and it is easy to see why we cannot. For any sentence σ, σ will be valid exactly when σ & $\bar{\sigma}$ is semivalid. Thus determining semivalidity is at least as hard (and, in fact, is exactly as hard) as determining validity.

Finally, it should be noted that the semivalid sentences based upon Ramsey theory or being nonplanar or the like are going to be quite complicated if for no other reason than the requirements of sufficiently large order. And because of the inherent lack of decision procedures for semivalidity, this approach cannot be expected to help in answering extremal questions such as the determination of Ramsey numbers.

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Compactness = JEP in any logic

by

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Abstract. L-elementary embeddings are for logic L what elementary embeddings are for first-order logic. If the Joint Embedding Property holds for L-elementary embeddings (for short: L has JEP), then the latter become a fundamental model, as well as an arrow-theoretical feature of L. Assuming Constructibility, $\neg 0^{\#}$, or even $\neg L^{\mu}$, we prove that in any small extension of first-order logic JEP is equivalent to compactness. We further give a characterization of Craig's interpolation along the same lines, by making use of a strong notion of amalgamation.

Preliminaries. The reader is referred to [MSS] for everything unexplained here; following [Fe2], for τ a (similarity) type, $\mathrm{Str}(\tau)$ is the class of all structures of type τ; if L is a (many-sorted) logic then $\mathrm{Stc}_L(\tau)$ is the class of all sentences of L of type τ; given $\mathfrak{N}, \mathfrak{M} \in \mathrm{Str}(\tau)$ we let

$$\mathrm{th}_L \mathfrak{M} = \{\varphi \in \mathrm{Stc}_L(\tau) | \mathfrak{M} \models \varphi\}$$

and we let $\mathfrak{M} \equiv_L \mathfrak{N}$ mean that $\mathrm{th}_L\mathfrak{M} = \mathrm{th}_L\mathfrak{N}$. For $\Gamma \subseteq \mathrm{Stc}_L(\tau)$ we let $\mathrm{mod}_L\Gamma = \{\mathfrak{A} \in \mathrm{Str}(\tau) | \mathfrak{A} \models \Gamma\}$. In logic L we allow *relativization*, e.g., relativization of formula ψ to formula $\varphi(x, y_1, \ldots, y_q)$ where $y_1, \ldots, y_q$ act as parameters, and we write

$$\psi\{x|\varphi(x, y_1, \ldots, y_q)\}$$

to denote the formula obtained by this process. If $\mathfrak{B} \in \mathrm{Str}(\tau)$ and $B' \subseteq B$ (with B the universe of $\mathfrak{B}$) where B' is nonempty on each sort of τ, then $\mathfrak{B}|B'$ is the substructure of $\mathfrak{B}$ generated by B', see [Fl]. For the definition of (λ, ω)-compactness, see [MSS] or [MS]. *(Full) compactness* is (λ, ω)-compactness for all $\lambda \geqslant \omega$; an important related notion is given by the following (see [MS]):

Definition. Logic L is *μ-relatively compact* (for short: μ-r.c.) with $\mu \geqslant \omega$, iff for any classes of sentences Σ, Γ with $|\Sigma| = \mu$, if for each $\Sigma' \subseteq \Sigma$ with $|\Sigma'| < \mu$, $\Sigma' \cup \Gamma$ is consistent, then $\Sigma \cup \Gamma$ is consistent.

For the definition of L having *Craig's interpolation property* (or theorem), see [Fe1], [Ba], [MSS]. An important related notion is given by the following:

Definition. We say that in L *Robinson's consistency theorem* holds (or: L has the *Robinson property*) iff given any types τ, τ_1 and τ_2 and classes of sentences T, T_1 and T_2, if T is complete in τ and T_1, T_2 are consistent extensions of T in type τ_1 and τ_2 respectively, with $\tau = \tau_1 \cap \tau_2$, then $T_1 \cup T_2$ is consistent.

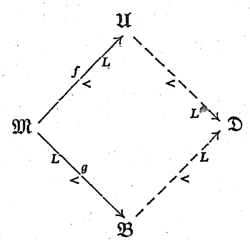
The *Löwenheim number* of L is (when it exists) the least λ such that any sentence of L having a model has a model of cardinality $\leqslant\lambda$.

