω-Luzin spaces are just the completely regular "Luzin" spaces in the terminology of N. Bourbaki.

If follows easily from the proof of [F-H$_2$, Prop. 3.1] that a space is point-$\varkappa$-analytic iff it is $\varkappa$-analytic and point-analytic.

3.1. THEOREM. *The following three conditions on an analytic (Luzin) space X are equivalent:*

(a) *X is point-analytic (point-Luzin);*

(b) *there exists a 1-1 σ-dd-preserving continuous mapping of X onto a metric space S;*

(c) *there exists a 1-1 continuous mapping g of X onto a metric space S.*

Obviously (b) $\Rightarrow$ (c).

Proof of (c) $\Rightarrow$ (a). Let f: $M\to X$ be an analytic (a Luzin) parametrization of X. Let T be the graph of f (given the subspace topology from $M\times X$), and let p be the restriction to T of the projection $M\times X\to M$, and let k be the restriction to T of the projection $M\times X\to X$. Then p: $T\to M$ is a perfect mapping, k: $T\to X$ is a continuous mapping, and put $f = k\circ p^{-1}$. Moreover, k is σ-dd-preserving by [F-H$_2$, L. 2.5 and Th. 3.1(a)]. It is enough to show that T is metrizable, because then T is complete in some metric because it is a Čech complete space as a perfect pre-image of a Čech complete space [F$_4$], and k: $T\to X$ is a point-analytic (point-Luzin) parametrization of X. The reduced product l of p and $g\circ k$ is a perfect mapping (because the former mapping is perfect and the latter mapping is continuous). On the other hand, it is obvious that l is 1-1. Hence l is a homeomorphism of T into $M\times S$.

Proof of (a) $\Rightarrow$ (b) follows from

LEMMA (b). *Let $\mathscr{D} = \bigcup\{\mathscr{D}_n\}$ be a σ-discrete base for the collection of all open sets in a uniform space X (such a base exists if there exists a continuous map f from a metric space M onto X such that $f[\mathrm{open}(M)]$ has a σ-discrete base). Then there exists a continuous 1-1 σ-dd-preserving mapping g onto a metric space S.*

Proof. Let g_1: $X\to S_1$ be a uniformly continuous mapping into a metric space S_1 such that each $g_1[\mathscr{D}_n]$ is discrete. We construct continuous g_2: $X\to S_2$ such that S_2 is a separable metric space, and

$$g = g_1\times_r g_2:\ X\to S_1\times S_2$$

is 1-1.

For each $n, m\in\omega$ let g_2^{nm}: $X\to R$ be a continuous function such that (put $D_n = \bigcup\mathscr{D}_n$):

$$g_2^{nm}x = \begin{cases} = 0 & \text{if } x\in D_n, \\ = 1 & \text{if } x\in D\in\mathscr{D}_m, \text{ and } D \text{ is distant to } D_n.\end{cases}$$

Put

$$g_2 = \textstyle\prod_r\{g_2^{nm}\}:\ X\to R^{\omega\times\omega}(= S_2).$$

We shall prove that g is 1-1.

Assume that $g_1x = g_1y$, and $x\neq y$. We shall find $n, m\in\omega$ such that $g_2^{nm}x\neq g_2^{nm}y$. Choose disjoint open sets U and V such that $x\in U$, $y\in V$. Choose $D\in\mathscr{D}$ such that $x\in D\subset U$. If $D\in\mathscr{D}_n$, then $y\notin\overline{D}_n$ and we can choose $D'\in\mathscr{D}$ such that

$$y\in D',\quad D' \text{ is distant to } \overline{D}_n.$$

Now D' belongs to some $\mathscr{D}_m$, and clearly

$$g_2^{nm}x = 0,\quad g_2^{nm}y = 1.$$

It is easy to check that $g_1\times_r g_2$ is σ-odd-preserving.

EXAMPLE. There exists a point-Luzin space X such that there exists no 1-1 uniformly continuous mapping of X onto a metric space. Let $X = 2\times\omega_1$ with the uniformity having the following covers $\mathscr{U}_\alpha$, $\alpha\in\omega_1$, for a basis:

$$\mathscr{U}_\alpha = \{\{\langle i,\beta\rangle\}|\ \beta\leqslant\alpha,\ i\in 2\}\cup\{2\times\{\gamma\}|\ \gamma>\alpha\}.$$

If f: $X\to S$ is uniformly continuous, and S is metric then there exist $\alpha_n\in\omega_1$ such that if $x\in\mathrm{st}(y,\mathscr{U}_{\alpha_n})$ ([1]) for each n, then $fx = fy$. Choose $\alpha>\sup\{\alpha_n\}$. Then $f\langle 0,\gamma\rangle = f\langle 1,\gamma\rangle$ for $\gamma>\alpha$.

3.2. Here we derive several consequences of Theorem 3.1.

PROPOSITION (a). *A metrizable space X is $\varkappa$-analytic ($\varkappa$-Luzin) iff it is point-$\varkappa$-analytic (point-$\varkappa$-Luzin, resp.).*

Proof. Consider the identity mapping $X\to X$, and apply Theorem 3.1.

PROPOSITION (b). (α) *Let $\{X_a|\ a\in A\}$ be a family of point-$\varkappa$-analytic (or point-$\varkappa$-Luzin) spaces, and $|A|\leqslant\varkappa$. Then the uniform sum $X = \sum\{X_a|\ a\in A\}$ of $\{X_a\}$ is also point-$\varkappa$-analytic (point-$\varkappa$-Luzin).*

([1]) The symbol $\mathrm{st}(y, \mathscr{M})$ stands for $\bigcup\{M\in\mathscr{M}|\ y\in M\}$.

