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Periodic homeomorphisms of chainable continua

by

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Abstract. We prove that for every positive integer n there exists a chainable continuum M_n which admits a homeomorphism of period n. While each M_n can be made to be a pseudo-arc, there also exist for each n non-hereditarily indecomposable chainable continua with homeomorphisms of period n.

Introduction. It is an easy exercise to show that every periodic homeomorphism of an arc must either have period 2 or be the identity. Beverly Brechner [Br–1] has constructed a homeomorphism of the wedge of two pseudo-arcs of period 4. (The pseudo-arc itself has an obvious homeomorphism of period 2.) Until now, all known periodic homeomorphisms of chainable continua had periods 1, 2, or 4.

Michel Smith and Sam Young [SY–1] have shown that if a chainable continuum admits a homeomorphism of period greater than 2, then the continuum must contain an indecomposable continuum. Since the pseudo-arc is hereditarily indecomposable [M–1], [B–1], and in many other ways is at the opposite end of the spectrum from an arc, it would seem a natural place to try to construct a homeomorphism of period greater than 2. It will follow from our results that the pseudo-arc has such homeomorphisms of high period, but our construction is more easily described for non-hereditarily indecomposable continua. (The author earlier [L–1] announced the existence of homeomorphisms of prime period for the pseudo-arc. The results here, in addition to being stronger, have what are hopefully much more readily understandable proofs.)

Throughout, by saying that a homeomorphism h has period n we will mean that n is the smallest positive integer such that h^n is the identity.

Construction of M_n. Let T_n be the continuum resulting from $[0,1]\times\{0,1,2,\ldots,n-1\}$ by identifying $\{0\}\times\{0,1,2,\ldots,n-1\}$ to a point. (Thus T_3 is the standard triod.)

Let M be an integer greater than n. Let f_i be a map from T_n onto T_n satisfying the following conditions. Let

$$\delta_i = \frac{1}{2^{2(2^i-1)}M^{(2^i)}}.$$

Then

$$[2\sum_{j=0}^{K} j\delta_i^2,\ 2\sum_{j=0}^{K} j\delta_i^2+(K+1)\delta_i^2]\times\{0\}$$

is mapped linearly, order preserving, onto $[0, (K+1)\delta_i]\times\{p\}$ where p is congruent to K (mod n) and $K = 0, \ldots, 1/\delta_i-1$. The interval

$$[2\sum_{j=0}^{K} j\delta_i^2+(K+1)\delta_i^2,\ 2\sum_{j=0}^{K+1} j\delta_i^2]\times\{0\}$$

is mapped linearly, order reversing, onto $[0, (K+1)\delta_i]\times\{p\}$. If $f_i(\langle x, 0\rangle) = \langle y, p\rangle$, then $f_i(\langle x, r\rangle) = \langle y, p+r\rangle$, where addition in the second co-ordinate is done modulo n.

Let $M_n = \varprojlim\langle T_n^i, f_i\rangle$ where each T_n^i is homeomorphic to T_n and $f_i\colon T_n^{i+1}\to T_n^i$ is the f_i defined above.

Let $h_n^i\colon T_n^i\to T_n^i$ be the homeomorphism defined by $h_n^i(\langle x, r\rangle) = \langle x, r+1\rangle$. (Addition in the second co-ordinate is done modulo n.) It is easily checked that $h_n^i f_i = f_i h_n^{i+1}$ for each i, so there is an induced homeomorphism $h_n\colon M_n\to M_n$.

Proof of periodic homeomorphisms. We are now ready to prove the existence of period n homeomorphisms for chainable continua. This will be done in two steps.

THEOREM 1. *For each positive integer n, h_n is a homeomorphism of M_n having period n.*

Proof. Each $h_n^i\colon T_n^i\to T_n^i$ clearly has period n. Since the h_n^i's commute with the bonding maps f_i, the induced h_n also has period n. ∎

Of course, since M_n was defined as an inverse limit of trees (n-ods), we need to show that M_n is arc-like, not just tree-like. The following theorem will suffice to prove this.

THEOREM 2. *For each $\varepsilon>0$ there exists a map $f\colon M_n\to[0,1]$ such that $\operatorname{diam}(f^{-1}(x))<\varepsilon$ for each $x\in[0,1]$.*

Proof. Define $g_j\colon T_n^j\to[0,1]$ to satisfy the following conditions. The interval $[2\sum_{p=0}^{K} p\delta_{j-1}^2,\ 2\sum_{p=0}^{K+1} p\delta_{j-1}^2]\times\{r\}$ is mapped linearly, order preserving, onto $[2\sum_{p=0}^{K+r-n} p\delta_{j-1}^2,\ 2\sum_{p=0}^{K+r-n+1} p\delta_{j-1}^2]$ for $n-r\leqslant K\leqslant 1/\delta_i-2$. The interval $[1-\delta_{j-1}, 1]\times\{r\}$ is mapped linearly, order preserving onto $[2\sum_{p=0}^{K+r-n} p\delta_{j-1}^2,\ 2\sum_{p=0}^{K+r-n} p\delta_{j-1}^2+(K+r-n+1)\delta_{j-1}^2]$, where $K+1 = 1/\delta_{j-1}$. The interval $[0, 2\sum_{p=0}^{n-r} p\delta_{j-1}^2]\times\{r\}$ is mapped onto 0.