Let $\mathfrak{M}\in \mathrm{Str}(\tau)$; the *diagram type* (often called: *diagram language*) of $\mathfrak{M}$ is the expansion τ_M of τ formed by adding a new constant c_m for each $m\in M$. The *diagram expansion* of $\mathfrak{M}$ is the structure $\mathfrak{M}_M\in \mathrm{Str}(\tau_M)$ in which each c_m is interpreted by m; more generally, for $Y\subseteq M$, expansion τ_Y and structure $\mathfrak{M}_Y$ are defined in the natural way; if f maps Y into M, then $\mathfrak{M}_{fY}$ is the structure of type τ_Y in which each c_y (with $y\in Y$) is interpreted by $f(y)$. Recall that the *elementary diagram* of $\mathfrak{M}$ is the complete theory $\mathrm{th}_{L_{\omega\omega}}\mathfrak{M}_M$ in type τ_M. It is well known that given structures $\mathfrak{M}$ and $\mathfrak{N}$ and $f\colon M\to N$, f is an *elementary embedding* of $\mathfrak{M}$ into $\mathfrak{N}$ iff $\mathfrak{N}_{fM}\models \mathrm{th}_{L_{\omega\omega}}\mathfrak{M}_M$, in symbols: $f\colon \mathfrak{M}\underset{<}{\to}\mathfrak{N}$ (see [Ke], p. 53). Both these notions are naturally generalized as follows:

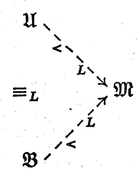
DEFINITION. Given logic L, the *L-elementary diagram* of structure $\mathfrak{M}$ is the complete theory $\mathrm{th}_L\mathfrak{M}_M$ in type τ_M; function $f\colon M\to N$ is an *L-elementary embedding* of $\mathfrak{M}$ into $\mathfrak{N}$ iff $\mathfrak{N}_{fM}\models \mathrm{th}_L\mathfrak{M}_M$; in symbols:

$$f\colon \mathfrak{M}\underset{<}{\to}_L\mathfrak{N}\quad (\text{or } \mathfrak{M}\overset{f}{\underset{<}{\to}}_L\mathfrak{N}).$$

We let $\mathfrak{M}\underset{<}{\to}_L\mathfrak{N}$ mean that there exists an L-elementary embedding f of $\mathfrak{M}$ into $\mathfrak{N}$ (read: $\mathfrak{M}$ is *L-elementarily embedded* into $\mathfrak{N}$). Following [Ke, p. 60] we say that the *amalgamation property* holds for L-elementary embeddings iff given $f\colon \mathfrak{M}\underset{<}{\to}_L\mathfrak{A}$ and $g\colon \mathfrak{M}\underset{<}{\to}_L\mathfrak{B}$ there is some structure $\mathfrak{D}$ and a diagram:



Equivalently, we say that *L has* AP. We say that the *Joint Embedding Property* holds for L-elementary embeddings (for short, *L has* JEP) iff, given $\mathfrak{A},\mathfrak{B}\in\mathrm{Str}(\tau)$ with $\mathfrak{A}\equiv_L\mathfrak{B}$ there is a structure $\mathfrak{M}$ such that $\mathfrak{A}\underset{<}{\to}_L\mathfrak{M}$ and $\mathfrak{B}\underset{<}{\to}_L\mathfrak{M}$; in other words, any two elementarily equivalent (w.r.t. logic L) structures can be L-elementarily embedded in a third one; we have the following diagram:



If $\mathrm{Stc}_L(\tau)$ is a set whenever τ is a set, and L is compact, then L has JEP ([CK 3.1.4] works as well for L); further, L has JEP iff $\equiv_L$ can be expressed in terms of $\underset{<}{\to}_L$ via the following equivalence:

$$\forall\mathfrak{A},\mathfrak{B},\ \mathfrak{A}\equiv_L\mathfrak{B}\quad\text{iff}\quad \exists\mathfrak{D}\text{ s.t. }\mathfrak{A}\underset{<}{\to}_L\mathfrak{D}\text{ and }\mathfrak{B}\underset{<}{\to}_L\mathfrak{D}.$$

Thus, if L has JEP, then $\underset{<}{\to}_L$ becomes a more fundamental notion than $\equiv_L$ itself. Our main theorem is the following:

THEOREM (Assuming Constructibility, $\neg 0^{\#}$, or $\neg L^{\mu}$). *Let in logic L, $\mathrm{Stc}_L(\tau)$ be a set whenever τ is a set, or even, let the Löwenheim number of L exist. Then*

L is fully compact iff L has JEP.

For the proof we argue as follows:

CLAIM 1. *Under the hypotheses of the theorem, if L is not (ζ,ω)-compact for some $\zeta\geqslant\omega$, then there exists $\varkappa$ such that L is not μ-r.c. for all regular μ such that $\mu\geqslant\varkappa$.*

Proof. Then, by [MS 6.6 (i)], L is not $\varkappa$-r.c. for some $\varkappa\geqslant\omega$; by [MS 6.6 (iv)], L is not μ-r.c. for all μ such that $\mathrm{cf}(\mu)\geqslant\varkappa$ (here Constructibility, $\neg 0^{\#}$ or $\neg L^{\mu}$ are used, see [DJK], p. 91); now take μ to be regular.

