affine space over GF(p) is polynomially equivalent to some medial idempotent quasigroup and the variety of all affine spaces over GF(p) is equationally complete. So, we infer that S_p is equationally complete because of the following fact: if $\mathfrak{A}$ is an algebra of a fixed type τ_1 and if the algebra $\mathfrak{A}$ can also be considered as an algebra of a type τ_2 (with same algebraic operations), then HSP$(\mathfrak{A})$ is equationally complete with respect to τ_1 if and only if it is equationally complete with respect to τ_2.

It follows from [1] that for different primes p and q the varieties S_p and S_q are different atoms in the lattice of subvarieties of all idempotent medial quasigroups.

Now we are in a position to complete the proof of the theorem. Suppose $\mathfrak{M}_n$ is equationally complete and suppose that $M_n = 2^n-1$ is not prime. Then there exist two different primes p and q such that $p|2^n-1$ and $q|2^n-1$. By Lemma 1 we infer that $G(p,n)$ and $G(q,n)$ belong to the variety $\mathfrak{M}_n$. Therefore the varieties S_p and S_q are contained as non-zero subvarieties in $\mathfrak{M}_n$, which contradicts the fact that $\mathfrak{M}_n$ is equationally complete.

Assume now that M_n is prime. To prove that $\mathfrak{M}_n$ is equationally complete it is enough to show $\mathfrak{M}_n = \mathrm{HSP}((G;\cdot))$ for every nontrivial groupoid $(G;\cdot)$ from $\mathfrak{M}_n$.

Let $(G;\cdot)\in\mathfrak{M}_n$. Then by Lemma 2 there exists an abelian group $(G;+)$ of exponent $d|2^n-1$, where $d>1$ and

$$(G;xy) = \left(G;\frac{d+1}{2}(x+y)\right).$$

Since 2^n-1 is prime, we have $d = 2^n-1$ and hence

$$\mathrm{HSP}((G;\cdot)) = \mathrm{HSP}((G;2^{n-1}(x+y))).$$

The latter variety is equal to the variety S_{2^n-1} since the sets of identities of the groupoid $(G;2^{n-1}(x+y))$ and $(\{o,\ldots,2^n-2\};\ 2^{n-1}(x+y))$ are equal (the latter groupoid is polynomially equivalent to the affine space over GF(2^n-1)). By Lemma 1 we find that $S_{2^n-1}\subset\mathfrak{M}_n$ and $S_{2^n-1} = \mathrm{HSP}((G;\cdot))$ for all $(G;\cdot)\in\mathfrak{M}_n$ with card $G\geqslant 2$. Using the well-known Birkhoff theorem, we infer that $\mathfrak{M}_n = S_{2^n-1}$ and hence $\mathfrak{M}_n$ is equationally complete.

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Solution of a problem of Ulam on countable sequences of sets

by

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Abstract. Let E be a set of cardinality 2^ω and $\{A_n\colon n\in\omega\}$ an arbitrary sequence of subsets of E. Let $\mathscr{B}$ denote the σ-algebra of subsets of E generated by the family $\{A_n\colon n\in\omega\}$ and $\mathscr{B}^*$ the σ-algebra of subsets of E^2 generated by the family $\{A_n\times A_m\colon n,m\in\omega\}$. S. M. Ulam stated a problem (see [3]), whether there exists an injection $\Phi\colon E\to E^2$ transforming $\mathscr{B}$ into $\mathscr{B}^*$ and conversely.

We give a negative answer to this question and formulate a condition on $\{A_n\colon n\in\omega\}$ under which the answer is positive.

§ 0. We use standard set theoretical notation and terminology.

By E we always denote a set of cardinality 2^ω. If $A\subset E$ then we put $A^1 = A$, $A^0 = E\setminus A$. If $\mathscr{A} = \{A_n\colon n\in\omega\}$ is a sequence of subsets of E then the function $\varphi_{\mathscr{A}}\colon E\to 2^\omega$ such that $\varphi_{\mathscr{A}}(x)(n) = 1 \equiv x\in A_n$ is called the *characteristic function of* $\mathscr{A}$. For every $f\in 2^\omega$ the set $\mathscr{A}(f) = \varphi_{\mathscr{A}}^{-1}*\{f\} = \bigcap_n A_n^{f(n)}$ is called a *component of* $\mathscr{A}$ and f the *index of* $\mathscr{A}(f_i)$. If $e\in E$ then $S(e)$ denotes the component containing e. Clearly the components are pairwise disjoint and their union is E. Conversely, every pairwise disjoint family of cardinality 2^ω with union E is the set of components of some sequence $\mathscr{A}$.

