The relationship between algebraic numbers and expansiveness of automorphisms on compact abelian groups

by

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Abstract. The structure of a compact group which admits an expansive automorphism has been investigated by several authors. The purpose of this paper is to characterize such a compact group. We firstly examine the relation between an automorphism σ of a d-dimensional solenoidal group X and the automorphism γ of the real vector space $\mathbf{R}^d$ extended by the dual (G,γ) of (X,σ). Then it will be shown that (X,σ) is expansive if and only if all the eigenvalues of γ are off the unit circle and G is finitely generated under γ. We shall prove that every expansive automorphism of a compact abelian group is densely periodic, and that every factor is also expansive. Let X be solenoidal and σ be an expansive automorphism of X. Then we shall prove that X contains σ-invariant subgroups S and T where S is a solenoidal group without torus subgroups when $S \neq \{0\}$ and T is a torus when $T \neq \{0\}$, such that both (S,σ) and (T,σ) are expansive and (X,σ) is a factor of a direct product of (S,σ) and (T,σ). It seems likely that these results will play a role for the study of the structure of automorphisms with the specification property.

§ 0. Introduction. The notion of expansive homeomorphisms of a compact metric space was introduced by Utz [26] with the term "unstable homeomorphisms". The properties of such homeomorphisms have been investigated in many papers, including the papers [6], [7], [9], [10], [15], [17], [18], [28] and [29]. It is known in the ergodic theory that a symbolic flow is expansive, and conversely if a discrete flow of a compact metric space is expansive then it admits a finite topological generator (see [15]). However it is not true that every compact metric space admits expansive homeomorphisms. This follows from the results in [17] and [18]; i.e. if a compact connected group admits an expansive automorphism, then it is finite-dimensional and abelian, and its dual group is finitely generated under the dual automorphism. For this case, the automorphism is densely periodic (see [19]). However, when a compact group with an expansive automorphism is not connected, it is unknown yet whether the automorphism is densely periodic. It will be interesting to examine what kind of automorphisms of a solenoidal group are expansive. This problem was studied partially in [9]; i.e. let E be a separable finite-dimensional real (complex) topological vector space and let γ be an automorphism of E, then γ is expansive if and only if all the eigenvalues of γ are off the unit circle. It was proved in [10] that an automorphism σ of a d-dimensional torus T^d is expansive if and only if all the eigenvalues of the automorphism of $\mathbf{R}^d$ corresponding to σ are off the unit circle. From this result, we see that every expansive automorphism of a torus is ergodic under the Haar measure (p. 53 of [12]). This was extended in [1] to a solenoidal group.

The purpose of this paper is to investigate the structure of expansive automorphisms on solenoidal groups. It seems likely that results obtained here will furnish with a tool for the study of the topological structure of automorphisms with the spesification property introduced in [5], [23] and [24].

The contents of this paper will be divided into four sections. In § 1 we shall prepare definitions and notations. In particular, we shall mention the relation between an automorphism σ of a solenoidal group X and the automorphism γ of the real vector space associated with (X, σ). We shall prove in § 2 that an automorphism of a solenoidal group is expansive if and only if all the eigenvalues of the corresponding matrix (Def. 5) are off the unit circle and its dual group is finitely generated under the dual automorphism (Theorem 1). We shall show in § 3 that every factor automorphism of an expansive automorphism of a compact metric abelian group is expansive (Theorem 2), and that the expansive automorphism is densely periodic (Theorem 3). In § 4 it will be proved that a solenoidal group X which admits an expansive automorphism σ contains σ-invariant subgroups S and T where S is a solenoidal group without torus subgroups when $S \neq \{0\}$ and T is a torus when $T \neq \{0\}$, such that both (S, σ) and (T, σ) are expansive and (X, σ) is a factor of a direct product of (S, σ) and (T, σ) (Theorem 4).

§ **1. Definitions.** Throughout this paper, we shall deal with a compact metric abelian group and its dual group, and write the group operation by addition. Subgroups of a compact metric abelian group will be closed. Sometimes non-closed subgroups are said to be *algebraic subgroups*. The identity of the group will be denoted by "0". To distinguish the direct sum of subgroups from the sum we denote them by the symbols "$\oplus$" and "$+$" respectively. The direct product of two groups will be denoted by the symbol "$\times$". It will be assumed that all maps used here are continuous. We shall call simply *automorphisms* group automorphisms. Given an automorphism of a group, its restrictions, its factors and its extensions will be denoted by the same symbols if there is no possibility of confusion.

In the remainder of this section, we shall give the definitions and the notations which are used in the proofs of the theorems.

DEFINITION 1. Let X be a compact metric abelian group and σ be an automorphism of X, then we call that (X, σ) is *expansive* (*positively expansive*) if there exists an open ball U of the identity such that $\bigcap_{n=-\infty}^{\infty} \sigma^n U$[$\bigcap_{n=0}^{\infty} \sigma^{-n} U$] consists only of the identity. We call an *expansive neighborhood* (a *positively expansive neighborhood*) *for* (X, σ) such a neighborhood U.

DEFINITION 2. Let X be a compact connected metric abelian group and σ be an automorphism of X. Denote by (G, γ) the dual of (X, σ) ($(\gamma g)(x) = g(\sigma x)$, $g \in G$ and $x \in X$). Since X is connected, G is torsion free (cf. see p. 148 of [21]). A finite set of elements $g_1, \ldots, g_d$ of G is called *linearly independent* if, for integers $a_1, \ldots, a_d$, $a_1 g_1 + \ldots + a_d g_d = 0$ implies $a_1 = \ldots = a_d = 0$. An infinite set of elements of G is called *linearly independent* if all finite subsets are linearly independent. The maximal cardinal number of a linearly independent set of G is called the *rank* of G (cf. see p. 19 of [21]).

DEFINITION 3. Let G be the dual group of a compact metric abelian group X. Then X is said to be *rank*(G)-*dimensional* (p. 148 of [21]). Obviously, X is zero-dimensional if and only if X is totally disconnected. We say that X is *solenoidal* if X is connected and finite-dimensional. A finite-dimensional torus is clearly solenoidal.