Let $p_j\colon M_n\to T_n^j$ be the projection of M_n onto the jth co-ordinate. Define $\tilde g_j\colon M_n\to[0,1]$ by $\tilde g_j = g_j p_j$.

If $g_{j+1}(x) = g_{j+1}(y)$, then $\operatorname{dist}(f_j(x), f_j(y))<2n\delta_j$. If $k>j$ and $g_k(x) = g_k(y)$, then

$$\operatorname{dist}(f_jf_{j+1}\ldots f_{k-1}(x), f_jf_{j+1}\ldots f_{k-1}(y))<\frac{1}{2^{k-1-j}}n\delta_j.$$

Thus if $\tilde g_j(x) = \tilde g_j(y)$, then

$$\operatorname{dist}(x, y)<\sum_{i=0}^{j-1}\frac{1}{2^i}\cdot\frac{1}{2^{j-i-1}}n\delta_i+\sum_{i=j}^{\infty}\frac{1}{2^{i-1}} = \sum_{i=0}^{j-1}\frac{1}{2^{j-1}}n\delta_i+\frac{1}{2^{j-2}}<\frac{1}{2^{j-3}}$$

(since $\sum_{i=0}^{\infty} n\delta_i<2$). Hence, if we choose j sufficiently large, $\operatorname{diam} g_j^{-1}(x)<1/2^{j-3}<\varepsilon$ for each $x\in[0,1]$. Let $f = \tilde g_j$.

COROLLARY 1. *For each n, M_n is a chainable continuum and h_n is a period n homeomorphism of M_n.*

Note that by our construction M_n is indecomposable for every n (even $n = 1, 2$). In a sense, M_n is a minimal example in that any subcontinuum of M_n which is mapped onto itself by h_n must be homeomorphic to M_n. (M_n also has other subcontinua which are arcs, Knaster continua, etc.) One can modify M_n, introducing crookedness into the f_i's, to make M_n a pseudo-arc.

COROLLARY 2. *For each positive integer n, there exists a homeomorphism of the pseudo-arc of period n.*

One can modify M_n, increasing the branching at each of the defining stages (making T_n^i homeomorphic to $T_{i!}$) and adjusting the f_i's accordingly, to produce a non-hereditarily indecomposable chainable continuum admitting homeomorphisms of every period.

By making T_p^i homeomorphic to $T_{(p^i)}$ and further adjusting the f_i's one can construct a non-trivial p-adic Cantor group action on a chainable continuum, including one on the pseudo-arc.

QUESTIONS. The arc has no homeomorphisms of period greater than 2, the pseudo-arc has homeomorphisms of every period, and there are other chainable continua with no periodic homeomorphisms other than the identity.

QUESTION 1. What conditions on a chainable continuum permit (guarantee) the existence of homeomorphisms of period n? p-adic Cantor group actions?

As we have constructed it, M_n has only one fixed point, 0, under h_n. The fixed point is also an endpoint of M_n, as the maps in Theorem 2 show. Thus we can attach any other chainable continuum N with an endpoint P to M_n, by identifying P with 0. $M_n\cup N$, with this identification, is a decomposable chainable continuum with a homeomorphism of period n. If N admits a homeomorphism of period s fixing P, then

$M_n \cup N$ admits a homeomorphism of period t, where t is the least common multiple of n and s. If N is homeomorphic to M_n, then $M_n \cup N$ admits a homeomorphism of period $2n$ (e.g. Brechner's period 4 homeomorphism of the wedge of two pseudo-arcs). However, all of these homeomorphisms in a sense factor into periodic homeomorphisms on indecomposable continua.

QUESTION 2. Is there a decomposable chainable continuum admitting a homeomorphism of period greater than 2 which does not factor into periodic homeomorphisms on indecomposable continua?

The following two questions are suggested by Brechner's work.

QUESTION 3. Are all period 2 (period n) homeomorphisms of the pseudo-arc conjugate?

QUESTION 4. For which periodic homeomorphisms of the pseudo-arc and which embeddings of the pseudo-arc in E^2 can the homeomorphisms be extended to periodic homeomorphisms of E^2?

Added in proof. Juan Toledo has recently extended these results by constructing periodic homeomorphisms on the pseudo-arc with nondegenerate subcontinua as fixed point sets.

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