We define generalized Borel classes over $\mathscr{A}$:

$$\Sigma_1^0(\mathscr{A}) = \{\bigcup X\colon X\subset\mathscr{A}\},$$

$$\Sigma_\xi^0(\mathscr{A}) = \{\bigcup X\colon |X|\leqslant\omega,\ X\subset\bigcup_{\eta<\xi}(\Sigma_\eta^0(\mathscr{A})\cup\Pi_\eta^0(\mathscr{A}))\},$$

$$\Pi_\xi^0(\mathscr{A}) = \{E\setminus X\colon X\in\Sigma_\xi^0(\mathscr{A})\},$$

$$\mathscr{B}(\mathscr{A}) = \bigcup_{\xi<\omega_1}(\Sigma_\xi^0(\mathscr{A})\cup\Pi_\xi^0(\mathscr{A})).$$

$\mathscr{B}(\mathscr{A})$ is the σ-algebra generated by $\mathscr{A}$. If $\mathscr{B}_1$ is a σ-algebra of subsets of E_1 and $\mathscr{B}_2$ a σ-algebra of subsets of E_2 then a function $\Phi\colon E_1\to E_2$ is called $(\mathscr{B}_1,\mathscr{B}_2)$-*preserving* iff $B\in\mathscr{B}_1 \Rightarrow \Phi*(B)\in\mathscr{B}_2$ and $B\in\mathscr{B}_2 \Rightarrow \Phi^{-1}*(B)\in\mathscr{B}_1$. In case when E_1 and E_2 are subsets of 2^ω and $\mathscr{B}_i$ is the family of Borel subsets of E_i $(i = 1,2)$, we say that Φ is Borel preserving instead of saying $(\mathscr{B}_1,\mathscr{B}_2)$-preserving.

It is well known (cf. [1]) that if E_1 and E_2 are uncountable Borel sets then there exists a Borel preserving bijection $\Phi\colon E_1 \to E_2$. E. Szpilrajn proved in [2] that the function $\varphi_{\mathscr{A}}\colon E \to \mathrm{Rg}(\varphi_{\mathscr{A}})$ is $(\mathscr{B}(\mathscr{A}), \mathrm{Bor})$-preserving, where Bor denotes the family of Borel subsets of $\mathrm{Rg}(\varphi_{\mathscr{A}})$.

Throughout this paper we fix a pairing function J for natural numbers: $J\colon \omega\times\omega \xrightarrow[\text{onto}]{1-1} \omega$ and functions K, L such that $K\colon \omega \to \omega$, $L\colon \omega \to \omega$ and $\forall n \in \omega$ $J(K(n), L(n)) = n$. $\mathbf{0}$ denotes the function $f \in 2^\omega$ constantly equal 0. Given a sequence $\mathscr{A} = \{A_n\colon n \in \omega\}$ of subsets of E we define a sequence $\mathscr{A}^* = \{A_n^*\colon n \in \omega\}$ of subsets of E^2, $A_n^* = A_{K(n)} \times A_{L(n)}$, for $n \in \omega$. A function $\Phi\colon E \to E^2$ is called *preserving for* $\mathscr{A}$ iff it is $(\mathscr{B}(\mathscr{A}), \mathscr{B}(\mathscr{A}^*))$-preserving.

In this terminology we can formulate Ulam's problem as follows: Does there exist a preserving injection for every sequence $\mathscr{A}$ of subsets of E?

§ 1. First we prove some technical lemmas.