(β) *Let* $\{X_n|\ n\in\omega\}$ *be a family of point-*$\varkappa$*-analytic (point-*$\varkappa$*-Luzin) spaces. Then the countable product* $X=\prod\{X_n|\ n\in\omega\}$ *is also point-*$\varkappa$*-analytic (point-*$\varkappa$*-Luzin).*

(γ) *Let* $\{Y_n|\ n\in\omega\}$ *be a family of* $\varkappa$*-analytic (or* $\varkappa$*-Luzin) subspaces of a space* Y*, and let* Y_0 *be point-*$\varkappa$*-analytic (or point-*$\varkappa$*-Luzin). Then the intersection* $Y=\bigcap\{Y_n|\ n\in\omega\}$ *is a point-*$\varkappa$*-analytic (or point-*$\varkappa$*-Luzin) subspace of* Y.

Proofs. If we omit "point" from Proposition (b) then we get statements that are proved in [F-H$_2$, § 3.1]. According to Theorem 3.1 above there are continuous 1-1 mappings $f_a\colon X_a\to M_a$, $g_n\colon X_n\to M_n$, and $h_n\colon Y_n\to M$ where M_a, M_n and M are metric spaces. It is easy to define 1-1 continuous mappings $f\colon \sum\{X_a\}\to\sum\{M_a\}$, $g\colon\prod\{X_n\}\to\prod\{M_n\}$, and $h\colon\bigcap\{Y_n\}\to M$, and apply Theorem 3.1.

Remark. In a similar way one can prove that a discrete union of $\varkappa$ point-$\varkappa$-analytic (point-$\varkappa$-Luzin) spaces is point-$\varkappa$-analytic (or point-$\varkappa$-Luzin) if the union is analytic or σ-dd-simple.

It follows also that Suslin (d-Suslin) subsets of point-$\varkappa$-analytic (point-$\varkappa$-Luzin) spaces are point-$\varkappa$-analytic (point-$\varkappa$-Luzin). This corollary to Theorem 3.1 will be strenghtened in Theorem 3.4.

On the other hand, further easy properties of point-analytic (point-Luzin) spaces, like invariance under taking countable disjoint unions, do not reduce to the corresponding properties of analytic (Luzin) spaces and Theorem 3.1. One must repeat the proofs of [F-H$_2$, 3.1], and check that the resulting correspondences are single-valued. The invariance under the Suslin operation is left to 3.4, and here we just note a result which will be needed in 3.3 (Cor. (c)).

PROPOSITION (c). *Countable union of point-analytic subspaces is point-analytic, countable disjoint union of point-Luzin subspaces is point-Luzin.*

3.3. By 1.4 Cor. 1 (a) in an analytic space the Baire sets are just the bi-Suslin sets, and the collection of all Borel sets may be much larger.

THEOREM. *If* X *is point-analytic then*

$$\mathbf{Ba}(X)=\mathbf{Bo}(X)\,.$$

Proof. It is enough to show that each open set U in X is analytic, because then U is bi-Suslin, hence Baire. Let $f\colon M\to X$ be a point-analytic parametrization of X, and let $S=f^{-1}[U]\subset M$ has the subspace topology. Since S is open in M, S is necessarily completely metrizable. The restriction g of f to a mapping of S onto U is an analytic (in fact, point-analytic) parametrization of U.

COROLLARY (a). *Every point-analytic space is hereditarily paracompact.*

Proof is standard. Let Y be a subspace of a point-analytic uniform space, and let $\mathscr{V}$ be an open cover of Y. For each V in $\mathscr{V}$ let U_V be an open set in X such that $V=Y\cap U_V$, and put $U=\bigcup\{U_V\}$. Since U is analytic, hence paracompact by [F-H$_2$, 3.1(a)], there is a σ-discrete refinement $\mathscr{U}'$ of $\{U_V\}$; the trace of $\mathscr{U}'$ on Y is a σ-discrete refinement of $\mathscr{V}$.

Remark. Note that paracompactness of a uniform space implies paracompactness of the induced topological space, see [F-H$_2$, § 2].

COROLLARY (b). *If* X *is point-Luzin then the following collections coincide:* $\mathbf{Ba}(X)$, $\mathbf{Bo}(X)$, *the collection of d-Suslin sets, the collection of all Luzin subsets.*

COROLLARY (c). *In any space* X*, the collection of point-Luzin subsets is closed under countable unions.*

Proof. Firstly assume that L_1 is a Luzin set and L_2 is a point-Luzin subsets of a space X. The set $L_1\cap L_2$ is Luzin in L_2, hence a Baire set in L_2 by Corollary (b), and hence $L_2\setminus(L_1\cap L_2)$ is a Baire set in L_2, hence a Luzin set by 2.2 Lemma (a).

Now if $\{L_n\}$ is a sequence of point-Luzin subsets of X, then all

$$L'_n=L_n\setminus\bigcup\{L_k|\ k<n\}$$

are point-Luzin, and

$$\bigcup\{L'_n\}=\bigcup\{L_n\}\,.$$

Since $\{L'_n\}$ is a disjoint sequence, $\bigcup\{L'_n\}$ is clearly point-Luzin.

Remark. The union of a countable family of Luzin sets does not need to be Luzin, see [F$_4$, 7.9].

3.4. Suslin operation. The proof of the following result follows the pattern of the proof of the corresponding result with "point-" omitted in [F-H$_2$, 3.1].

THEOREM. *The class of all point-analytic (point-Luzin) subsets of a space* X *is closed under Suslin operation (d-Suslin operation).*

Proof. Let

$$A=\bigcup\{\bigcap\{A_{\sigma|n+1}|\ n\in\omega\}|\ \sigma\in\omega^\omega\}$$

where all $A_{\sigma|n+1}$ are analytic (or Luzin and the union is disjoint).