DEFINITION 4. Let G be a discrete countable abelian group and γ be an automorphism of G. Then G is said to be *finitely generated under* γ if G contains a finite set $\{g_1, \ldots, g_n\}$ such that $G = \mathrm{gp}\{\gamma^{-j} g_i\colon -\infty < j < \infty,\ 1 \leqslant i \leqslant n\}$ (the notation "gp$E$" means a subgroup generated by a subset E of G).

Let σ be an automorphism of a d-dimensional solenoidal group X and as before (G, γ) denote the dual of (X, σ). Since $\operatorname{rank}(G) = d$ (by Def. 3), G contains a linearly independent set $\{g_1, \ldots, g_d\}$. Hence $0 \neq g \in G$ is expressed as $ag = a_1 g_1 + \ldots + a_d g_d$ for some integers $0 \neq a$ and $a_1, \ldots, a_d$ with $(a_1, \ldots, a_d) \neq (0, \ldots, 0)$. We now define an into isomorphism $\varphi\colon G \to \mathbf{Q}^d$ (the notation $\mathbf{Q}^d$ denotes the d-dimensional rational vector space) by the equality $\varphi(g) = (a_1/a, \ldots, a_d/a)$ (notice that φ is continuous since G is discrete). To simplify the notations, we identify g with $(a_1/a, \ldots, a_d/a)$ under the map φ. Then $G \subset \mathbf{Q}^d$ $(\subset \mathbf{R}^d)$ and so $g_1 = (1, 0, \ldots, 0), \ldots, g_d = (0, \ldots, 0, 1)$. By p. 166 of [21], we can define a homomorphism from $\mathbf{R}^d$ into X by

$$\psi(t) g = t_1 a_1/a + \ldots + t_d a_d/a \quad (\text{addition mod}\,1)\,,$$

$$t = (t_1, \ldots, t_d) \in \mathbf{R}^d, \qquad g = (a_1/a, \ldots, a_d/a) \in G\,,$$

and define the adjoint map $\hat{\gamma}$ of $\mathbf{R}^d$ by

$$\psi(\hat{\gamma} t) g = \psi(t) \gamma g\,, \qquad g \in G \text{ and } t \in \mathbf{R}^d\,.$$

Then γ is similar to $\hat{\gamma}$. The annihilator F of $\mathrm{gp}\{g_1, \ldots, g_d\} = \mathbf{Z}^d$ in X is totally disconnected. If in particular X is a torus, then it follows that F is finite. We denote by π_F the projection $X \to X/F$. In order to clarify the relation between the maps σ, γ, $\hat{\gamma}$, ψ and π_F obtained above, we set the following diagram

$$\begin{tikzcd}
(\mathbf{R}^d, \hat{\gamma}) \arrow[r, "\text{isomorphic}"] \arrow[d, "\psi"'] & (\mathbf{R}^d, \gamma) \\
(X, \sigma) \arrow[d, "\pi_F"'] & (G, \gamma) \arrow[u, "\text{inclusion}"'] \\
X/F & \mathrm{gp}\{g_1, \ldots, g_d\} \arrow[u, "\text{inclusion}"']
\end{tikzcd}$$

DEFINITION 5. Let X be a d-dimensional solenoidal group and σ be an automorphism of X. Let $\mathrm{gp}\{g_1, \dots, g_d\}$, ψ, $\hat{\gamma}$, f and π_F be as above. Then we call $(\mathrm{gp}\{g_1, \dots, g_d\}, \psi, \hat{\gamma}, F, \pi_F)$ to be the *system induced by the linearly independent set* $\{g_1, \dots, g_d\}$.

The followings will be easily obtained.

(i) $\psi(\mathbf{R}^d)$ is dense in X.

This follows from the fact that if $g \in G$ and $\psi(t)g = 0$ for all $t \in \mathbf{R}^d$, then $g = 0$.

(ii) $\{t \in \mathbf{R}^d: \psi(t) \in F\} = \mathbf{Z}^d$.

This is clear from the definition of ψ.

(iii) $\pi_F\psi(\mathbf{R}^d) = X/F$ and $\pi_F\psi$ is an open map.

From (ii) we have

$$\pi_F\psi(\mathbf{R}^d) = \pi_F\psi(E) \quad \text{where} \quad E = \{(t_1, \dots, t_d) \in \mathbf{R}^d: 0 \leqslant t_i \leqslant 1, 1 \leqslant i \leqslant d\},$$

which implies that $\pi_F\psi(\mathbf{R}^d)$ is compact. Hence, by (i) we get $\pi_F\psi(\mathbf{R}^d) = X/F$. Obviously $\pi_F\psi: \mathbf{R}^d \to X/F$ is open since it is an onto homomorphism.

(iv) $X = \psi(\mathbf{R}^d) + F$. If in particular X is a torus, then $X = \psi(\mathbf{R}^d)$.

These follow from (iii).

Hereafter the eigenvalues of $\gamma|_G$ mean those of $\gamma: \mathbf{R}^d \to \mathbf{R}^d$, since $G \subset \mathbf{R}^d$ and the linear span of G is equal to $\mathbf{R}^d$.

§ 2. Expansive solenoidal automorphisms. The aim of this section is to prove the following

THEOREM 1. *Let σ be an automorphism of a solenoidal group X and as before let (G, γ) be the dual of (X, σ). Then (X, σ) is expansive if and only if all the eigenvalues of γ are off the unit circle and G is finitely generated under γ.*

For the proof we need the following Lemmas 1, 2 and 3.

LEMMA 1. *Let γ be an automorphism of the d-dimensional real vector space $\mathbf{R}^d$. Then $(\mathbf{R}^d, \gamma)$ is expansive if and only if all the eigenvalues of γ are off the unit circle. Further, $(\mathbf{R}^d, \gamma)$ is positively expansive if and only if all the eigenvalues of γ are out of the unit circle.*

The result of Lemma 1 is extended in [9] to a more general vector space. We give here a simple proof of Lemma 1 for completeness. It follows that $(\mathbf{R}^d, \gamma)$ is expansive if and only if the complexification of $(\mathbf{R}^d, \gamma)$ is expansive. And further the complexification is expansive if and only if the linear map given by the Jordan normal form is expansive. Finally we can show that the normal form is expansive if and only if $|\lambda| \neq 1$ for each eigenvalue λ. The second statement is obtained from the similar proof to the above one.