LEMMA 1.1. *Let $\mathscr{A}$ be a sequence of subsets of E. If $B \in \mathscr{B}(\mathscr{A})$ and $e \in B$ then $S(e) \subset B$.*

Proof. By induction on the hierarchy $\Sigma_\xi^0(\mathscr{A})$, $\Pi_\xi^0(\mathscr{A})$. Let $B \in \Sigma_1^0(\mathscr{A})$. In this case

$$B = \bigcup_{n\in\omega} A_{i_n} = \bigcup_{n\in\omega} \bigcup \{\mathscr{A}(f)\colon f \in 2^\omega \,\&\, f(i_n) = 1\}$$

and the conclusion follows. Assume that the lemma is true for

$$B \in \bigcup \{\Sigma_\eta^0(\mathscr{A}) \cup \Pi_\eta^0(\mathscr{A})\colon \eta < \xi\}.$$

Take $B \in \Sigma_\xi^0(\mathscr{A})$, $B = \bigcup X$, $X \subset \bigcup_{\eta<\xi} \{\Sigma_\eta^0(\mathscr{A}) \cup \Pi_\eta^0(\mathscr{A})\}$. Let $e \in B$, hence $e \in B_1$ for some $B_1 \in X$. It follows from the inductive hypothesis that $S(e) \subset B_1 \subset B$.

Assume finally that the lemma is true for $B \in \Sigma_\xi^0(\mathscr{A})$. Take $B_1 \in \Pi_\xi^0(\mathscr{A})$, $B_1 = E \setminus B$. Let $e \in B_1$. If $S(e) \subset B_1$ then we are done. If not, there exists $e' \in S(e)$ s.t. $e' \in B$, but in this case $S(e) = S(e') \subset B$, hence $e \in B$, contradiction. ∎

Remark 1.2. Notice that Lemma 1.1 remains true for $\varkappa$-complete algebras generated by $\mathscr{A}$, where $\varkappa$ is arbitrary.

Let E_1 and E_2 have cardinality 2^ω and $\mathscr{A}_i$ be a sequence of subsets of E_i, $(i = 1, 2)$.

LEMMA 1.3. *The following pairs of statements are equivalent:*

a) (i) *There exists a $(\mathscr{B}(\mathscr{A}_1), \mathscr{B}(\mathscr{A}_2))$-preserving function $\Phi\colon E_1 \to E_2$.*
(ii) *There exists a Borel preserving function $\Psi\colon \mathrm{Rg}(\varphi_{\mathscr{A}_1}) \to \mathrm{Rg}(\varphi_{\mathscr{A}_2})$ such that $|\mathscr{A}_2(\Psi(f))| \leqslant |\mathscr{A}_1(f)|$ for every $f \in \mathrm{Rg}(\varphi_{\mathscr{A}_1})$.*

b) (i′) *There exists a $(\mathscr{B}(\mathscr{A}_1), \mathscr{B}(\mathscr{A}_2))$-preserving injection $\Phi\colon E_1 \to E_2$.*
(ii′) *There exists a Borel preserving injection $\Psi\colon \mathrm{Rg}(\varphi_{\mathscr{A}_1}) \to \mathrm{Rg}(\varphi_{\mathscr{A}_2})$ such that $|\mathscr{A}_2(\Psi(f))| = |\mathscr{A}_1(f)|$ for every $f \in \mathrm{Rg}(\varphi_{\mathscr{A}_1})$.*

c) (i″) *There exists a $(\mathscr{B}(\mathscr{A}_1), \mathscr{B}(\mathscr{A}_2))$-preserving bijection $\Phi\colon E_1 \to E_2$.*
(ii″) *There exists a Borel preserving bijection $\Psi\colon \mathrm{Rg}(\varphi_{\mathscr{A}_1}) \to \mathrm{Rg}(\varphi_{\mathscr{A}_2})$ such that $|\mathscr{A}_2(\Psi(f))| = |\mathscr{A}_1(f)|$ for every $f \in \mathrm{Rg}(\varphi_{\mathscr{A}_1})$.*

Proof. We prove only part a). The proofs of part b) and c) are similar.

(i) → (ii). It follows from Lemma 1.1 that images of components are components. Let $\Psi\colon \mathrm{Rg}(\varphi_{\mathscr{A}_1}) \to \mathrm{Rg}(\varphi_{\mathscr{A}_2})$ be such that $\Phi * (\mathscr{A}_1(f)) = \mathscr{A}_2(\Psi(f))$. We show that Ψ is Borel preserving. Let B be a Borel subset of $\mathrm{Rg}(\varphi_{\mathscr{A}_1})$. By Szpilrajn's theorem $\varphi_{\mathscr{A}_1}^{-1} * (B) \in \mathscr{B}(\mathscr{A}_1)$, hence $\Phi * (\varphi_{\mathscr{A}_1}^{-1} * (B)) \in \mathscr{B}(\mathscr{A}_2)$ and