CLAIM 2. *Let $\{S_m\}_{m<\omega}$ be a set of symbols over a finite number of sorts $\{s_0,\ldots,s_k\}$; under the hypotheses of the theorem, there exists θ such that, up to logical equivalence, there are at most θ many theories (i.e. classes of sentences) in L of type $\{S_m\}_{m<\omega}$.*

Proof. It suffices to prove that there is θ' such that, up to logical equivalence, there are at most θ' many sentences in L of type $\{S_m\}_{m<\omega}$; as a matter of fact, if $\mathrm{Stc}_L(\tau)$ is a set for any τ a set, then we have nothing to prove; on the other hand, assuming that Löwenheim number of $L=\lambda$, then for any two logically inequivalent sentences α and β of type $\{S_m\}_{m<\omega}$ there must be some isomorphism class I of structures of this type and of cardinality $\leqslant\lambda$ such that for any $\mathfrak{A}\in I$, $\mathfrak{A}\models\alpha$ and not-$\mathfrak{A}\models\beta$ or vice versa; now, for some cardinal θ'', the number of such classes I is less than θ''. Hence, the number of sentences of type $\{S_m\}_{m<\omega}$ is, up to logical equivalence, less than θ' with $\theta'=2^{\theta''}$.

CLAIM 3. *Assume that L is not μ-r.c.; then there is a counterexample Σ, Γ to μ-relative compactness in L such that the type of $\Sigma\cup\Gamma$ is two-sorted, has no n-ary function symbol (for all $n\geqslant 1$) and has only countably many relation symbols.*

Proof. Let Σ', Γ' be a counterexample to μ-relative compactness in L; it is no loss of generality to assume that the type τ' of $\Sigma'\cup\Gamma'$ has no n-ary function symbols (for all $n\geqslant 1$); further, by mapping each sort s of τ' into a new unary relation symbol U_s and by relativizing to U_s all variables of sort s in each formula of $\Sigma'\cup\Gamma'$ (with the U_s all over one common sort) we can transform Σ', Γ' into a pair of theories Σ'', Γ'' whose type τ'' is single-sorted, in such a way that Σ'', Γ'' is still a counterexample to μ-relative compactness in L (for this sort-reducing relativization process in $L_{\omega\omega}$ see, e.g. [Mo], p. 484). Now, from Σ'', Γ'' of type τ'' over sort s'' we construct the required Σ, Γ as follows: let $\{R_i''\}_{i\in I}$ display all the relation symbols of τ''; let s^0 be a new sort, and $\{c_i\}_{i\in I}$ be new constant symbols over sort s^0; finally let $\{R^a\}_{1<a<\omega}$ be new relation symbols, where R^a is a-ary.

Let ϱ map R_i'' into R^{a+1} where a = arity of R_i''; let χ map R_i'' into c_i; notice that χ is one-one and the range of ϱ is countable. Given formula φ'' in $\Sigma''\cup\Gamma''$, let $\delta(\varphi'')$ be obtained by replacing each occurrence of $R_i''(x_1,\ldots,x_a)$ in φ'' by $R^{a+1}(c_i,x_1,\ldots,x_a)$ where $c_i=\chi(R_i'')$ and $R^{a+1}=\varrho(R_i'')$. For each subset Δ'' of $\Sigma''\cup\Gamma''$ let $\delta(\Delta'')=\{\delta(\varphi'')|\ \varphi''\in\Delta''\}$. Then it is straightforward to verify that Δ'' has a (single-sorted) model $\mathfrak{A}''$ iff $\delta(\Delta'')$ has a (two-sorted) model $\mathfrak{A}$. (This passage from n-ary relations to $(n+1)$-ary relations is analogous to the one described in [CK] exercise 7.2.16; here we use a new sort for the indexes.) Now $\delta(\Sigma'')$, $\delta(\Gamma'')$ are a counterexample to μ-relative compactness in L satisfying all the requirements of our claim.

Having proved Claim 3, we fix a set of two sorts $\{s_1,s_2\}$ and a collection of relation symbols $\{R_m^a\}_{a,m<\omega}$ such that R_m^a is a-ary and $R_m^a\neq R_n^a$ for $m\neq n$. If we assume that L is not (ζ,ω)-compact, then by Claims 1 and 3 we can consider a counterexample Σ_μ, Γ_μ (of type τ_μ) to μ-relative compactness in L for each regular $\mu\geqslant\varkappa$, such that the relation symbols of τ_μ are among the R_m^a's, there are no n-ary function symbols ($n\geqslant 1$) and the sorts of τ_μ are exactly s_1,s_2.

We shall now construct a class $\{T_\mu\}_{\mu\ \mathrm{regular}\geqslant\varkappa}$ of complete consistent theories which will provide the decisive step to show that, under the assumption that L is not (ζ,ω)-compact, L has not JEP.