LEMMA 2. *Let (X, σ) and (G, γ) be as in Theorem 1. If (X, σ) is expansive, then all the eigenvalues of γ are off the unit circle and G is finitely generated under γ.*

Proof. It is proved in [18] that G is finitely generated under γ. Thus it only remains to show that all the eigenvalues of γ are off the unit circle. Let X be d-dimensional, then $\mathrm{rank}(G) = d$ (Def. 3). Hence G contains a linearly independent set $\{g_1, \dots, g_d\}$. We construct a system $(\mathrm{gp}\{g_1, \dots, g_d\}, \psi, \hat{\gamma}, F, \pi_F)$ as in Def. 5. Let U be an expansive neighborhood for (X, σ). Choose a small neighborhood V of the zero vector of $\mathbf{R}^d$ such that $\psi(V) \subset U$ and ψ is one-to-one on V. The existence of such a neighborhood V is a consequence of Def. 5 (ii). We now have

$$\{0\} = \bigcap_{n=-\infty}^{\infty} \sigma^{-n}U \supset \bigcap_{n=-\infty}^{\infty} \sigma^{-n}\psi(V) = \psi\Big(\bigcap_{n=-\infty}^{\infty} \hat{\gamma}^{-n}V\Big)$$

(see Def. 5 (iv)),

which implies that $(\mathbf{R}^d, \hat{\gamma})$ is expansive. Hence all the eigenvalues of $\hat{\gamma}$ are off the unit circle by Lemma 1. Since $\hat{\gamma}$ and γ are similar, we get the conclusion of Lemma 2.

Remark 1. From p. 53 of [12] together with Lemma 2, it follows that every expansive automorphism of a solenoidal group is ergodic (under the Haar measure). However there exists an ergodic automorphism which is not expansive. For example, let γ be the automorphism of the 2-dimensional rational vector space, corresponding to a matrix $\begin{pmatrix} 0 & -1 \\ 1 & \frac{1}{2} \end{pmatrix}$. Let us put $G = \sum_{j=-\infty}^{\infty} \gamma^j(\mathbf{Z} \times \{0\})$ where $\mathbf{Z}$ denotes the set of all integers. If G is imposed with the discrete topology, then the dual (X, σ) of (G, γ) satisfies our requirement.

It is known that every expansive automorphism σ of a 2-dimensional torus X has the specification property (see [5]). If in general X is solenoidal and (X, σ) is expansive, then it is unknown yet whether (X, σ) has the specification property. It will be interesting to examine the existence of dynamical systems with the specification property.

LEMMA 3. *Let (X, σ) and (G, γ) be as in Theorem 1. Assume that*

$$G = \mathrm{gp}\{\gamma^{-j}g: -\infty < j < \infty\}$$

for some $0 \neq g \in G$. If all the eigenvalues of γ are off the unit circle, then (X, σ) is expansive.

Proof. Since X is solenoidal, $\mathrm{rank}(G) = d$ for some integer $d > 0$. Hence $\{g, \gamma g, \dots, \gamma^{d-1}g\}$ is linearly independent. Construct

$$(\mathrm{gp}\{g, \gamma g, \dots, \gamma^{d-1}g\}, \psi, \hat{\gamma}, F, \pi_F)$$

as in Def. 5 and take a bounded closed set

$$D = \{(t_1, \dots, t_d) \in \mathbf{R}^d: -\tfrac{1}{3} \leqslant t_i \leqslant \tfrac{1}{3}, 1 \leqslant i \leqslant d\}.$$

Then $\psi(D) + F$ is a closed neighborhood of the identity in X since $\pi_F\psi: \mathbf{R}^d \to X/F$ is an open map (Def. 5(iii) and (iv)). Obviously $\psi(D) \cap F = \{0\}$ (Def. 5(ii)). Put $D_1 = \bigcap_{j=-1}^{1} \hat{\gamma}^{-j}D$ and $F_1 = \bigcap_{j=-1}^{1} \sigma^{-j}F$. It is easy to see that $\psi(D_1) + F_1$ is a closed neighborhood of the identity in X. Indeed, D_1 is a closed neighborhood of the zero vector in $\mathbf{R}^d$. On the other hand, since the dual group of F/F_1 is the finite group

$$\sum_{j=-1}^{1} \gamma^{-j}\mathrm{gp}\{g, \gamma g, \ldots, \gamma^{d-1}g\}/\mathrm{gp}\{g, \gamma g, \ldots, \gamma^{d-1}g\},$$

F/F_1 is finite (cf. see p. 140 of [21]), whence F_1 is open in F. To see that $\psi(D_1)+F_1$ is a closed neighborhood of X, it is enough to show that $\psi(D)\times F$ is homeomorphic to $\psi(D)+F$. Define a map $f(\psi(d), x) = \psi(d)+x$, $d \in D$ and $x \in F$. Then f is onto and continuous. It is easy to see that f is one-to-one. Indeed, if $f(\psi(d), x) = f(\psi(d'), x')$ for some d, $d' \in D$ and some x, $x' \in F$, then $\psi(d-d') \in \psi(\mathbf{R}^d) \cap F$ and so $d-d' \in \mathbf{Z}^d$ since $\psi(\mathbf{R}^d) \cap F = \psi(\mathbf{Z}^d)$ and ψ is one-to-one on D (Def. 5(ii)). Thus the distance of each component of $d-d'$ is not more than $\frac{2}{3}$. This implies that $d-d' = 0$. Therefore f must be one-to-one.

We shall show that (X, σ) is expansive. Since all the eigenvalues of γ are off the unit circle and γ is similar to $\hat{\gamma}$, $(\mathbf{R}^d, \hat{\gamma})$ is expansive by Lemma 1, from which we get $\bigcap_{j=-\infty}^{\infty} \hat{\gamma}^{-j}D_1 = \{0\}$. Since $\sum_{j=-\infty}^{\infty} \gamma^j\mathrm{gp}\{g, \gamma g, \ldots, \gamma^{d-1}g\} = G$ and F is the annihilator of $\mathrm{gp}\{g, \gamma g, \ldots, \gamma^{d-1}g\}$ in X,

$$\bigcap_{j=-\infty}^{\infty} \sigma^{-j}F_1 \subset \bigcap_{j=-\infty}^{\infty} \sigma^{-j}F = \{0\}.$$

We now have

$$\bigcap_{j=-n}^{n} \sigma^{-j}(\psi(D_1)+F_1) = \bigcap_{j=-n}^{n} \psi(\hat{\gamma}^{-j}D_1) + \bigcap_{j=-n}^{n} \sigma^{-j}F_1$$

for all $n \geqslant 0$, and so $\bigcap_{j=-\infty}^{\infty} \sigma^{-j}(\psi(D_1)+F_1) = \{0\}$. By definition (X, σ) is expansive.