$$\Psi * (B) = \varphi_{\mathscr{A}_2} * \big(\Phi * (\varphi_{\mathscr{A}_1}^{-1} * (B))\big)$$

is a Borel subset of $\mathrm{Rg}(\varphi_{\mathscr{A}_2})$ again by Szpilrajn's theorem. Similarly if B is Borel in $\mathrm{Rg}(\varphi_{\mathscr{A}_2})$ then $\varphi_{\mathscr{A}_1} * \big(\Phi^{-1} * (\varphi_{\mathscr{A}_2}^{-1} * (B))\big) = \Psi^{-1} * (B)$ is Borel in $\mathrm{Rg}(\varphi_{\mathscr{A}_1})$.

(ii) → (i). The family $\{\mathscr{A}_i(f)\colon f \in \mathrm{Rg}(\varphi_{\mathscr{A}_i})\}$ is a disjoint partition of E_i (for $i = 1, 2$). Let $\Phi_f\colon \mathscr{A}_1(f) \xrightarrow{\text{onto}} \mathscr{A}_2(\Psi(f))$. The existence of such a function follows from $|\mathscr{A}_2(\Psi(f))| \leqslant |\mathscr{A}_1(f)|$. Put $\Phi = \bigcup \{\Phi_f\colon f \in \mathrm{Rg}(\varphi_{\mathscr{A}_1})\}$. Observe that $\Phi\colon E_1 \to E_2$ and is $(\mathscr{B}(\mathscr{A}_1), \mathscr{B}(\mathscr{A}_2))$-preserving. ∎

LEMMA 1.4. *Let $\mathscr{A} = \{A_n\colon n \in \omega\}$ be a sequence of subsets of E. For an arbitrary component $S = \mathscr{A}^*(f)$ of $\mathscr{A}^*$ and $e = \langle x, y\rangle \in S$ the following holds:*

a) *if $f = \mathbf{0}$ then $S = \{\langle z, t\rangle \in E^2\colon \varphi_{\mathscr{A}}(z) = \mathbf{0}$ or $\varphi_{\mathscr{A}}(t) = \mathbf{0}\}$,*

b) *if $f \neq \mathbf{0}$ then $S = S(x) \times S(y)$.*

Proof. a) Take $\langle z, t\rangle$ such that $\varphi_{\mathscr{A}}(z) = \mathbf{0}$ or $\varphi_{\mathscr{A}}(t) = \mathbf{0}$. Assume e.g. $\varphi_{\mathscr{A}}(z) = \mathbf{0}$. The proof in the other case is similar. We have $z \notin A_n$ for all $n \in \omega$ and $\langle z, t\rangle \notin A_n \times A_m$ for all $t \in E$, $m \in \omega$, $n \in \omega$. It follows that $\langle z, t\rangle \in S$ and one inclusion is proved. Take $\langle z, t\rangle$ such that $\varphi_{\mathscr{A}}(z) \neq \mathbf{0}$ and $\varphi_{\mathscr{A}}(t) \neq \mathbf{0}$. There exist $n, m \in \omega$ such that $z \in A_n$, $t \in A_m$, hence $\langle z, t\rangle \in A_n \times A_m = A_{J(n,m)}^*$. In case $\langle z, t\rangle \in S$ we would have $f(J(n, m)) = 1$, contradiction with $f = \mathbf{0}$. This proves the equality.

b) First we prove $S \subset S(x) \times S(y)$. Take n such that $f(n) = 1$, thus $x \in A_{K(n)}$, $y \in A_{L(n)}$. Let $\langle z, t\rangle \in S$ and assume $\langle z, t\rangle \notin S(x) \times S(y)$, e.g. $z \notin S(x)$. We have $\langle z, t\rangle \in A_{K(n)} \times A_{L(n)}$. If there exists $m \in \omega$ s.t. $z \in A_m$ and $x \notin A_m$ then, in view of $z \in A_{K(n)}$, $t \in A_{L(n)}$ we would have $\langle z, t\rangle \in A_m \times A_{L(n)}$ and $\langle x, y\rangle \notin A_m \times A_{L(n)}$. Contradiction because $\langle x, y\rangle$ and $\langle z, t\rangle$ are in the same component. In case $z \notin A_m$, $x \in A_m$ we argue similarly. Thus the inclusion is proved.