Construction of T_μ. Starting from Σ_μ,Γ_μ as described above, we let $\Sigma_\mu=\{\varphi_\beta|\ \beta<\mu\}$ and $\Sigma_\mu^\nu=\{\varphi_\beta|\ \beta<\nu\}$ for all ordinals $\nu<\mu$; we also let $\mathfrak{A}_\mu^\nu$ be a model of $\Sigma_\mu^\nu\cup\Gamma_\mu$ (such a model existing by hypothesis); without loss of generality we assume that $\mathfrak{A}_\mu^\alpha$ and $\mathfrak{A}_\mu^\beta$ have disjoint universes for $\alpha\neq\beta$; we let type τ_μ^* be obtained from τ_μ by adding one more sort s_3 and new symbols $<,f,m_\nu$ ($<$ binary relation, f unary function, m_ν constant symbols for each ordinal $\nu<\mu$); we let $\mathfrak{M}_\mu\in\mathrm{Str}(\tau_\mu^*)$ be any structure having the following properties:

$\mathfrak{M}_\mu\restriction\{<,m_\nu\}_{\nu<\mu}=\langle\mu,<,\nu\rangle_{\nu<\mu}$ ($<,m_\nu$ on sort s_3),
domain of f = (disjoint) union of the universes of the $\mathfrak{A}_\mu^\nu$'s (μ fixed, $\nu<\mu$),
range of $f=\mu$,
$f^{-1}(\nu)$ = universe of $\mathfrak{A}_\mu^\nu$ ($\nu<\mu$),
$\mathfrak{M}_\mu\restriction\tau_\mu|f^{-1}(\nu)=\mathfrak{A}_\mu^\nu$ ($\nu<\mu$).

Roughly, $\mathfrak{M}_\mu$ is a "disjoint union" of the $\mathfrak{A}_\mu^\nu$'s together with a function f indexing each universe. (1) Define $T_\mu=\mathrm{th}_L\mathfrak{M}_\mu$ (in type τ_μ^*) and notice that the following sentences are in T_μ (here we use symbol ε instead of x^{s_3} for a variable of third sort; we also let x^1 and x^2 denote variables over s_1 and s_2 respectively):

$(1)_{\alpha<\beta<\mu}$ $\qquad<$ is a linear ordering over s_3 and $m_\alpha<m_\beta$,

(2) $$\forall\varepsilon\big(\exists x^1 f(x^1)=\varepsilon\wedge\exists x^2 f(x^2)=\varepsilon\big),$$

so that each inverse image $f^{-1}(\varepsilon)$ is nonempty on both sorts s_1 and s_2

$(3)_{\beta<\mu}$ $$\forall\varepsilon(m_\beta<\varepsilon\to\varphi_\beta^{\{x|f(x)=\varepsilon\}}),$$

so that each $f^{-1}(\varepsilon)$ is the universe of a model of Σ_μ^ε;

$(4)_{\psi\in\Gamma_\mu}$ $$\forall\varepsilon\,\psi^{\{x|f(x)=\varepsilon\}},$$

so that $f^{-1}(\varepsilon)$ is the universe of a model of Γ_μ.

Claim 4. *Given any model $\mathfrak{N}$ of T_μ, the m_β's are unbounded, i.e.*

$$\neg\exists n\in N \text{ such that } \mathfrak{N}\models m_\beta<n\quad(\forall\beta<\mu).$$

Proof. Otherwise, any counterexample $\langle\mathfrak{N},n\rangle$ has the following properties:
(a) $f^{-1}(n)\neq\varnothing$ on each sort of τ_μ by (2),
(b) for any $\beta<\mu$, φ_β holds in $\mathfrak{N}\restriction\tau_\mu|f^{-1}(n)$ by (3),
(c) Γ_μ holds in $\mathfrak{N}\restriction\tau_\mu|f^{-1}(n)$ by (4),
so that $\Sigma_\mu\cup\Gamma_\mu$ has a model, a contradiction.

Claim 5. *Given any model $\mathfrak{N}$ of T_μ and any increasing chain $a_0<a_1<\ldots<a_\varrho<\ldots$ ($\varrho<\nu$) of cardinality $\nu<\mu$, there exists some $\alpha<\mu$ such that m_α is a bound for the chain, i.e.*

$$\mathfrak{N}\models a_\varrho<m_\alpha\quad(\forall\varrho<\nu).$$

Proof. Otherwise, any counterexample $\langle\mathfrak{N},a_\varrho\rangle_{\varrho<\nu}$ has the following properties:
(i) $\forall\alpha\,\exists\varrho$ such that $\mathfrak{N}\models m_\alpha<a_\varrho$, by absurdum hyp.
(ii) $\forall\varrho\,\exists\alpha$ such that $\mathfrak{N}\models a_\varrho<m_\alpha$, by Claim 4.

Letting $m(\varrho)$ be the least among the m_β's satisfying (ii) (such $m(\varrho)$ existing by axiom (1) being in T_μ), we have exhibited a subset $\{m(\varrho)\}_{\varrho<\nu}$ of cardinality $\nu<\mu$ which is cofinal in $\{m_\beta\}_{\beta<\mu}$, by (i), thus contradicting the assumed regularity of μ.