For $s\in\omega^{n+1}$ denote by Bs the corresponding Baire interval

$$\{\sigma|\ \sigma|n+1=s\}\,,$$

and consider the subset of $\omega^\omega\times X$

$$S=\bigcap\{\bigcup\{\mathrm{Bs}\times A_s|\ s\in\omega^{n+1}\}|\ n\in\omega\}\,.$$

The subspace S is point-analytic (point-Luzin) by 3.2 Propositions (b) and (c). The projection $\pi_X\colon S\to X$ is continuous and σ-dd-preserving (by [F-H$_2$, 2.5]), (one-to-one in the case of disjoint-Suslin operation), and hence $\pi_X[S]=A$ is point-analytic (point-Luzin).

COROLLARY. *Suslin (d-Suslin) subsets of a point-analytic (point-Luzin) space* X *are point-analytic (point-Luzin).*

Proof. The assertion for a closed set F is trivial because it suffices to intersect the images of some point-analytic (point-Luzin) parametrization of X with F.

3.5. Point-Suslin sets. The subset S of a uniform space X is said to be *point-Suslin (point-d-Suslin)* if it is the image of some Baire space $\varkappa^\omega$ under a σ-dd-preserving closed-graph (one-to-one) mapping.

One can prove that any point-Suslin or point-d-Suslin set is Suslin or d-Suslin, respectively, Theorem 3.4 thus implies that point-analytic (point-Luzin) subsets are closed under point-Suslin (point-d-Suslin) operation.

The main results about point-Suslin sets are contained in 3.6 (Theorem 2, Proposition).

LEMMA. *Point-Suslin (point-d-Suslin) subsets of an analytic (Luzin) space A are point-analytic (point-Luzin).*

Proof. Let f be a σ-dd-preserving closed-graph (one-to-one) mapping from the Baire space $\varkappa^\omega$ onto the set $S \subset A$. Let $g\colon M \to A$ be some analytic (Luzin) parametrization of A. Then, the map $h\colon M \times \omega^\omega \to \operatorname{gr} f$ defined by $h\langle m, \sigma\rangle = f\sigma$ for $f\sigma \in gm$ is a point-analytic (point-Luzin) parametrization, and the projection of $\operatorname{gr} f$ onto S is a continuous σ-dd-preserving (one-to-one) map by [F-H$_2$, 2.5].

3.6. Characterization Theorems. Here we adapt the characterization of analytic and Luzin spaces in [F-H$_2$, § 4] to obtain characterizations of point-analytic and point-Luzin spaces. An analytic (Luzin) sequence $\{\mathscr{C}_n\}$ of covers is called *point-analytic (point-Luzin)* if the cardinality of $\bigcap_n \{\overline{\bigcap_{k\leqslant n}\{C_k\}}\}$, $C_k \in \mathscr{C}_k$, is at most one.

The proof of the following lemma corresponding to [F-H$_2$, Lemma 4.2(a)] is omitted.

LEMMA. *Let $\{\mathscr{C}_n\}$ be a point-analytic (point-Luzin) sequence of covers of $S \subset X$. Then S is a point-Suslin (point-d-Suslin) subset of X.*

THEOREM 1. *The following assertions concerning a space X are equivalent*:

(a) *X is point-analytic (point-Luzin),*

(b) *X is the image of a complete metric space M under a σ-dr-preserving (one-to-one) continuous mapping,*

(c) *X is the image of a complete metric space M under a (one-to-one) continuous mapping f such that $\{f[G] \mid G$ open in $M\}$ has a σ-discrete refinement,*

(d) *there is a point-analytic (point-Luzin) sequence of covers of X,*

(e) *there is a uniformly continuous (w.r.t. the fine uniformily in the case when X is a topological space) homeomorphism h from X to $P \times K$ where P is a complete metric space, and K a compact space such that $h[X]$ is point-Suslin (point-d-Suslin).*

The proof follows that of [F-H$_2$, Th. 4.1]. For (d) $\Rightarrow$ (e) use the above lemma, and for (e) $\Rightarrow$ (a) use Lemma 3.5.

PROPOSITION. *Let S be point-analytic (point-Luzin) in a uniform space X. Then S is point-Suslin (point-d-Suslin) in X.*

The proof follows from Theorem 1 ((a) $\Rightarrow$ (d)), and the Lemma.

We conclude with a consequence of Lemma 3.5 and the above proposition:

THEOREM 2. *The class of point-Suslin (point-d-Suslin) sets in an analytic (Luzin) space coincides with the class of its point-analytic (point-Luzin) subsets.*

§ 4. Measurable correspondences and mappings. If F is a usco-compact σ-dd-preserving correspondence of a $\varkappa$-analytic space X onto a space Y, then Y is $\varkappa$-analytic.

In this section we are going to weaken "usco" to Baire-measurability. It would be interesting to know, if in the case of a mapping and under the additional assumption that the spaces are metrizable one can relax "σ-db-preserving" to "the image of open sets has a σ-discrete baselike refinement". This seems to be an important problem.

It should be noted that Theorem 4 below generalizes the classical Luzin theorem for separable metric spaces, and also the main result in [F$_3$] for ω-analytic spaces, which has a conclusion that the weight of the range is countable. On the other hand the result in [F$_3$] was used in the proof of the main result of [F–H$_1$], and therefore in the proofs of Theorems 3 and 4 below.