For the proof of $S(x) \times S(y) \subset S$ take $\langle z, t\rangle \in S(x) \times S(y)$. For all $m, n \in \omega$ we have $\langle z, t\rangle \in A_m \times A_n \equiv (z \in A_m \,\&\, t \in A_n) \equiv (x \in A_m \,\&\, y \in A_n) \equiv \langle x, y\rangle \in A_m \times A_n$, hence $\langle z, t\rangle \in S$ and we are done. ∎

§ 2. The following theorem gives a negative answer to the problem of Ulam.

THEOREM 2.1. *Let $k > 1$ be a natural number and $\mathscr{A}$ a sequence of subsets of E. Assume that for every $f \in 2^\omega |\mathscr{A}(f)| = 0$ or $k \leqslant |\mathscr{A}(f)| < k^2$. Then there does not exist a function $\Phi\colon E \to E^2$ preserving for $\mathscr{A}$.*

Proof. If follows from Lemma 1.4 that all non-void components of $\mathscr{A}^*$ have cardinalities $\geqslant k^2$. The conclusion follows now from Lemma 1.3 part a). ∎

Remark 2.2. Let $E = 2^\omega$ and $A_n = \{f \in 2^\omega\colon f(n) = 1\}$. Then clearly the set $\{A_n\colon n \in \omega\}$ generates the σ-algebra of Borel subsets of 2^ω. However

$$\{A_n \times A_m\colon n, m \in \omega\}$$

does not generate all Borel subsets of $2^\omega \times 2^\omega$. Namely it does not generate the singleton $\{\langle \mathbf{0}, \mathbf{0}\rangle\}$, because in view of Lemma 1.4 its component has cardinality 2^ω.

The next result shows that for a wide class of sequences the answer to Ulam's problem is positive.

THEOREM 2.3. *Let $\mathscr{A} = \{A_n: n \in \omega\}$ be a sequence of subsets of E. Assume that every non-void component of $\mathscr{A}$ is either an infinite or one element set. If $\mathscr{A}(\mathbf{0})$ is non-void assume that there exist $f_1 \neq f_2$, $f_i \neq \mathbf{0}$ $(i = 1, 2)$ such that $|\mathscr{A}(f_1)| \leqslant |\mathscr{A}(\mathbf{0})|$, $|\mathscr{A}(f_2)| = |\mathscr{A}(\mathbf{0})|$. Then there exists a preserving injection $\Phi\colon E \to E^2$ for $\mathscr{A}$.*

Proof. Consider the following function $\Psi\colon 2^\omega \to 2^\omega$.

$$\Psi(f)(n) = 1 \equiv f(K(n)) = 1 \,\&\, f(L(n)) = 1\,,$$

for all $f \in 2^\omega$, $n \in \omega$. Clearly Ψ is continuous and one-to-one, hence it is a Borel preserving injection.

First we prove that $\Psi * (\mathrm{Rg}(\varphi_{\mathscr{A}})) \subset \mathrm{Rg}(\varphi_{\mathscr{A}} *)$. Take $f \in \mathrm{Rg}(\varphi_{\mathscr{A}})$ and let $x \in \mathscr{A}(f)$. Hence $\langle x, x\rangle \in \mathscr{A}^*(\Psi(f))$ and thus $\Psi(f) \in \mathrm{Rg}(\varphi_{\mathscr{A}} *)$, which proves the inclusion.

Next we prove $\Psi^{-1} * (\mathrm{Rg}(\varphi_{\mathscr{A}} *)) = \mathrm{Rg}(\varphi_{\mathscr{A}})$. "$\supset$" follows from the above. To show "$\subset$" take $f \in 2^\omega$ such that $\Psi(f) \in \mathrm{Rg}(\varphi_{\mathscr{A}^*})$. Let $\langle x, y\rangle \in \mathscr{A}^*(\Psi(f))$. Consider two cases:

1⁰ $\Psi(f) = \mathbf{0}$. Then $f = \mathbf{0}$ and either $\varphi_{\mathscr{A}}(x) = \mathbf{0}$ or $\varphi_{\mathscr{A}}(y) = \mathbf{0}$. Hence $\mathbf{0} \in \mathrm{Rg}(\varphi_{\mathscr{A}})$ and we are done.