Claim 6. *For $\mu'\neq\mu''$, $T_{\mu'}\cup T_{\mu''}$ is inconsistent.*

Proof. Otherwise, let $\mathfrak{N}\in\mathrm{Str}(\tau_{\mu'}^*\cup\tau_{\mu''}^*)$ be a model of $T_{\mu'}\cup T_{\mu''}$; assume $\mu'<\mu''$; $\mathfrak{N}$ can be written as

$$\mathfrak{N}=\langle N,\ldots,<,m_\alpha',m_\beta''\rangle_{\alpha<\mu',\beta<\mu''};$$

$<$ is a linear ordering over sort s_3 and the m_α''s as well as the m_β'''s are well ordered by axiom (1). First consider that $\mathfrak{N}\models T_{\mu''}$ and that $\{m_\alpha'\}_{\alpha<\mu'}$ is an increasing chain of cardinality $\mu'<\mu''$ so, by Claim 5, $\exists\beta<\mu''$ such that m_β'' bounds the chain; but this contradicts Claim 4, if we now regard $\mathfrak{N}$ as a model of $T_{\mu'}$.

After the proof of Claim 6, observe that for all regular $\mu\geqslant\varkappa$ the type τ_μ^* of T_μ has three sorts s_1, s_2 and s_3, has its relation symbols among set

$$\{<\}\cup\{R_m^a\}_{a,m<\omega}$$

(1) Strictly speaking, τ_μ^* should also be equipped with a binary function k (independent of μ) taking care of the different interpretations of the constants of τ_μ in the components of $\mathfrak{M}_\mu$; however, this would only result in burdening the notation of the rest of our proof, so we prefer to neglect k.

and has just one function symbol f; let τ_μ° be the three-sorted type obtained from τ_μ^* by eliminating all the constant symbols of τ_μ^*; given theory T_μ, let T_μ° be defined by

$$T_\mu^\circ = T_\mu \cap \mathrm{Stc}_L \tau_\mu^\circ$$

so that T_μ° is obtained by taking those sentences of T_μ of type $\tau_\mu^* \cap \{<, f, R_m^a\}_{a,m<\omega}$. T_μ° will be complete and consistent, as so is T_μ, for each regular $\mu \geqslant \varkappa$; hence, by Claim 2 and the pigeon hole principle, at least two theories T_α° and T_β° (α, β regular $\geqslant \varkappa$ with $\alpha \neq \beta$) are logically equivalent, i.e.

$$\tau_\alpha^\circ = \tau_\beta^\circ \text{ (both types over sorts } s_1, s_2, s_3) \quad \text{and} \quad \mathrm{mod}_L T_\alpha^\circ = \mathrm{mod}_L T_\beta^\circ .$$

In particular

$$\mathfrak{M}_\alpha \restriction \tau_\alpha^\circ \equiv_L \mathfrak{M}_\beta \restriction \tau_\beta^\circ ;$$

notice that $\mathfrak{M}_\alpha$ and $\mathfrak{M}_\alpha \restriction \tau_\alpha^\circ$ have the same (three-sorted) universe, since no sort is lost in the reduction process from τ_α^* to τ_α°; the same holds for $\mathfrak{M}_\beta$ and $\mathfrak{M}_\beta \restriction \tau_\beta^\circ$. It is no loss of generality to assume that no two constant symbols of τ_α^* (resp. of τ_β^*) have the same interpretation in $\mathfrak{M}_\alpha$ (resp. in $\mathfrak{M}_\beta$). We also assume, without loss of generality, that τ_α^* and τ_β^* have no common constant symbols.

CLAIM 7. *There is no* $\mathfrak{M}$ *such that*

$$\mathfrak{M}_\alpha \restriction \tau_\alpha^\circ \underset{<}{\rightarrow}_L \mathfrak{M} \quad \text{and} \quad \mathfrak{M}_\beta \restriction \tau_\beta^\circ \underset{<}{\rightarrow}_L \mathfrak{M} .$$

Proof. Let M_α (resp. M_β) be the three-sorted universe of $\mathfrak{M}_\alpha \restriction \tau_\alpha^\circ$ (resp. of $\mathfrak{M}_\beta \restriction \tau_\beta^\circ$); let τ_{M_α} be an expansion of τ_α° formed by adding a new constant for each $m \in M_\alpha$ in such a way that $\mathrm{th}_L(\mathfrak{M}_\alpha \restriction \tau_\alpha^\circ)_{M_\alpha} \supseteq \mathrm{th}_L \mathfrak{M}_\alpha$ $(= T_\alpha)$; such τ_{M_α} exists since the function and relation symbols of type τ_α^* are also in τ_α°, and since M_α is also the universe of $\mathfrak{M}_\alpha$. Let τ_{M_β} similarly expand τ_β° so that $\mathrm{th}_L(\mathfrak{M}_\beta \restriction \tau_\beta^\circ)_{M_\beta}$ has more sentences than T_β. Assume (absurdum hypothesis), that f and g jointly L-embed $\mathfrak{M}_\alpha \restriction \tau_\alpha^\circ$ and $\mathfrak{M}_\beta \restriction \tau_\beta^\circ$ respectively into $\mathfrak{M}$, with $\mathfrak{M} \in \mathrm{Str}(\tau_\alpha^\circ)$ $(= \tau_\beta^\circ)$; by definition