Proof of Theorem 1. Assume that (X, σ) is expansive. By Lemma 2 we get that all the eigenvalues of γ are off the unit circle and G is finitely generated under γ. It remains only to show the converse.

Since G is finitely generated under γ, G contains a finite set $\{g_1, \ldots, g_n\}$ such that $G = \sum_{i=1}^{n} G_i$ where $G_i = \mathrm{gp}\{\gamma^{-j}g_i\colon -\infty < j < \infty\}$ for $i = 1, \ldots, n$. Let K_i denote the annihilator of G_i in X for $i = 1, \ldots, n$. Then each G_i is the dual group of X/K_i. Using Lemma 3, we see that $(X/K_i, \sigma)$ is expansive. Denote by π_i the projection $X \to X/K_i$ for $i = 1, \ldots, n$, and put $U = \pi_1^{-1}(U_1) \cap \ldots \cap \pi_n^{-1}(U_n)$ where each U_i is an expansive neighborhood for $(X/K_i, \sigma)$. Since $\sum_{i=1}^{n} G_i = G$, we have $\bigcap_{i=1}^{n} K_i = \{0\}$. Hence U is an expansive neighborhood for (X, σ). The proof of Theorem 1 is completed.

COROLLARY 1. *Let X be solenoidal and σ be an expansive automorphism of X. If H is a σ-invariant subgroup of X, then $(X/H, \sigma)$ is also expansive.*

Proof. As before let (G, γ) denote the dual of (X, σ). Then the rank of G is finite and G is torsion free. Since (X, σ) is expansive, G is finitely generated under γ

by Theorem 1. Let G_H be the annihilator of H in G, then it follows that each eigenvalue of $\gamma|_{G_H}$ is off the unit circle. We claim that G_H is finitely generated under γ. Indeed, since G is finitely generated under γ, G contains a finite set $\{g_1, \ldots, g_k\}$ such that $G = \sum_{j=-\infty}^{\infty} \gamma^{-j}\mathrm{gp}\{g_i\colon 1 \leqslant i \leqslant k\}$. Under the action of γ, we can consider G to be a $\mathbf{Z}[x, x^{-1}]$-module (the notation $\mathbf{Z}[x, x^{-1}]$ denotes the ring of all polynomials in x and x^{-1} with integral coefficients). Then G_H is clearly a $\mathbf{Z}[x, x^{-1}]$-submodule of G. Since $\mathbf{Z}[x, x^{-1}]$ is Noetherian, we have that $G_H = \sum_{-\infty}^{\infty} \gamma^{-j}G'$ where $G'_1 = \mathrm{gp}\{f_1, \ldots, f_n\}$ for some $f_1, \ldots, f_n \in G_H$. Using Theorem 1 again, we see that $(X/H, \sigma)$ is expansive.

Remark 2. We remark (Theorems 8 and 9 of [7]) that every solenoidal group does not admit positively expansive automorphisms. Let σ be an expansive automorphism of a solenoidal group X. Assume that there exists a totally disconnected subgroup H such that the following conditions hold; (1) $\sigma H \subsetneqq H$, (2) X/H is a torus and (3) $(X/H, \sigma)$ is positively expansive ($\sigma\colon X/H \to X/H$ is an endomorphism). We write

$$X_s = \{x \in X\colon \sigma^j x \to 0 \text{ as } j \to \infty\},$$

$$X_u = \{x \in X\colon \sigma^{-j} x \to 0 \text{ as } j \to \infty\}.$$

Then X_s and X_u are algebraic subgroups and they are dense in X. For, since X is solenoidal, $\dim(X) = d < \infty$. As before let (G, γ) be the dual of (X, σ), and denote by G_H the annihilator of H in G. Then G/G_H is a torsion group (since H is totally disconnected) and $\gamma G_H \subsetneqq G_H$ by (1). Since $\mathrm{rank}(G_H) = d$, G_H is isomorphic to $\mathbf{Z}^d$ for some integer $d > 0$ by (2). We can find a polynomial $p(x) = x^d + a_{d-1}x^{d-1} + \ldots + a_0 \in \mathbf{Z}[x]$ such that $\gamma^d g = -(a_{d-1}\gamma^{d-1} + \ldots + a_1\gamma + a_0)g$, $g \in G$ (notice that the polynomial $p(x)$ is monic). Since σ is expansive, G is finitely generated under γ (by Theorem 1). Hence G contains a finite set $\{g_1, \ldots, g_n\}$ such that

$$G = \mathrm{gp}\{\gamma^{-j}g_i\colon 0 \leqslant j < \infty,\ 1 \leqslant i \leqslant n\}.$$

Choose a maximal linearly independent set $\{f_1, \ldots, f_d\}$ such that

$$\mathrm{gp}\{f_1, \ldots, f_d\} = \mathrm{gp}\{\gamma^{-j}g_i\colon 0 \leqslant j \leqslant d-1,\ 1 \leqslant i \leqslant n\}.$$

Then we get $\gamma(\mathrm{gp}\{f_1, \ldots, f_d\}) \subset \mathrm{gp}\{f_1, \ldots, f_d\}$ and $G = \bigcup_{j=0}^{\infty} \gamma^{-j}\mathrm{gp}\{f_1, \ldots, f_d\}$. Construct $(\mathrm{gp}\{f_1, \ldots, f_d\}, \psi, \hat{\gamma}, F, \pi_F)$ as in Def. 5. Since F is the annihilator of $\mathrm{gp}\{f_1, \ldots, f_d\}$ in X (see Def. 5), $\sigma F \subset F$ and $\bigcap_{j=0}^{\infty} \sigma^j F = \{0\}$. Hence the diameter of $\sigma^j F$ converges to 0 as $j \to \infty$, which implies that $\bigcup_{j=0}^{\infty} \sigma^{-j}F \subset X_s$. By (3), $(X/H, \sigma)$ is positively expansive and H is totally disconnected. Thus all the eigenvalues of γ are out of the unit circle (see Lemma 1), so that $\bigcap_{j=0}^{\infty} \gamma^j\mathrm{gp}\{f_1, \ldots, f_d\} = \{0\}$. Hence

$\bigcup_{j=0}^{\infty} \sigma^{-j}F$ is dense in X, and so X_s is dense in X. Since γ and $\hat{\gamma}$ are similar (Def. 5(v)), we have that for all $t \in \mathbf{R}^d$, $\hat{\gamma}^{-j}t \to 0$ as $j \to \infty$. Hence $\psi(\mathbf{R}^d) \subset X_u$ since $\psi\hat{\gamma} = \sigma\psi$. Since $\psi(\mathbf{R}^d)$ is dense in X (Def. 5(i)), X_u is also dense in X. It is easy to check that X_s and X_u are algebraic subgroups of X.