Let $f\colon X \to Y$ be a correspondence, and let $\mathscr{M} \subset \exp X$, $\mathscr{N} \subset \exp Y$. Then f is called *upper-$(\mathscr{M} \leftarrow \mathscr{N})$-measurable* if $f^{-1}[N] \in \mathscr{M}$ for each N in $\mathscr{N}$, and f is called *upper-$(\mathscr{M} \to \mathscr{N})$*-measurable if $f^{-1}\colon \langle Y, \mathscr{N}\rangle \to \langle X, \mathscr{M}\rangle$ is upper-$(\mathscr{N} \to \mathscr{M})$-measurable. Lower measurability is defined in an obvious way, and will be not used here. Therefore no confusion is likely to arise when "upper" is omitted; and if both $\mathscr{M}$ and $\mathscr{N}$ are Suslin sets, Baire sets or Borel sets we say Suslin or Baire or Borel measurable to mean $(\mathscr{M} \leftarrow \mathscr{N})$-measurable.

4.1. LEMMA. *Let F be a closed-valued correspondence of X into Y, and let $\{\mathscr{C}_n\}$ be a sequence of disjoint covers of Y such that if $y \in C_n \in \mathscr{C}_n$ for $n \in \omega_0$, then $\{C_n\}$ converges to y (i.e. for each neighborhood U of y we have $C_n \subset U$ for n large enough. Then*

$$(*) \qquad \operatorname{gr} F = \bigcap \{\bigcup \{F^{-1}[C] \times C \mid C \in \mathscr{C}_n\} \mid n \in \omega_0\}.$$

Remark. The condition on $\{\mathscr{C}_n\}$ is satisfied if $\{\mathscr{C}_n\}$ is a point-analytic sequence of disjoint covers (e.g. point-Luzin sequence or if Y is a metric space and the elements of $\mathscr{C}_n$ are of diameter $\leqslant 1/n+1$.

Proof. The inclusion $\subset$ is obvious. For the other inclusion suppose that $\langle x, y\rangle \notin \operatorname{gr} F$, i.e. $y \notin F[x]$. Since $\bigcup = Y - F[x]$ is a neighborhood of y, we have $C_n \subset U$ for some n (here $C_k \in \mathscr{C}_k$ is defined by $y \in C_k$), and then $\langle x, y\rangle \notin F^{-1}[C_n] \times C_n$ because $x \in F^{-1}[C_n]$. If $C \in \mathscr{C}_n$, and $C \neq C_n$, then $y \notin C$ (because $\mathscr{C}_n$ is disjoint), and hence $\langle x, y\rangle \notin F^{-1}[C] \times C$. Thus

$$\langle x, y\rangle \notin \bigcup \{F^{-1}[C] \times C \mid C \in \mathscr{C}_n\}.$$

4.2. If we take in Lemma 4.1 σ-discrete $\mathscr{C}_n$ such that each $F^{-1}[C] \times C$ is Baire, Borel or Suslin in $X \times Y$, then we get that the graph of F is, respectively, Baire, Borel or Suslin in $X \times Y$. In particular:

PROPOSITION. *Let F be an upper-$(\mathscr{M} \to$ Baire$)$-measurable closed-valued correspondence from a space X into a point-Luzin space Y. If $\mathscr{M}$ is the collection of all Suslin, Borel or Baire sets in Y, then the graph of F is, respectively, Suslin, Borel or Baire in $X \times Y$.*

Proof. There exists a point-Luzin sequence of covers $\{\mathscr{C}_n\}$ on Y; the elements of $\mathscr{C}_n$ are Baire sets, and the covers $\{\mathscr{C}_n\}$ are σ-discrete and disjoint. Lemma 4.1 applies, and the sets $F^{-1}[C]\times C$ are Suslin, Borel or Baire, respectively.

COROLLARY. *If F in the proposition is compact-valued (in particular, if F is a mapping), and if Y is assumed just metrizable (not point-Luzin) then the graph of F is Suslin, Borel or Baire in $X\times\hat{Y}$, where $\hat{Y}$ is a completion of Y.*

Proof. The correspondence F: $X\to\hat{Y}$ *is closed-valued, and the measurablities w.r.t. Y and $\hat{Y}$ are equivalent.*

4.3. THEOREM. *Let F: $X\to L$ be a closed-valued upper-(Suslin $\leftarrow$ Baire)-measurable σ-dd-preserving correspondence.*

(α) *If X is analytic, and if L is point-Luzin then the spaces* $\operatorname{gr}F(\subset X\times L)$, *and* $F[X](\subset L)$ *are analytic. Hence F^{-1} is upper-(Suslin $\to$ Suslin)-measurable (i.e. F is upper-(Suslin $\leftarrow$ Suslin)-measurable).*

(β) *If X is $\varkappa$-analytic, F is compact-valued and L is either point-Luzin or metric then the spaces* $\operatorname{gr}F$ *and* $F[X]$ *are $\varkappa$-analytic, and F^{-1} is σ-dd-preserving.*

(γ) *If X is Luzin, and if L is point-Luzin then the space* $\operatorname{gr}F(\subset X\times L)$ *is Luzin, and if moreover F is disjoint then $F[X](\subset L)$ is also Luzin and F^{-1} is a Baire measurable map when restricted to $F[X]$.*

(δ) *If X is $\varkappa$-Luzin, F is compact-valued and L is either point-Luzin or metric then the space* $\operatorname{gr}F$ *is $\varkappa$-Luzin, F is a σ-dd-isomorphism, and if moreover F is disjoint then $F[X]$ is also $\varkappa$-Luzin.*

Proof. (α) By 4.2 the set $\operatorname{gr}F$ is Suslin in $X\times L$, and since $X\times L$ is analytic as the product of two analytic spaces [F–H$_2$, Th. 3.1(c)], the subspace $\operatorname{gr}F$ of $X\times L$ is analytic by [F–H$_2$, Th.3.1(d)] as Suslin set in an analytic space. The projection $\operatorname{gr}F\to L$ is σ-dd-preserving by [F–H$_2$, 2.5(a)], and hence the image of $\operatorname{gr}F$ under this projection, which is $F[X]$, is analytic.