$$\mathfrak{M}_{fM_\alpha} \models \mathrm{th}_L(\mathfrak{M}_\alpha \restriction \tau_\alpha^\circ)_{M_\alpha} \quad \text{and} \quad \mathfrak{M}_{gM_\beta} \models \mathrm{th}_L(\mathfrak{M}_\beta \restriction \tau_\beta^\circ)_{M_\beta} .$$

Assuming, without loss of generality, that τ_{M_α} and τ_{M_β} have no common constant symbols, then

$$\mathfrak{M}_{fM_\alpha \cup gM_\beta} \models T_\alpha \cup T_\beta ,$$

a contradiction with Claim 6.

Claim 7 shows that L has not JEP under the hypothesis of Claim 1 that L is not (ζ, ω)-compact for some $\zeta \geqslant \omega$. This proves the harder direction of the theorem. To prove the other direction, first notice that, under the hypotheses of the theorem, for τ a set there exists a cardinal η such that in L there are no more than η logically inequivalent sentences of type τ (if $\mathrm{Stc}_L(\tau)$ is a set we have nothing to prove; if the Löwenheim number of L exists, then argue as in the proof of Claim 2). Now argue as in [CK 3.1.4]. ■

Remark. If one weakens the assumptions about logic L in such a way that the sort-reducing relativization used in Claim 3 is not allowed in L, then the argu-

ment of the above proof is still true, provided the class of sorts where one has counterexamples Σ_μ, Γ_μ to μ-relative compactness can be assumed to be a fixed set: but this is indeed the case for *any* logic L, as can be ascertained by inspecting the proof of 6.4(ii) in [MS]. Then one constructs a countable fixed set of relation symbols as in Claim 3, and applies the pigeon hole principle, via some suitable variant of Claim 2, after constructing the T_μ's just as in the above proof (see [Mu6] for details).

The proof of Claim 1 in [MS] is based upon a set-theoretical result due to [JK], see [DJK]; Claim 1 is the only step where $V = L$, $\neg 0^\#$ or $\neg L^\mu$ are needed.

COROLLARY. *Under the hypotheses of the above theorem, in logic L Robinson's consistency theorem holds iff Craig's interpolation theorem holds and L is fully compact.*

Proof. It is well known that Robinson's consistency theorem implies JEP, and that in compact logics satisfying our hypotheses Craig's interpolation is equivalent to Robinson's consistency. Now apply the above theorem. ■

To give an algebraic characterization of Craig's interpolation theorem or Robinson's consistency theorem, we need the following stronger notion of amalgamation:

DEFINITION. We say that logic L has the *strong amalgamation property* (for short: *L has* AP$^+$) iff for any structures $\mathfrak{M}, \mathfrak{A}, \mathfrak{B}$ resp. of type τ, τ' and τ'', if $\tau = \tau' \cap \tau''$ and $\mathfrak{M} \underset{<L}{\rightarrow} \mathfrak{A}$, $\mathfrak{M} \underset{<L}{\rightarrow} \mathfrak{B}$, then $\exists \mathfrak{D} \in \mathrm{Str}(\tau' \cup \tau'')$ such that $\mathfrak{A} \underset{<L}{\rightarrow} \mathfrak{D}$ and $\mathfrak{B} \underset{<L}{\rightarrow} \mathfrak{D}$.

Remark. Thus we have a diagram for $\mathfrak{M}, \mathfrak{A}, \mathfrak{B}$ and $\mathfrak{D}$ just as in the definition of AP: however, the latter only takes care of the case $\tau = \tau' = \tau''$. The strong amalgamation property was proved in [Mu4] to characterize elementary equivalence on the class of countable structures of finite type. Further properties are given by the following:

THEOREM. (A) *Let L have* AP$^+$; *then the following are equivalent*:

(i) *L has* JEP;

(ii) (*connectedness*): *for any* $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ *with* $\mathfrak{A} \equiv_L \mathfrak{B}$ *there is a finite path*:

$$\mathfrak{A} = \mathfrak{M}_0 \underset{(1)}{\text{—}} \mathfrak{M}_1 \underset{(2)}{\text{—}} \cdots \underset{(n)}{\text{—}} \mathfrak{M}_n = \mathfrak{B}$$