§ 3. Expansive automorphisms. In this section we shall study the structure of a compact metric abelian group X which admits an expansive automorphism σ. Obviously σ preserves the normalized Haar measure of X. We denote by $h(\sigma)$ the Kolmogorov entropy of σ with respect to the normalized Haar measure. The followings are well known in the ergodic theory.

PROPERTY (1). Let X be a totally disconnected compact metric abelian group and σ be an automorphism of X. Assume that K is an open subgroup of X. If $h(\sigma) = 0$, then $\bigcap_0^n \sigma^j K = \bigcap_1^n \sigma^j K$ for some $n>0$.

To see this, we assume that $\bigcap_0^n \sigma^j K \subsetneq \bigcap_1^n \sigma^j K$ for all $n>0$. Then it follows that $|X/\bigcap_0^n \sigma^j K| \geqslant 2^n$ for all $n>0$ (the notation "$|E|$" means the cardinality of a set E). Let $\varrho(K)$ be the partition of X consisting of the cosets of K, then $\varrho(K)$ is a finite measurable partition of X and $\bigvee_0^n \sigma^j\varrho(K) = \varrho(\bigcap_0^n \sigma^j K)$ holds. Since $|\varrho(\bigcap_0^n \sigma^j K)| > 2^n$, by definition we have $h(\sigma) \geqslant \lim_{n\to\infty}(1/(1+n))\log 2^n = \log 2$, which is a contradiction.

Let H be a compact metric group and X be the direct product of a doubly infinite sequence of replicaes of the group H. The elements of X will be the sequence $\{h_n\}$, $h_n \in H$, $n = 0, \pm 1, \pm 2, \ldots$ A *Bernoulli automorphism* with the groups of states H will mean the automorphism σ of G defined by $\sigma\{h_n\} = \{h_{n+1}\}$.

PROPERTY (2). Under the above notations, one has $h(\sigma) = \log|H|$ (p. 95 of [27]).

PROPERTY (3). Let X and σ be as in Property (1) and let $X = F_0 \supset F_1 \supset \ldots$ be a sequence of σ-invariant subgroups such that $\bigcap F_n = \{0\}$ and for every $n>0$, $\sigma_{F_n/F_{n+1}}$ is a Bernoulli automorphism. Then σ is densely periodic (Proposition 10.6 of [13]).

PROPERTY (4). Let X be a compact metric abelian group and σ be an automorphism of X. If H is a σ-invariant subgroup of X, then $h(\sigma) = h(\sigma_{X/H}) + h(\sigma_H)$ (§ 6 of [13]).

PROPERTY (5). Let X and σ be as in Property (4). If (X, σ) is expansive, then there exists a finite open covering $\mathfrak{p}$ such that the topology of X generated by $\bigcup_{-\infty}^{\infty} \sigma^j\mathfrak{p}$ equals the original topology (see [15]). Let $\mathfrak{p}'$ denote the partition of X consisting of the finitely many atoms of the algebra generated by $\mathfrak{p}$. Then we have $h(\sigma) = h(\sigma, \mathfrak{p}') < \infty$ since $\bigvee_{-\infty}^{\infty} \sigma^j\mathfrak{p}'$ is the partition into points of X (p. 87 of [27]).

PROPERTY (6). Let X be solenoidal and σ be an automorphism of X. As before let (G, γ) be the dual of (X, σ). Since $\operatorname{rank}(G) = d < \infty$, G is imbedded in $\mathbf{Q}^d$. Thus we can consider G to be a subset of $\mathbf{Q}^d$, and as before denote by the same symbol the extension on $\mathbf{Q}^d$ of γ. Let $p(x)$ be the characteristic polynomial of γ. Then we have

$$h(\sigma) = \sum_{|\lambda|>1} \log|\lambda| + \log\Delta$$

where λ's are the eigenvalues of γ and Δ is the smallest positive integer such that $\Delta p(x)$ has integral coefficients (see [14]).

THEOREM 2. *Let σ be an expansive automorphism of a compact metric abelian group X. Then for every σ-invariant subgroup H ($\sigma H = H$), $(X/H, \sigma)$ is expansive.*

Before beginning with the proof, we shall prepare the following Lemmas 4–7.

LEMMA 4. *Let σ be an automorphism of a totally disconnected compact metric abelian group X and as before (G, γ) be the dual of (X, σ). Then (X, σ) is expansive if and only if $G = \sum_{j=-\infty}^{\infty} \gamma^{-j}G_1$ where G_1 is a finite subgroup.*

Proof. If (X, σ) is expansive, by definition there is an open set F such that $\bigcap_{-\infty}^{\infty} \sigma^{-j}F = \{0\}$. Since X is totally disconnected, F contains an open subgroup F_0 and obviously $\bigcap_{-\infty}^{\infty} \sigma^{-j}F_0 = \{0\}$. Hence $\sum_{-\infty}^{\infty} \gamma^{-j}G_{F_0} = G$ where G_{F_0} is the annihilator of F_0 in G.

Conversely, if $G = \sum_{-\infty}^{\infty} \gamma^{-j}G_1$ where G_1 is a finite subgroup, then the annihilator F of G_1 in X is open and $\bigcap_{-\infty}^{\infty} \sigma^{-j}F = \{0\}$. This implies that (X, σ) is expansive.