To prove the last statement observe that the restriction of F to each Suslin subset of X satisfies the assumption of (α).

(β) We may and shall assume that L is point-Luzin (because in the case of the metric space we can take the completion of L without weakening the assumptions). By (α) the spaces $\operatorname{gr}F$ and $F[X]$ are analytic, and we have to prove that they are $\varkappa$-analytic. It is enough to show that $F[X]$ is $\varkappa$-analytic because $\operatorname{gr}F$ is then $\varkappa$-analytic as an analytic subspace of a $\varkappa$-analytic space. It is enough to show that if $\{D_a|\ a\in A\}$ is a discrete family of non-void Baire sets in $f[X]$ then the cardinal of A is $\leqslant\varkappa$. The family $\{F^{-1}[D_a]|\ a\in A\}$ is completely Suslin-additive and point-finite because F is compact-valued, and hence by the main result in [F–H$_1$], $\{F^{-1}[D_a]\}$ is σ-dd, and hence the cardinal of A is $\leqslant\varkappa$, because X is $\varkappa$-analytic. We just proved the last statement of (β).

(γ) The space $X\times L$ or $X\times\hat{L}$ is analytic for a Luzin or a metric space L, respectively. Thus they are σ-dd-simple by [F–H$_2$, Th. 3.1(a)], and therefore the σ-discrete disjoint unions in (*) {when $\{\mathscr{C}_n\}$ is a point-Luzin sequence in L or $\hat{L}$, respectively} are Luzin [F–H$_2$, Th. 3.1(d)]. Thus the graph of F is Luzin. The set $F[X]$ is a continuous σ-dd-preserving 1-1 projection of $\operatorname{gr}F$ by [F–H$_2$, Lemma 2.5(a)] for F disjoint, and hence $F[X]$ is Luzin.

The last statement follows by the first separation principle (Theorem 1.3).

(δ) The statement (γ) implies that $\operatorname{gr}F$ is Luzin. By (β) it is $\varkappa$-analytic, hence $\varkappa$-Luzin [F–H$_2$, Th. 3.1(b), Corollary (a)]. The statement concerning $F[X]$ follows easily from (γ) and (β). Finally F is "a σ-dd-isomorphism" by (β).

4.4. THEOREM. (α) *Let F be a (Suslin $\leftarrow$ Baire)-measurable σ-dr-preserving mapping of a $\varkappa$-analytic space into a point-Luzin (or metric) space L. Then the spaces* $\operatorname{gr}F$ *and $F[X]$ are $\varkappa$-analytic, and F: $X\to F[X]$ is a Baire-quotient (i.e. $B\subset F[X]$ is a Baire set iff $F^{-1}[B]$ is a Baire set), in particular F: $X\to L$ is Baire-measurable.*

(β) *Let F: $X\to L$ be a one-to-one Baire measurable σ-dr-preserving (i.e. σ-dd-preserving) mapping of a $\varkappa$-Luzin space X into a point-Luzin space L. Then $F[X]$ is point-$\varkappa$-Luzin, and F: $X\to F[X]$ is a Baire isomorphism, and also a σ-dd-isomorphism.*

Proof. (α) The first assertion is proved exactly like (α), (β) in Theorem 4.3, just for the conclusion that "$F[X]$ is analytic from $\operatorname{gr}[F]$ is analytic" one uses [F–H$_2$, L 2.5 (b)] instead of [F–H$_2$, L 2.5 (a)]. The second assertion follows from the first one and from the first separation principle (Th. 1.3) as follows: If B is a Baire set in $f[X]$, then so is $F[X]\setminus B$, hence both $f^{-1}[B]$ and $f^{-1}[f[X]\setminus B]=X\setminus f^{-1}[B]$ are Suslin, hence Baire. If $f^{-1}[B]$ is Baire, then so is $X\setminus f^{-1}[B]$, hence the sets are Suslin, hence by the first assertion, B and $f[X]\setminus B$ are Suslin, hence the two sets are Baire sets in $F[X]$ by the first separation principle.

(β) follows from Theorem 4.3 (δ) and Theorem 4.4 (α).

Theorem (β) implies:

COROLLARY. *If a metrizable space X is Baire-isomorphic to a $\varkappa$-Luzin space, and if X is analytic (or more generally Hansell in terminology of* [F–H$_1$]), *then X is $\varkappa$-Luzin, hence point-$\varkappa$-Luzin.*

Remark. Let Q be a Q-set, i.e. an uncountable subspace of the reals such that each subset of Q is a Baire set. Let X be Q with the discrete topology. Then Q and X are Baire isomorphic, X is point-Luzin, while Q is not even analytic.

§ 5. Bimeasurable mappings between complete metric spaces.

Let f: $X\to Y$ be a Baire-measurable σ-dd-preserving mapping. We know (Theorem 4.4 (β)) that if X and Y are point-Luzin and if f is 1–1, then the images of Baire sets are Baire sets. Here we relax the assumption "1-1" as much as possible (see Theorem 5.3), generalizing the classical results of Luzin [L] and of Purves [Pu]. It should be remarked that it is enough to study the case when X and Y are complete metric spaces.

We follow the proofs from Kuratowski [Ku, pp. 402–407] to prove Lemma 5.2. The statement (α) generalizes the well-known result of Mazurkiewicz-Sierpiński [M–S], and allows us to generalize [Pu]. The statement (δ) generalizes [L].

Let us start with the second separation principle.