2⁰ $\Psi(f) \neq \mathbf{0}$. Let $\Psi(f)(n) = 1$, hence $\langle x, y\rangle \in A_{K(n)} \times A_{L(n)}$. Let $\varphi_{\mathscr{A}}(x) = g$, $\varphi_{\mathscr{A}}(y) = h$. We show that $f = g = h$. Indeed $g(K(n)) = 1$ and $h(L(n)) = 1$. On the other hand $f(K(n)) = 1$ and $f(L(n)) = 1$. For every $m \in \omega$ we have $f(K(n)) = 1$ $\& f(m) = 1 \equiv \Psi(f)(J(K(n), m)) = 1 \equiv \langle x, y\rangle \in A_{K(n)} \times A_m \equiv g(K(n)) = 1 \,\&\, h(m)$ $= 1$. Hence $f(m) = 1 \equiv h(m) = 1$ and we get $f = h$. (Similarly $f = g$). It follows that $f \in \mathrm{Rg}(\varphi_{\mathscr{A}})$. We have proved $\Psi^*(\mathrm{Rg}(\varphi_{\mathscr{A}})) \subset \mathrm{Rg}(\varphi_{\mathscr{A}^*})$ and

$$\Psi^{-1} * (\mathrm{Rg}(\varphi_{\mathscr{A}^*})) = \mathrm{Rg}(\varphi_{\mathscr{A}})\,.$$

Now it is easy to show that

$$\overline{\Psi} = \Psi \restriction \mathrm{Rg}(\varphi_{\mathscr{A}})\colon \mathrm{Rg}(\varphi_{\mathscr{A}}) \to \mathrm{Rg}(\varphi_{\mathscr{A}^*})$$

is a Borel preserving injection. This is not obvious because in general

$$\mathrm{Rg}(\varphi_{\mathscr{A}^*}) \neq \Psi * (\mathrm{Rg}(\varphi_{\mathscr{A}}))\,.$$

Take an arbitrary Borel subset B of 2^ω.

$$\overline{\Psi}^{-1} * (B \cap \mathrm{Rg}(\varphi_{\mathscr{A}^*})) = \Psi^{-1} * (B) \cap \Psi^{-1} * (\mathrm{Rg}(\varphi_{\mathscr{A}^*})) = \Psi^{-1} * (B) \cap \mathrm{Rg}(\varphi_{\mathscr{A}})$$

is Borel in $\mathrm{Rg}(\varphi_{\mathscr{A}})$.

$$\overline{\Psi} * (B \cap \mathrm{Rg}(\varphi_{\mathscr{A}})) = \Psi * (B) \cap \Psi * (\mathrm{Rg}(\varphi_{\mathscr{A}})) = \Psi * (B) \cap \mathrm{Rg}(\varphi_{\mathscr{A}^*})$$

is Borel in $\mathrm{Rg}(\varphi_{\mathscr{A}^*})$. The last equality follows from

$$\Psi * (2^\omega) \cap \big(\mathrm{Rg}(\varphi_{\mathscr{A}^*}) \setminus \Psi * (\mathrm{Rg}(\varphi_{\mathscr{A}}))\big) = \emptyset\,.$$

Next take f_1 and f_2 as in the assumptions (in case when $\mathscr{A}(\mathbf{0}) \neq \emptyset$) and let $\tilde{f} = \varphi_{\mathscr{A}^*} * (\varphi_{\mathscr{A}}^{-1} * \{f_1\} \times \varphi_{\mathscr{A}}^{-1} * \{f_2\})$. Define $\tilde{\Psi}\colon \mathrm{Rg}(\varphi_{\mathscr{A}}) \to \mathrm{Rg}(\varphi_{\mathscr{A}^*})$:

$$\tilde{\Psi}(f) = \begin{cases} \overline{\Psi}(f) & \text{if } f \neq \mathbf{0}\,, \\ \tilde{f} & \text{if } f = \mathbf{0}\,. \end{cases}$$

Clearly $\tilde{\Psi}$ is Borel preserving since $\overline{\Psi}$ was. It is an injection. Moreover it follows from Lemma 1.4 and the assumptions that

$$|\mathscr{A}(f)| = |\mathscr{A}^*(\tilde{\Psi}(f))|\,.$$

Indeed if $f \neq \mathbf{0}$ we have

$$\mathscr{A}^*(\tilde{\Psi}(f)) = \mathscr{A}(f) \times \mathscr{A}(f)$$

and if $f = \mathbf{0}$ we have

$$\mathscr{A}^*(\tilde{\Psi}(f)) = \mathscr{A}(f_1) \times \mathscr{A}(f_2)\,.$$

The conclusion follows now from Lemma 1.3 part b). ∎

THEOREM 2.4. *Let $\mathscr{A} = \{A_n: n \in \omega\}$ be a sequence of subsets of E. Assume that $\mathrm{Rg}(\varphi_{\mathscr{A}})$ is an infinite Borel set and all non-void components of $\mathscr{A}$ have cardinality 2^ω. Then there exists a preserving bijection for $\mathscr{A}$.*