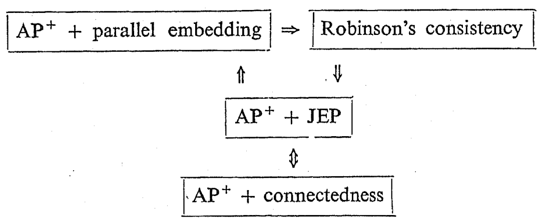
where each $\mathfrak{M}_i$ *is of type* τ *and each* $\underset{(i)}{\text{—}}$ *is either* $\underset{<L}{\rightarrow}$ *or* $\underset{L>}{\leftarrow}$ (*depending on* i);

(iii) (*parallel embedding*): *for any* $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ *with* $\mathfrak{A} \equiv_L \mathfrak{B}$, *if* $\mathfrak{B}^+ \in \mathrm{Str}(\tau^+)$ *with* $\tau^+ \supseteq \tau$ *and* $\mathfrak{B} \underset{<L}{\rightarrow} \mathfrak{B}^+$, *then there exists* $\mathfrak{A}^+ \in \mathrm{Str}(\tau^+)$ *such that* $\mathfrak{A}^+ \equiv_L \mathfrak{B}^+$ *and* $\mathfrak{A} \underset{<L}{\rightarrow} \mathfrak{A}^+$; *in other words we have the following diagram*:

$$\begin{array}{ccc} \mathfrak{A} & \underset{<}{\dashrightarrow}_L & \mathfrak{A}^+ \\ \equiv_L & & \equiv_L \\ \mathfrak{B} & \underset{<}{\longrightarrow}_L & \mathfrak{B}^+ \end{array}$$

(B) *Robinson's consistency theorem holds in* L *iff* L *has* AP^+ *together with either of* (i), (ii), (iii) *above.*

For the proof we shall proceed according to the following diagram:



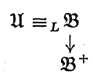
We shall write $\to$ instead of $\underset{<L}{\to}$.

Robinson's consistency $\Rightarrow \mathrm{AP}^+ + \mathrm{JEP}$.

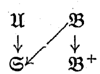
Proof. Immediate.

$\mathrm{AP}^+ + \mathrm{JEP} \Rightarrow \mathrm{AP}^+ +$ *parallel embedding.*

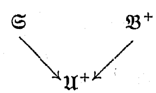
Proof. Let $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$, $\mathfrak{B}^+ \in \mathrm{Str}(\tau^+)$, $\tau^+ \supseteq \tau$ with



By JEP we have, for some $\mathfrak{S} \in \mathrm{Str}(\tau)$:



By AP^+ we have, for some $\mathfrak{A}^+ \in \mathrm{Str}(\tau^+)$:



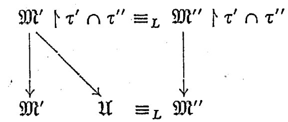
Now $\mathfrak{A}^+ \equiv_L \mathfrak{B}^+$ and $\mathfrak{A} \to \mathfrak{A}^+$ easily follows, which establishes the parallel embedding property for L.

$\mathrm{AP}^+ +$ *parallel embedding* $\Rightarrow$ *Robinson's consistency.*

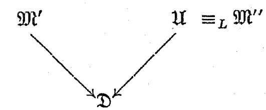
Proof. We first prove the following:

(*) $\forall \mathfrak{M}' \in \mathrm{Str}(\tau')$, $\mathfrak{M}'' \in \mathrm{Str}(\tau'')$, if $\mathfrak{M}' \restriction \tau' \cap \tau'' \equiv_L \mathfrak{M}'' \restriction \tau' \cap \tau''$ then $\exists \mathfrak{D} \in \mathrm{Str}(\tau' \cup \tau'')$ with $\mathfrak{D} \restriction \tau' \equiv_L \mathfrak{M}'$ and $\mathfrak{D} \restriction \tau'' \equiv_L \mathfrak{M}''$.

As a matter of fact, under the hypotheses of (*), by the assumed parallel embedding property we can write, for some $\mathfrak{A} \in \mathrm{Str}(\tau'')$:



By AP^+ we have, for some $\mathfrak{D} \in \mathrm{Str}(\tau' \cup \tau'')$:



Now notice that $\mathfrak{D} \restriction \tau' \equiv_L \mathfrak{M}'$ and $\mathfrak{D} \restriction \tau'' \equiv_L \mathfrak{A} \equiv_L \mathfrak{M}''$ so that (*) is proved. Now, to prove Robinson's consistency, let T, T_1 and T_2 and types τ, τ_1 and τ_2 be as in the definition of Robinson's consistency; let $\mathfrak{M}' \models T_1$ and $\mathfrak{M}'' \models T_2$; then $\mathfrak{M}' \restriction \tau \equiv_L \mathfrak{M}'' \restriction \tau$, since T is complete in $\tau = \tau_1 \cap \tau_2$. By (*) above $\exists \mathfrak{D} \in \mathrm{Str}(\tau_1 \cup \tau_2)$ such that $\mathfrak{D} \restriction \tau_1 \equiv_L \mathfrak{M}'$ and $\mathfrak{D} \restriction \tau_2 \equiv_L \mathfrak{M}''$, so that $\mathfrak{D}$ is a model of $T_1 \cup T_2$, which establishes Robinson's theorem for L.