5.1. Co-Suslin sets; Second separation principle. Recall that the complement of a Suslin set need not be Suslin. The Suslin sets which have Suslin complements in some space X are called *bi-Suslin in* X. The complements of Suslin sets are called *co-Suslin.*

We will use some assertions concerning co-Suslin sets.

PROPOSITION. *The class of co-Suslin subsets of a space X is closed under countable intersections, and under unions of families having a σ-discrete base.*

Proof. The first part of the proposition follows from the fact that countable union of Suslin sets is Suslin.

Let $\{C_a|\ a \in A\}$ be a family of co-Suslin subsets of X, which has a σ-discrete base $\mathscr{B} = \bigcup \{\mathscr{B}_n|\ n \in \omega\}$ with all $\mathscr{B}_n$ discrete. For any $B \in \mathscr{B}$ find some C_a such that $B \subset C_a$; denote this C_a by C_B.

There are zero sets $Z_B \supset B$ such that $\{Z_B|\ B \in \mathscr{B}_n\}$ are discrete, and $\{Z_B \cap C_B|\ B \in \mathscr{B}\}$ forms a σ-discrete cover of $\bigcup \{C_a|\ a \in A\}$. The complement of $\bigcup \{Z_B \cap C_B|\ B \in \mathscr{B}\}$ equals to

$$\bigcap \{\bigcup \{Z_B \setminus C_B|\ B \in \mathscr{B}_n\} \cup (X \setminus \bigcup \{Z_B|\ B \in \mathscr{B}_n\})|\ n \in \omega\}$$

which is a countable intersection of σ-discrete unions of Suslin sets, and therefore it is a Suslin set.

Recall that $\boldsymbol{S}$ (**bi**-$S(X)$) stands for the class of sets that arise from bi-Suslin sets by Suslin operation. The second separation principle [R–W, Theorem 14] says:

THEOREM. (α) *Let A and B be in S (bi-$S(X)$). Then there are co-Suslin sets C, D in X that satisfy*

$$A \setminus B \subset C, \quad B \setminus A \subset D, \quad C \cap D = \emptyset.$$

(β) *Let A_n be in $\boldsymbol{S}$ (**bi**-$S(X)$). Then there are co-Suslin sets H_n in X that satisfy*

$$A_n \setminus \bigcup \{A_m|\ m \neq n\} \subset H_n,$$

and

$$H_n \cap H_m = \emptyset \quad \textit{for} \quad m \neq n.$$

In fact only the assertion (α) is proved in [R–W]. However assertion (β) follows easily in the same way as in [Ku, p. 401] for the case of a Polish space. Let C_n, D_n separate A_n, and $\bigcup \{A_m|\ m \neq n\}$ as in (α). Put

$$H_n = C_n \cap \bigcap \{D_m|\ m \neq n\}.$$

COROLLARY (a). *Let $\{S_a|\ a \in A\}$ be a σ-dd-family of elements of $\boldsymbol{S}$ (**bi**-$S(X)$). Then there is a disjoint σ-discrete family $\{C_{an}|\ a \in A, n \in \omega\}$ of co-Suslin sets in X such that*

$$S_a \setminus \bigcup \{S_b|\ b \in A \setminus \{a\}\} \subset \bigcup \{C_{an}|\ n \in \omega\}.$$

Proof. Let $\{S_{an}|\ a \in A, n \in \omega\}$ be some σ-discrete decomposition of $\{S_a\}$. There are Baire sets $B_{an} \supset S_{an}$ with $\{B_{an}|\ a \in A\}$ discrete. Put $B^*_{an} = B_{an} \setminus \bigcup \{B_{am}|\ m < n\}$, and $S^*_{an} = S_a \cap B^*_{an}$.

Denote $A_n = \bigcup \{S^*_{an}|\ a \in A\}$, and find co-Suslin sets C_n like in Theorem (β). Put $C_{an} = C_n \cap B^*_{an}$. Since $\{C_n\}$ is disjoint the family $\{C_{an}|\ a \in A, n \in \omega\}$ is also disjoint. Let $x \in S_a \setminus \bigcup \{S_b|\ b \neq a\}$. Then $x \in A_n \setminus \bigcup \{A_m|\ m \neq n\}$ for some $n \in \omega$. It follows that $x \in C_n \cap B^*_{an} = C_{an}$.

COROLLARY (b). *Let $\{A_s|\ s \in \varkappa^{n+1}, n \in \omega\}$ be a family of elements of $\boldsymbol{S}$ (**bi**-$S(X)$) that satisfy*

$$A_s = \bigcup \{A_{s,i}|\ i \in \varkappa\} \quad \textit{for} \quad s \in \varkappa^{n+1},$$

and

$$\{A_s|\ s \in \varkappa^{n+1}\} \textit{ is } \sigma\textit{-dd} \quad \textit{for} \quad n \in \omega.$$

Then there are disjoint σ-dd families $\{C^*_s|\ s \in \varkappa^{n+1}\}$ of co-Suslin sets such that

$$A_s \setminus \bigcup \{A_t|\ t \neq s,\ t, s \in \varkappa^{n+1}\} \subset C^*_s,$$

and

$$C^*_s \subset C^*_{s|n+1} \quad \textit{for} \quad s \in \varkappa^{n+2},\ n \in \omega.$$

Proof. Find $C_{s,m}$, $s \in \varkappa^{n+1}$, $m, n \in \omega$, as in Corollary (a). Put

$$C_s = \bigcup \{C_{s,m}|\ m \in \omega\},$$

and

$$C^*_s = \bigcap \{C_{s|j}|\ j = 1, \dots, n+1\} \quad \text{for} \quad s \in \varkappa^{n+1}.$$

The family $\{C^*_s|\ s \in \varkappa^{n+1}\}$ is σ-dd because its members are in the family of finite meets of $n+1$ σ-dd families $\{C_{s|j}\}$ for $j = 1, \dots, n+1$ [F–H$_2$, 1.4(e)].