Proof. It follows from the assumptions that for every $f \in \mathrm{Rg}(\varphi_{\mathscr{A}})$, $g \in \mathrm{Rg}(\varphi_{\mathscr{A}^*})$ we have $|\mathscr{A}(f)| = |\mathscr{A}^*(g)|$. Hence in view of Lemma 1.3 part c) it suffices to prove the existence of a Borel preserving bijection $\Psi\colon \mathrm{Rg}(\varphi_{\mathscr{A}}) \to \mathrm{Rg}(\varphi_{\mathscr{A}^*})$. Clearly $|\mathrm{Rg}(\varphi_{\mathscr{A}})| = |\mathrm{Rg}(\varphi_{\mathscr{A}^*})|$ (easily by Lemma 1.4, because $\mathrm{Rg}(\varphi_{\mathscr{A}})$ is infinite) thus we are done if we prove that $\mathrm{Rg}(\varphi_{\mathscr{A}^*})$ is Borel. Let

$$\mathscr{F} = \Big\{f \in 2^\omega\colon \forall m, n \in \omega\big[f(m) = 1 \,\&\, f(n) = 1 \Rightarrow f\big(J(K(m), L(n))\big) = 1 \,\& \\ \& f\big(J(K(n), L(m))\big) = 1\big]\Big\}\,,$$

$$G = \Big\{f \in 2^\omega\colon \exists g \exists h\big[g \in \mathrm{Rg}(\varphi_{\mathscr{A}}) \,\&\, h \in \mathrm{Rg}(\varphi_{\mathscr{A}}) \,\& \\ \& \forall m \in \omega\big[\big(g(m) = 1 \equiv \exists k f(J(m, k)) = 1\big) \,\& \\ \big(h(m) = 1 \equiv \exists l f(J(l, m)) = 1\big)\big]\big]\Big\}\,.$$

$\mathscr{F}$ is a closed set and if $\mathrm{Rg}(\varphi_{\mathscr{A}})$ is Borel then G is Borel as well because another definition of G is

$$G = \Big\{f \in 2^\omega\colon \forall g \forall h\big[\forall m \in \omega\big[\big(g(m) = 1 \equiv \exists k f(J(m, k)) = 1\big) \,\& \\ \& \big(h(m) = 1 \equiv \exists l f(J(l, m)) = 1\big)\big]\big] \Rightarrow g \in \mathrm{Rg}(\varphi_{\mathscr{A}}) \,\&\, h \in \mathrm{Rg}(\varphi_{\mathscr{A}})\Big\}\,.$$

hence G is Δ_1^1 with a Borel parameter. Thus it suffices to prove that $\mathrm{Rg}(\varphi_{\mathscr{A}^*}) = \mathscr{F} \cap G$. Let $f \in \mathrm{Rg}(\varphi_{\mathscr{A}^*})$ and $\langle x, y\rangle \in \mathscr{A}^*(f)$. If $f(m) = f(n) = 1$ then $\langle x, y\rangle \in A_{K(m)} \times A_{L(m)}$ and $\langle x, y\rangle \in A_{K(n)} \times A_{L(n)}$ hence obviously

$$\langle x, y\rangle \in A_{K(m)} \times A_{L(m)}$$

and $\langle x, y\rangle \in A_{K(n)} \times A_{L(m)}$ which proves $f \in \mathscr{F}$. Now we prove that $f \in G$. If $f \in \mathbf{0}$ then in view of $\mathscr{A}^*(f) \neq \emptyset$ we have $\mathscr{A}(f) \neq \emptyset$, hence $f \in \mathrm{Rg}(\varphi_{\mathscr{A}})$ and we can take $g = h = f$. Assume $\exists n f(n) = 1$. Take g and h such that $x \in \mathscr{A}(g)$, $y \in \mathscr{A}(h)$. Let $g(m) = 0$, hence $x \notin A_m$ and for every k $\langle x, y\rangle \notin A_m \times A_k$. We infer that $f(J(m, k)) = 0$. Let $g(m) = 1$, hence $x \in A_m$. On the other hand we have $\langle x, y\rangle \in A_{K(n)} \times A_{L(n)}$, hence $y \in A_{L(n)}$ and we get $\langle x, y\rangle \in A_m \times A_{L(n)}$, so