LEMMA 5. *Let X and σ be as in Lemma 4. If (X, σ) is expansive, then for every σ-invariant subgroup L ($\sigma L = L$), $(X/L, \sigma)$ is expansive.*

Proof. Let (G, γ) be the dual of (X, σ) and G_L be the annihilator of L in G. Since (X, σ) is expansive, there is a finite subgroup G_1 such that $\sum_{-\infty}^{\infty} \gamma^{-j}G_1 = G$ (by Lemma 4). Under the action of γ, G is a $\mathbf{Z}[x, x^{-1}]$-module and so G_L is clearly a $\mathbf{Z}[x, x^{-1}]$-submodule. Hence G_L is expressed as $G_L = \sum_{-\infty}^{\infty} \gamma^{-j}G_1'$ where G_1' is a finite subgroup (since $\mathbf{Z}[x, x^{-1}]$ is Noetherian). By Lemma 4 again, we get the conclusion of the lemma.

LEMMA 6. *Let σ be an automorphism of a compact metric abelian group X and X_0 be the connected component of the identity in X. If σ is an automorphism of X, then there exists a σ-invariant totally disconnected subgroup K ($\sigma K = K$) such that $X = X_0 + K$.*

The lemma is proved in [4] or [2] and so we omit the proof.

LEMMA 7. *Let X and σ be as in Lemma 6 and H be a σ-invariant subgroup ($\sigma H = H$) of X. If (H, σ) and $(X/H, \sigma)$ are expansive, then (X, σ) is expansive.*

Proof. Let $\dot{U}$ and U' be expansive neighborhoods for $(X/H, \sigma)$ and (H, σ) respectively. Obviously, $\dot{U} = \{x+H: x \in V\}$ and $U' = H \cap V$ where U and V are suitable neighborhoods of X. Letting $W = U \cap V$, we have $\bigcap_{-\infty}^{\infty} \sigma^j W = \bigcap_{-\infty}^{\infty} \sigma^j W \cap H = \bigcap_{-\infty}^{\infty} \sigma^j (W \cap H) = \{0\}$ since $\bigcap_{-\infty}^{\infty} \sigma^j (W+H) = H$. Therefore W is an expansive neighborhood for (X, σ).

Proof of Theorem 2. Since X is a compact metric abelian group, X splits into a sum $X = X_0 + K$ as in Lemma 6. Since (X_0, σ) is expansive, the dimension of X_0 is finite; i.e. X_0 is solenoidal. Let H be a σ-invariant subgroup given in Theorem 2. Then we have $X/H = \{(X_0+H)/H\} + \{(K+H)/H\}$. Since $(X_0+H)/H$ is a factor group of X_0, $((X_0+H)/H, \sigma)$ is expansive by Corollary 1. On the other hand, $((K+H)/H, \sigma)$ is expansive by Lemma 5. Therefore $(X/H, \sigma)$ is expansive by Lemma 7. The proof is completed.

Let X be a compact metric abelian group and σ be an automorphism of X. Let us put

$$P_n(\sigma) = \{x \in X: \sigma^n x = x\}, \quad n \geqslant 1 .$$

It is obvious that $P_n(\sigma)$ is an algebraic subgroup of X for $n \geqslant 1$. The automorphism σ is said to be *densely periodic* if $\bigcup_{n=1}^{\infty} P_n(\sigma)$ is dense in X.

THEOREM 3. *Let σ be an expansive automorphism of a compact metric abelian group X, then σ is densely periodic.*

For the proof we need the following Lemmas 8 and 9.

LEMMA 8. *Let σ be an automorphism of a totally disconnected compact metric abelian group X. If (X, σ) is expansive and $h(\sigma) = 0$, then X is finite.*

Proof. Take an open subgroup K as an expansive neighborhood for (X, σ). By Property (1) we get $\bigcap_{n=0}^{k} \sigma^n K = \bigcap_{n=1}^{k} \sigma^n K$ for some k (since $h(\sigma) = 0$). Thus $K \supset \bigcap_{n=1}^{k} \sigma^n K$ for some integer k, because X is compact and each $\sigma^n K$ is closed and open. We have $\sigma K' = K'$ where $K' = \bigcap_{n=1}^{k} \sigma^n K$. Since K is an expansive neighborhood, $K' = \{0\}$ and K' is open. Therefore X is finite.

The following is proved in [2].

LEMMA 9. *Let σ be an automorphism of a compact metric abelian group X. If $h(\sigma) > 0$, then there exist σ-invariant subgroups X_1 and X_2 such that the following conditions hold*; (i) *(X_1, σ) has zero entropy*, (ii) *(X_2, σ) is ergodic and* (iii) *$X = X_1 + X_2$.*

Proof of Theorem 3. As before let X_0 be the connected component of the identity of the group X. By Lemmas 6 and 9, X splits into a sum $X = X_0 + K_1 + K_2$ of subgroups where K_1 and K_2 are σ-invariant totally disconnected subgroups such that (K_1, σ) has zero entropy and (K_2, σ) is ergodic. Expansiveness implies that K_1 is finite by Lemma 8. Obviously σ_{K_1} is densely periodic. Since (X_0, σ) is expansive, the dual group of X_0 is finitely generated under the dual of σ_{X_0}. Hence σ_{X_0} is densely periodic (see [19]).

To show that σ_{K_2} is densely periodic, let (G_{K_2}, γ) be the dual of (K_2, σ). Obviously G_{K_2} is a torsion group and γ on G_{K_2} has no periodic points except the identity. Since (K_2, σ) is expansive, $G_{K_2} = \sum_{-\infty}^{\infty} \gamma^{-j} G_1$ where G_1 is a finite group (by Lemma 4), and G_1 splits into a direct sum $G_1 = \bigoplus_{m=1}^{n} G_{1,m}$ of primary groups $G_{1,m}$ (p. 137 of [16]). Hence it follows that

$$G = \bigoplus_{m=1}^{n} G'_m \quad \text{where} \quad G'_m = \sum_{-\infty}^{\infty} \gamma^{-j} G_{1,m} .$$