5.2. Sets of "λ-values".

LEMMA. *Let f be a Borel (i.e. Baire) measurable σ-dd-preserving mapping from a complete metric space M into a complete metric space P. Then*

(α) *the set $A = \{y|\ f^{-1}[y]$ is uncountable$\}$ is Suslin (= analytic) in P,*

(β) *the set $Z = \{y \in P|\ f^{-1}[y]$ is a singleton$\}$ is co-Suslin (= "co-analytic") in P,*

(γ) *the set $I = \{y|\ f^{-1}[y]$ has an isolated point in $M\}$ is co-Suslin in P,*

(δ) *the set $C = \{y|\ f^{-1}[y]$ is non-empty at most countable$\}$ is co-Suslin in P.*

Proof. (α) The graph of f is analytic (Th. 4.4(a)). There are some Baire space $\varkappa^\omega$, and a continuous σ-dd-preserving mapping $\varphi = \langle \varphi_1, \varphi_2 \rangle$ from a closed subset F of $\varkappa^\omega$ onto $\operatorname{gr} f \subset M \times P$.

According to [F–H$_2$, 2.5(a)] the "projection" π_P: $\operatorname{gr} f \to P$ is σ-dd-preserving. Thus the $\pi_P^{-1}[y]$ are closed and separable in $\operatorname{gr} f$, and $\varphi^{-1}[\pi_P^{-1}[y]]$ are separable closed subsets of $F \subset \varkappa^\omega$. Use [Ku, p. 353] for $\varphi|\varphi^{-1}[\pi_P^{-1}[y]]$ where y is fixed. Thus the set $\{x|\ f[x] = y\}$ is uncountable if and only if there is a sequence S of elements of $\varkappa^\omega$ which is dense in itself, and $\varphi_2[d] = y$ for $d \in S$, and $\varphi_1|S$ is one-to-one.

Now the set

$$A = \{y|\ \exists (\xi \in H) \forall (n \in \omega) y = \varphi_2[\xi^n] \text{ and } \forall (m \neq n) \varphi_1[\xi^{n+1}] \neq \varphi_1[\xi^n]\},$$

where H is the set of all sequences $\xi = (\xi^n)$ of elements of $\varkappa^\omega$ $(\xi^n = (\xi_0^n, \xi_1^n, \ldots))$ which are dense in itself in the product topology of $(\varkappa^\omega)^\omega$. It is easy to show that

$$Y = \{\langle y, \xi\rangle \in P\times H \mid \forall (n\in\omega)[y = \varphi_2[\xi^n]] \text{ and } \forall (m\neq n) g[\xi^m] \neq g[\xi^m]\}$$

is a G_δ subset of $P\times(\varkappa^\omega)^\omega$. According to 3.3 Corollary (b) Y is analytic, and our statement is proved by showing that the projection π_P^* of Y to P is σ-dd-preserving. In fact it suffices to consider the set

$$Y' = \{\langle y, \xi\rangle \in P\times H \mid \forall (n\in\omega) y = \varphi_2[\xi^n]\}$$

in the space $P\times H$.

Let $\{D_a\}$ be discrete in Y'. We can easily reduce this general case to the case when the projection π_H: $Y'\to H$ maps $\{D_a\}$ to a discrete family.

In fact we find σ-discrete covers $\mathscr{U}$, $\mathscr{V}$ of P and H, respectively such that $U\times V\cap D_a \neq \emptyset$ for at most one $a\in A$, whenever $U\in\mathscr{U}$, $V\in\mathscr{V}$. According to [F–H$_2$, 1.4(g)] we can consider the family $\{\pi_P^{*-1}[U]\cap D_a \mid a\in A\}$ for U fixed.

Let r_n: $(\varkappa^\omega)^\omega\to\varkappa^\omega$ denote the projection to the nth coordinate $(r_n[(\xi^m)] = \xi^n)$. The projection

$$\pi_P^*(y, \xi) = \bigcap\{\varphi_2 r_n \pi_H(y, \xi) \mid n\in\omega\}\,.$$

Since $\{\pi_H[D_a]\}$ is discrete it suffices to prove that the map

$$\alpha[\xi] = \bigcap\{\varphi_2 r_n[\xi] \mid n\in\omega\}$$

is σ-dd-preserving. Put α is a countable "meet" of α_n: $X_n\to P$ where $\alpha_n = \varphi_2$, $X_n = \varkappa^\omega$ for $n\in\omega$, and we use 2.4(d$_2$) from [F–H$_2$] that says that countable meets of σ-dd-preserving maps from metric spaces are σ-dd-preserving.

Remark. Notice that we have only used the analyticity of $\operatorname{gr} f$, however it is Luzin by Theorem 4.4(b).

(β) The graph of f is Luzin, and there is a 1-1 continuous σ-dd-preserving mapping φ from a closed subset F of some Baire space $\varkappa^\omega$ onto $\operatorname{gr} f$ ([F–H$_2$, 3.2]). Let π_P be the projection of $\operatorname{gr} f\subset M\times P$ to P again. We can consider $\pi_P\circ\varphi$ instead of f, and therefore we can suppose that f is continuous 1-1 σ-dd-preserving map with domain F.