$$f(J(m, L(n))) = 1 .$$

For h we argue similarly and thus one inclusion is proved. Now take $f \in \mathscr{F} \cap G$ and appropriate functions $g, h \in \mathrm{Rg}(\varphi_{\mathscr{A}})$. We claim that if $x \in \mathscr{A}(g)$, $y \in \mathscr{A}(h)$ then $\langle x, y\rangle \in \mathscr{A}^*(f)$ and hence $f \in \mathrm{Rg}(\varphi_{\mathscr{A}^*})$. Indeed for every $m \in \omega$ we have:

$$\begin{aligned}\langle x, y\rangle \in A_m^* &\equiv \langle x, y\rangle \in A_{K(m)} \times A_{L(m)} \equiv x \in A_{K(m)} \,\&\, y \in A_{L(m)} \\ &\equiv g(K(m)) = 1 \,\&\, h(L(m)) = 1 \\ &\equiv \exists k f(J(K(m), k)) = 1 \,\&\, \exists l f(J(l, L(m))) = 1 \\ &\equiv f(J(K(m), L(m))) = 1 \equiv f(m) = 1.\end{aligned}$$

This proves that $\mathrm{Rg}(\varphi_{\mathscr{A}^*}) = \mathscr{F} \cap G$ and finishes the proof of our theorem. ∎

It would be interesting to find a necessary and sufficient condition for the existence of a preserving injection for a sequence in terms of its components. We would like to state this problem as a natural remainder of Ulam's question.

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Monotone decompositions of hereditarily smooth continua

by

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Abstract. It is proved that if a Hausdorff compact continuum X is hereditarily smooth at a certain point (see below), then there is an upper semi-continuous decomposition $\mathfrak{D}$ of X into continua such that the quotient space $X/\mathfrak{D}$ is arcwise connected and hereditarily smooth and $\mathfrak{D}$ is minimal with respect to these properties. This result generalizes theorems obtained by Gordh [3] and by Maćkowiak [6].

1. Introduction. A *continuum* is a compact connected Hausdorff space. A continuum I is *irreducible between its points a and b* if no proper subcontinuum of I contains them. The symbol $I(a, b)$ always denotes a continuum irreducible between a and b. We use the following notation: $\mathrm{cl}\,A$ ($\mathrm{int}\,A$) denotes the closure (the interior) of A. A continuum X is *smooth at a point p* [4], [7] provided that for each subcontinuum K of X such that $p \in K$ and for each open set V which includes K, there is an open connected set U such that $K \subset U \subset V$. The following is well known [7].

PROPOSITION 1. *Let p be a point of a continuum X. Then the following conditions are equivalent:*

(i) *X is smooth at p,*

(ii) *for each convergent net $x_n \in X$ with* $\lim x_n = x$ *and for each continuum $I(p, x)$ irreducible between p and x there are continua $I(p, x_n)$ each one irreducible between p and x_n such that* $\mathrm{Lim}\, I(p, x_n) = I(p, x)$,

(iii) *for each subcontinuum K of X containing p and for each convergent net $\{x_n,\ n \in D\}$ with* $\lim x_n = x \in K$ *there is a net $\{K_i,\ i \in E\}$ of subcontinua of X such that each K_i contains a certain x_n and p and* $\mathrm{Lim} K_i = K$ *(if K is irreducible, then it is possible to have each K_i irreducible also).*

A continuum X is *hereditarily unicoherent at a point p* [3] if the intersection of any two subcontinua of X, each of which contains p, is connected. Any Hausdorff compactification αJ of the set J consisting of the interval $[0, 1)$ of reals and of a circle S such that $[0, 1) \cap S = \{0\}$ is a continuum which is smooth at each point of J but not hereditarily unicoherent at any point of $\alpha J \setminus J$. A continuum X is *hereditarily smooth at a point p* if each subcontinuum of X containing p is smooth at p.