If $(K_{2,m}, \sigma)$ denotes the dual of (G'_m, γ) for $1 \leqslant m \leqslant n$, then we have $(K_2, \sigma) = \bigoplus_{1}^{n} (K_{2,m}, \sigma)$ Thus it will be enough to show that each $\sigma_{K_{2,m}}$ is densely periodic. To see this, let G'_m be a p_m-primary group for $1 \leqslant m \leqslant n$. Denote by W_i the subgroup of G'_m annihilated by multiplication by p_m^i for $i \geqslant 1$. Then there is an integer $a > 0$ such that

$$W_1 \subset W_2 \subset \ldots \subset W_a = G'_m$$

since G'_m is finitely generated under γ. It is easy to see that for $1 \leqslant i \leqslant a$, $\gamma|_{W_{i+1}/W_i}$ has no periodic points except the identity. Considering W_{i+1}/W_i to be a $\mathbf{Z}/p_m\mathbf{Z}[x, x^{-1}]$-module (the notion $\mathbf{Z}/p_m\mathbf{Z}[x, x^{-1}]$ denotes the ring of all polynomials in x and x^{-1} with coefficients in the field $\mathbf{Z}/p_m\mathbf{Z}$), we see that W_{i+1}/W_i is finitely generated under $\mathbf{Z}/p_m\mathbf{Z}[x, x^{-1}]$. Since $\mathbf{Z}/p_m\mathbf{Z}[x, x^{-1}]$ is a principal ideal domain, we have

$$\begin{aligned} W_{i+1}/W_i &= \mathbf{Z}/p_m\mathbf{Z}[\gamma, \gamma^{-1}]\dot{g}_1 \oplus \ldots \oplus \mathbf{Z}/p_m\mathbf{Z}[\gamma, \gamma^{-1}]\dot{g}_k \\ &= \Big(\bigoplus_{-\infty}^{\infty} \gamma^j \langle \dot{g}_1 \rangle\Big) \oplus \ldots \oplus \Big(\bigoplus_{-\infty}^{\infty} \gamma^j \langle \dot{g}_k \rangle\Big) \end{aligned}$$

for some $\dot{g}_1, \ldots, \dot{g}_k \in W_{i+1}/W_i$. Let V_i be the annihilator of W_i in $K_{2,m}$ for $1 \leqslant i \leqslant a$. Obviously, $V_1 \supset V_2 \supset \ldots \supset V_a = \{0\}$ and each V_i is σ-invariant. Since V_i/V_{i+1} has the dual group W_{i+1}/W_i, $\sigma_{V_i/V_{i+1}}$ is a Bernoulli automorphism. Using Property (3), we see that $(K_{2,m}, \sigma)$ is densely periodic, and therefore so is (K_2, σ). Since (X, σ) is a factor of a direct product system $(X_0 \times K_1 \times K_2, \sigma \times \sigma \times \sigma)$, (X, σ) is densely periodic. The proof is completed.

Remark 3. Let G be a countable discrete group of type p^{∞} (see p. 56 of [16]). We remark that p is a prime number. Assume that m is a prime number such that $m > p$. We can define an automorphism γ of G by $\gamma g = mg$ for $g \in G$. Let (X, σ) be the dual of (G, γ). Then X is totally disconnected and σ has no periodic points except the identity. This shows that (X, σ) is not expansive by Theorem 3. To get this, it will be enough to show that $(\gamma^k - I)G = G$ for all $k>0$ (the notation "I" means the identity map). When there is $k>0$ such that p does not divide m^k-1, we get easily that $(\gamma^k-I)G = G$ since G is a group of type p^∞. When p divides m^k-1 for some $k>0$, we can write $m^k-1 = p^n q$ where $n>0$ and q is prime to p unless $q = 1$. Then for every $a>0$ and a generator $g \in G$ with order p^{a+n}, it follows that $(\gamma^k-I)g$ is a generator with order p^a. Hence we get $(\gamma^k-I)G = G$. In any case we see that $(\gamma^k-I)G = G$ for all $k>0$; i.e. σ has no periodic points except the identity.

If in particular $p = 2$ and $m = 3$, then $\gamma g = g$ for $0 \neq g \in G$ with $2g = 0$. This shows that (X, σ) is non-ergodic (see [12]). It is clear from the above conclusion that σ is not densely periodic.

The first author proved the existence of an ergodic automorphism of a totally disconnected compact metric abelian group which has no periodic points except the identity. He will treat it in a future paper [1].

§ 4. The structure of expansive solenoidal automorphisms. Let X be a solenoidal group and σ be an automorphism of X. Then (X, σ) is said to satisfy *condition* $(*)$ if X has not a σ-invariant subgroup H such that X/H is a torus.

The aim of this section is to show the following

THEOREM 4. *Let X and σ be as above. If (X, σ) is expansive, then there exist in X σ-invariant subgroups T and S where T is a torus when $T \neq \{0\}$ and S is a solenoidal group without torus subgroups when $S \neq \{0\}$, such that (T, σ) and (S, σ) are expansive and (X, σ) is a factor of the direct product of (T, σ) and (S, σ).*

For the proof we use the following lemmas.

LEMMA 10. *Let X and σ be as above. If $h(\sigma) < \infty$, then there exist a totally disconnected subgroup N and connected subgroups T and S such that the following conditions hold*; (a) *N, S and T are σ-invariant*, (b) $h(\sigma_N) = 0$, (c) $N = S \cap T$, (d) *if $S \neq \{0\}$ then it is a solenoidal group with condition* $(*)$, (e) *if $T \neq \{0\}$ then it is a torus and* (f) $X/N = T/N \oplus S/N$.

For the proof the reader may refer [3].

LEMMA 11. *Let σ be an automorphism of a solenoidal group S. If (S, σ) has condition $(*)$, then S has no torus subgroups.*

Proof. As before let (G, γ) be the dual of (S, σ). We assume that S contains a σ-invariant torus subgroup A. If G_A is the annihilator of A in G, then $\gamma G_A = G_A$ and G/G_A is the dual group of A, so that G/G_A is torsion free and finitely generated. Hence there are $0 \neq f \in G$ and $p(x) \in \mathbf{Z}[x]$ such that $p(\gamma) f \in G_A$. We remark that $p(x)$ is monic and its constant term is 1 or -1. Let $p(x)$ be a polynomial with minimal degree satisfying $p(\gamma) f \in G_A$. Since $\operatorname{rank}(G_A) < \infty$, it follows that $q(\gamma)p(\gamma)f = 0$ for some $0 \neq q(x) \in \mathbf{Z}[x]$ with minimal degree.