Let $I(s)$ denote the set $\{d\in F \mid d|n+1 = s\}$. The set

$$Z = \bigcup\{f[d]\setminus f[F\setminus\{d\}] \mid d\in F\}$$
$$= \bigcup\{\bigcap\{f[I(d|n+1)]\setminus f[F\setminus I(d|n+1)] \mid n\in\omega\} \mid d\in F\}\,.$$

Denote

$$A(s) = f[I(s)\cap F] \quad\text{and}\quad B(s) = f[F\setminus I(s)] \quad\text{for}\quad s\in\varkappa^{n+1}.$$

Thus

$$Z = \bigcup\{\bigcap\{A(d|n+1)\setminus B(d|n+1) \mid n\in\omega\} \mid d\in F\}\,,$$

and

$$B(d|n+1) = \bigcup\{A(e|n+1) \mid e|n+1 \neq d|n+1\}\,.$$

Using Corollary 5.1(b) we find co-Suslin sets $C^*(s)$ for $s\in\varkappa^{n+1}$ such that

$$Z = \bigcap\{\bigcup\{C^*(d|n+1) \mid d|n+1\in\varkappa^{n+1}\} \mid n\in\omega\}\,,$$

where $C^*(d|n+1)\subset f[I(d|n+1)]$. Thus $\{C^*(s) \mid s\in\varkappa^{n+1}\}$ is σ-dd, and Z is co-Suslin by Proposition 5.1.

(γ) Let $\mathscr{B}$ be some σ-discrete base for open subsets of M. Denote f_B the restriction of f to B for $B\in\mathscr{B}$.

The set

$$I = \bigcup\{\{y \mid f_B^{-1}[y] \text{ is a singleton}\} \mid B\in\mathscr{B}\}\,.$$

The assertion (γ) follows because f is σ-dd-preserving, and (β) can be applied to f_B (use Proposition 5.1).

(δ) The set $C = \{y\in P \mid f^{-1}[y]$ is not uncountable, and has an isolated point$\}$ $= I\setminus A$, and this is a co-Suslin set.

5.3. Characterization of bimeasurable σ-dd-preserving mappings.

The main result reads:

THEOREM. *The Borel (= Baire)-measurable σ-dd-preserving map f from a point-Luzin space L' into a point-Luzin space L'' is (Borel) bimeasurable iff*

(*) $\{y\in L'' \mid f^{-1}[y]$ *is uncountable*$\}$ *is σ-discrete.*

Proof. Any point-Luzin parametrization φ of a point-Luzin space is 1-1, (Borel) bimeasurable [F–H$_2$, Th. 2(a)], and φ and φ^{-1} map σ-discrete sets to σ-discrete sets again. Therefore we can consider complete metric spaces M, P instead of L', L'', respectively.

Let the condition (*) be fulfilled, and B be a Borel subset of M. According to Lemma 5.2 (δ) the set

$$\{y\in P \mid f^{-1}[y]\cap B \text{ is non-empty at most countable}\}$$

is co-Suslin thus co-analytic [F–H$_2$, Th. 3.1(d)], and in the same time according to Theorem 4.4(α), analytic. Use the first separation principle [Th. 1.3] to finish this part of the proof. Assume that f does not fulfil the condition (*). It means that the set

$$A = \{y\in P \mid f^{-1}[y] \text{ is uncountable}\}$$

is not σ-discrete. The set A is point-analytic according to Lemma 5.1(α), and thus it contains an uncountable compact set K (even Cantor discontinuum — see Lemma bellow). Now we apply [Pu, Theorem] to $f|f^{-1}[K]$ to obtain a contradiction.

The following lemma is a generalization of the classical result of Hausdorff and Alexandrov; it should be noted that for the case of a Suslin set in a complete metric space the result is due to Elkin. However, all proofs are essentially the same.

LEMMA. *Assume that a point-analytic space A is not σ-discrete. Then there is a subset C of A which is homeomorphic to the Cantor space 2^ω.*

Proof. Let $\varphi\colon F(\subset\varkappa^{\omega})\to A$ be some point-analytic parametrization of A. Let $I(s)$ have the meaning as in the proof of Lemma 5.2(β). We use the following two facts:

1. If M is a non-σ-discrete subset of A then there are $m, n\in\omega$, $s\in\varkappa^{m+1}$ and $t\in\varkappa^{n+1}$ such that $M\cap f[I(s)]$ and $M\cap f[I(t)]$ are non-σ-discrete, and $\overline{f[I(s)]}\cap\overline{f[I(t)]}=\emptyset$ (Sketch of the proof: Find open $U, V\subset A$ with $U\cap M$, $V\cap M$ non-σ-discrete, and $\overline{U}\cap\overline{V}=\emptyset$. $\{f[I(s)]|\ s\in\varkappa^{n+1}, n\in\omega\}$ form a σ-dd base for open sets in A).

2. If $M\cap f[I(s)]$ is non-σ-discrete for $s\in\varkappa^{m}$, and $n>m$ then there is an $s'\in\varkappa^{n}$ such that $M\cap f[I(s')]$ is non-σ-discrete.

Now we construct by induction a set $C\subset\varkappa^{\omega}$ of sequences

$$d=(d^{i_0}, d^{i_0 i_1}, d^{i_0 i_1 i_2}, \ldots),$$

and sequence $n_j\in\omega$ such that $d^{i_0,\ldots,i_j}\in\varkappa^{n_j}$ for $j\in\omega$, $i_j\in\{0, 1\}$, and

$$\overline{f[I(d^{i_0}, \ldots, d^{i_0,\ldots,i_j})]}\cap\overline{f[I(d^{i'_0}, \ldots, d^{i'_0,\ldots,i'_j})]}=\emptyset$$

for $(i_0, \ldots, i_j)\neq(i'_0, \ldots, i'_j)$.

Obviously the mapping $\varphi(i_0, i_1, \ldots)=(d^{i_0}, d^{i_0 i_1}, \ldots)$ is 1-1, and continuous. The same is true about $f\circ\varphi\colon\{0, 1\}^{\omega}\to C$, and thus $f\circ\varphi$ is a homeomorphism.

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