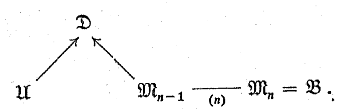
$\mathrm{AP}^+ + \mathrm{JEP} \Leftrightarrow \mathrm{AP}^+ +$ *connectedness.*

Proof. ($\Rightarrow$) is trivial. To prove ($\Leftarrow$), assume $\mathfrak{A}, \mathfrak{B} \in \mathrm{Str}(\tau)$ with $\mathfrak{A} \equiv_L \mathfrak{B}$; by hypothesis we can write

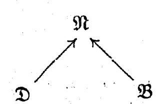
$$\mathfrak{A} = \mathfrak{M}_0 \underset{(1)}{\text{———}} \mathfrak{M}_1 \underset{(2)}{\text{———}} \ldots \underset{(n)}{\text{———}} \mathfrak{M}_n = \mathfrak{B}$$

as in the definition of connectedness, for some $n \in \omega$.

If $n = 1$ then clearly either $\mathfrak{A}$ or $\mathfrak{B}$ jointly embeds $\mathfrak{A}$ and $\mathfrak{B}$. Proceeding by induction on n we can write, for some $\mathfrak{D} \in \mathrm{Str}(\tau)$:



Now, if $\underset{(n)}{\text{—}}$ is $\leftarrow$, then $\mathfrak{D}$ jointly embeds $\mathfrak{A}$ and $\mathfrak{B}$. If $\underset{(n)}{\text{—}}$ is $\to$, then by AP (hence, a fortiori, by AP^+) we can write, for some $\mathfrak{N} \in \mathrm{Str}(\tau)$:



so that $\mathfrak{N}$ jointly embeds $\mathfrak{A}$ and $\mathfrak{B}$ in this case, and JEP holds in any case. ■

Remark. The proof of the first theorem of this paper was found by the author during Spring 1979. Announcements in Notiziario della Unione Matematica Italiana, August–September 1979, N. 8-9, p. 19; and in Atti Accademia Nazionale dei Lincei (Rome), Rendiconti Cl. Sc. Fis. Mat. Nat., Ser. VIII, 67.6 (1979), pp. 383–386. We refer the reader to [MS1], as well as [MS] and [Mu1–6], for further information about Robinson's theorem, amalgamation, JEP and other soft model theoretical properties.

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A note on the isomorphic classification of spaces of continuous functions defined on intervals of ordinal numbers

by

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Abstract. Let ω_1 denote the first uncountable number, and let $\Gamma(\alpha)$ denote the interval of ordinal numbers not exceeding α, endowed with the order topology. For each natural number n an isomorphic classification of the space of continuous functions $C(\Gamma(\omega_1\cdot n))$ is given among the spaces $C(S)$ for which every point of S is either a P-point or a G_δ-point. For $n = 1$, this classification yields a characterization of $\Gamma(\omega_1)$.

Introduction. For each ordinal number α, let $\Gamma(\alpha)$ denote the topological space of non-zero ordinal numbers not exceeding α, endowed with the interval topology (cf. [5], p. 57). Let ω and ω_1 denote the smallest infinite ordinal number and the smallest uncountable ordinal number respectively. As customary, for any compact Hausdorff topological space S, $C(S)$ denotes the supremum-normed Banach space of continuous complex-valued functions defined on S. Two Banach spaces are said to be *isomorphic* provided there is a one-one bounded linear operator from one space onto the other space. A point p in a compact Hausdorff space S is called a *P-point* provided every G_δ-set containing p is a neighborhood of p (cf. [4], p. 63).

In [10], Semadeni showed that the Banach spaces $C(\Gamma(\omega_1\cdot n))$ for $1\leqslant n<\omega$ were mutually non-isomorphic. In this paper, we obtain an extension of this result by giving an isomorphic classification of these spaces among the spaces $C(S)$ for compact Hausdorff topological spaces S in which every point is either a G_δ-point or a P-point. A characterization of $\Gamma(\omega_1)$ in terms of isomorphisms of spaces of continuous functions is also thereby obtained.

Before stating our first result, we need to introduce a few more notions. A topological space is said to be *dispersed* (*scattered*) provided it contains no dense-in-itself non-empty subset (cf. [11], p. 147). Let S be a compact Hausdorff dispersed space and let $m(S)$ denote the Banach space of bounded complex-valued functions on S equipped with the supremum norm. Then, according to a theorem of Rudin [9], the conjugate space of $C(S)$ is isometric to the Banach space $l_1(S) = \{g : g \in m(S)$ and $\sum_s |g(s)| < \infty\}$ equipped with the usual l_1-norm so that the second conjugate