If $q(\gamma)f = 0$, then $q(\gamma) f \in G_A$ and hence $p(x)$ divides $q(x)$ over $\mathbf{Q}$ (the notation $\mathbf{Q}$ denotes the rational field); i.e. $q(x) = p_1(x)p(x)$ for some $0 \neq p_1(x) \in \mathbf{Q}[x]$. When $p_1(\gamma)f = 0$, by the same reason we have $p_1(x) = p_2(x)p(x)$ for some $p_2(x) \in \mathbf{Q}[x]$. Repeating this process, we see that there is $p_n(x) \in \mathbf{Q}[x]$ with $q(x) = p_n(x)p(x)^n$ such that $p_n(\gamma)f \neq 0$. Put $g = p(\gamma)^{n-1}p_n(\gamma)f$. Then $g \neq 0$ and $p(\gamma)g = 0$. Since $p(x)$ is monic and its constant term is 1 or -1, the subgroup G_B generated by $\{\gamma^j g\colon -\infty < j < \infty\}$ is γ-invariant and finitely generated. Therefore the annihilator B of G_B in S is σ-invariant and S/B is a torus, which is a contradiction.

For the case when $q(\gamma)f \neq 0$, by the same way we get that S has no σ-invariant torus subgroups.

Assume that S contains a torus subgroup C to get the conclusion. For $n \geq 1$, $C_n = \sum_0^n \sigma^i C$ is also a torus subgroup of S. Since $\dim(S) < \infty$, we have $C_n = C_{n+1}$ for some $n>0$, and so $\sigma C_n \subset C_n$. Put $C_{n,m} = \sum_{i=0}^m \sigma^{-i} C_n$ for $m>0$, then $C_{n,m} = C_{n,m+1}$ for some $m>0$. Hence $\sigma^{-1} C_{n,m} \subset C_{n,m}$. Since $\sigma C_{n,m} = \sigma C_n + \ldots + \sigma^{-m+1} C_n \subset C_{n,m}$, $C_{n,m}$ is a σ-invariant torus subgroup of S. This can not happen by the above conclusion. The proof of the lemma is completed.

Proof of Theorem 4. Since X is solenoidal, $h(\sigma) < \infty$ by Property (6). Hence X splits as in Lemma 10, and so (X, σ) is a factor of a direct product of (T, σ) and (S, σ). Since $h(\sigma_N) = 0$ (Lemma 10 (b)) and (N, σ) is expansive, N is finite by Lemma 4. Hence the duals of (S, σ) and (T, σ) are finitely generated under the dual automorphisms and they are hyperbolic. By Theorem 1, (S, σ) and (T, σ) are expansive. It follows from Lemma 11 that S has no torus subgroups. The proof is completed.

Remark 4. Let X and σ be as in Theorem 4. For the case when (X, σ) is expansive, it is not always true that X splits into a direct sum of T and S which are the subgroups in Theorem 4. For, let γ be an automorphism of $\mathbf{Q}^3$ induced by a matrix

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix}.$$

Define by G the discrete subgroup of $\mathbf{Q}^3$ generated by $\{\gamma^j g\colon -\infty < j < \infty\}$ for $g = \{\frac{1}{5}, \frac{1}{5}, \frac{3}{5}\} \in \mathbf{Q}^3$. Then G is finitely generated under γ and all the eigenvalues of γ are off the unit circle. Therefore the dual (X, σ) of (G, γ) is expansive. We assume that $X = S \oplus T$ to get a contradiction. Then we have $G = G_S \oplus G_T$ where G_T and G_S are the annihilators of S and T in G, respectively. It is not difficult to see that $G_T \subset \{0\} \oplus \mathbf{Z}^2$ and $G_S \subset \mathbf{Q} \oplus \{0\} \oplus \{0\}$. Hence we have $G = G_S \oplus G_T \subset \mathbf{Q} \oplus \mathbf{Z}^2$. But $g \notin \mathbf{Q} \oplus \mathbf{Z}^2$, which is a contradiction.

Remark 5. Let X be a compact metric abelian group and σ be an expansive automorphism of X. Denote by X_0 the connected component of the identity in X. If X/X_0 is finite, then it is a problem whether X splits into a direct sum of X_0 and

[1] See the paper by N. Aoki, J. Math. Soc. Japan 33 (1981), pp. 693–700.

a σ-invariant subgroup. But this answer is negative. We have an easy example. Let G_C be the subgroup $\{m/3^k\colon k, m \in \mathbf{Z}\}$ of $\mathbf{Q}$ and G_F be the abelian group of order 2 consisting of $\{0, 1\}$. We denote by G the direct product group of G_C and G_F, and define an automorphism γ of G by

$$\gamma g = \begin{cases} (3g_1, g_2) & \text{if } g_1 \in \{2g\colon g \in G_C\}, \\ (3g_1, g_2') & \text{if } g_1 \notin \{2g\colon g \in G_C\} \end{cases}$$

for $g = (g_1, g_2) \in G = G_C \times G_F$ and $g_2' \neq g_2$. Let (X, σ) be the dual of (G, γ). Then it is easy to see that (X, σ) is expansive. Indeed, let C denote the annihilator of $G_C \times \{0\}$ in X and F denote that of $\{0\} \times G_F$ in X. Then C is finite and F is connected. We have the direct sum splitting $X = C \oplus F$. Since $\gamma(\{0\} \times G_F) = \{0\} \times G_F$ and $\gamma(G_C \times \{0\}) \neq G_C \times \{0\}$, $\sigma C \neq C$ and $\sigma F = F$. Using Theorem 1, we see that (F, σ) is expansive. Since C is finite, F is open in X and hence (X, σ) is expansive. To get the conclusion, we assume that there exists a γ-invariant subgroup G' such that $G = G' \oplus (\{0\} \times G_F)$. Then there is an element $0 \neq g \in G'$ such that $g \notin \{2g\colon g \in G_C\} \times G_F$. But we have $0 \neq \gamma g - 3g \in \{0\} \times G_F$. This can not happen. Therefore the group X does not split into a direct sum of a finite subgroup and a connected subgroup, invariant with respect to σ.

Added in proof. The specification property was studied in Monatsh. Math. 93 (1982), pp. 79–110 for solenoidal automorphisms, and in Lecture Notes 729, Springer, pp. 93–104 for toral automorphisms.

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