Boolean spectra and model completions

by

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Abstract. By combining Cole's theory of spectra and localizations with Comer's version of the Feferman–Vaught theorem, we give a new version of Macintyre's theorem on the existence of model completions of certain theories of (von Neuman) regular rings. The proof is topos-theoretic and in it, the two ingredients of a regular ring, regarded as a field in a topos of sheaves on a Boolean algebra, are clearly separated. Results of Macintyre, Carson, and Weispfenning, are immediately recovered.

Introduction. In this paper, we use topos theoretic methods to give a new version of Macintyre's theorem on the existence of model completions of certain theories of (von Neumann) regular rings (cf. [10] for historical remarks and further information). Since several versions now exist, we shall point out some specific features of ours (considered desirable by Macintyre in loc. cit.).

We take the point of view, common to workers in topos theory, that the basic notions of the theory are those arising from the internal logic of a topos, rather than those tied down to the set-theoretical scaffold such as stalks, points, etc. (used in previous proofs). In particular, ours is a stalk-free approach. Our framework is the theory of spectra and localizations (cf. § 1), framework which allows us to state those features of the theories of ring representations used in previous proofs, (such as Pierce's [12] or Keimel's [7]), as axioms on morphisms arising from spectra. As a consequence, it is possible to isolate the two ingredients of, say, a regular ring (viewed as a field in a topos of sheaves over a Boolean algebra) rather neatly and follow their contributions to the truth of the main theorem. This theorem, as well as a few related results, is proved in § 2 by using Robinson's test on model-completeness (as in [8]). This possibility arises thanks to the functoriality of ring representations (built in our spectra). The missing link between truth in a sheaf (on a Boolean algebra) and truth in its global sections is provided by Comer's version of the Feferman–Vaught theorem (cf. [2]) rather than Carson's lemma used in previous proofs.

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While the second author has profited from discussions with André Joyal (partly contained in [6]) the first acknowledges the influence of George Loullis through conversations as well as her reading of a manuscript of his, based on his Ph. D. thesis at Yale and completed shortly before his untimely death in the fall 1978. We are also grateful to M. Barr, M. Coste, P. Johnstone, M. Makkai, R. Seely and H. Volger for useful comments on the preprint mentioned above.

Our standard references will be [5] for topos theory and [11] for categorical logic.

We dedicate this work to the memory of George Loullis.

§ 1. Spectra and Boolean representation theories. Our framework, as mentioned in the Introduction, is the theory of spectra or localizations (cf. [1], [3], [4]) for a quotient $T_0 \to T$ of geometric theories with an admissible class A of morphisms of models of T. Only Grothendieck toposes will be considered and we shall assume the following condition throughout:

(∗) If Sets $\underset{\Delta}{\overset{\Gamma}{\leftrightarrows}} \mathscr{E}$ is a topos and $\tilde{M}$ is an $\mathscr{E}$-model of T, then $\Gamma\tilde{M}$ is a model of T_0.

We let Spec: T_0–Top $\to A$-Top be the right adjoint to the inclusion functor. Let us recall, that, whenever M is a set-theoretical model of T_0, Spec(Sets, M) $= (\mathscr{E}_M, \tilde{M})$ is the classifying topos of the theory of extremal morphisms (in the factorization given by extremal-admissible maps) of models of T_0 with "domain" M and codomain a model of T in some topos. We let $\Delta M \xrightarrow{k_M} \tilde{M} \in \mathscr{E}_M$ be the generic extremal morphism (or localization).

To obtain Pierce's representation via spectra, we let T_0 = theory of non-trivial regular (commutative) rings with 1; T = theory of fields, and A = monomorphism. Similarly, the choice T_0 = theory of non-trivial regular f-rings with 1; T = theory of ordered fields and A = order preserving monomorphisms, yields Keimel's representation specialized to regular f-rings (cf. [7]). Notice that (∗) is satisfied in these cases.

Our first axiom requires $\mathscr{E}_M$ to be a spatial topos (as in most applications of spectra) but whose underlying topological space is the space of a Boolean algebra (as in both cases above: cf. [9]).

Axiom (B). Given a set-theoretical model M of T_0, there is a Boolean algebra B_M such that $\mathscr{E}_M = \mathrm{Sh}_{\mathrm{fc}}(B_M)$, i.e., the topos of sheaves on B_M for the finite cover topology (a finite family covers $b \in B_M$ if its supremum is b).

Notice that B_M can be recovered from $\mathscr{E}_M$ as the complemented subobjects of 1 (since $\mathscr{E}_M$ is coherent). Furthermore $M \mapsto B_M$ is functorial. In fact, if $M \xrightarrow{\alpha} N \in \mathrm{Mod}_{\mathrm{Sets}}(T_0)$, the composite $\Delta M \xrightarrow{\Delta\alpha} \Delta N \xrightarrow{\alpha} \tilde{N} \in \mathrm{Mod}_{\mathscr{E}_N}(T_0)$ gives (by universality) a morphism

$$(\mathrm{Sh}_{\mathrm{fc}}B_N, \tilde{N}) \xrightarrow{(p_\alpha, h_\alpha)} \mathrm{Sh}_{\mathrm{fc}}(B_M, \tilde{M}) \quad \text{in } A\text{-Top},$$

with $p_\alpha^* \dashv (p_\alpha)_*$, p_α^* left exact (i.e., p_α is a geometric morphism, and h_α: $p_\alpha^*\tilde{M} \to \tilde{N}$ admissible. Obviously, p_α^* induces a Boolean homomorphism $B_M \xrightarrow{g_\alpha} B_N$.

For later applications, we need to know that g_α is a monomorphism whenever α is an *extension*, i.e. a morphism of models of T_0 which reflects (as well as preserves) the primitive relations (including =) in the sense that (in Sets, say)

$$\langle \alpha(a_1), \ldots, \alpha(a_n) \rangle \in N(R) \Leftrightarrow \langle a_1, \ldots, a_n \rangle \in M(R).$$

We state the definition as follows: let T be a geometric theory and $\mathscr{E}$ an arbitrary topos. The map $M \xrightarrow{\alpha} N \in \mathrm{Mod}_{\mathscr{E}}T$ is an *extension* if for every primitive relation R (including =) of the language of T, the diagram

$$\begin{array}{ccc} M^n & \xrightarrow{\alpha^n} & N^n \\ \uparrow & & \uparrow \\ M(R) & \longrightarrow & N(R) \end{array}$$

obtained by the definition of α being a morphism, is a pull-back.

Our next axiom states this condition (the "conformality" condition of [12, Lemma 6.3] or [8, Definition 5.5]).

Axiom (C). If $M \xrightarrow{\alpha} N \in \mathrm{Mod}_{\mathrm{Sets}}(T_0)$ is an extension, then p_α is a surjection (i.e., p_α^* is faithful).

To obtain a genuine representation by global sections, we require the following condition on sectional representation for any set-theoretical model M of T_0.

Axiom (SR). The canonical morphism $M \xrightarrow{\sigma_M} \Gamma\tilde{M}$, obtained from $\Delta M \xrightarrow{k_M} \tilde{M}$ by adjunction, is an isomorphism.

For Pierce's representation this is [12, Theorem 4.4], while for Keimel's representation this is [7, Theorem 7.4], specialized to the case of non-trivial regular f-rings.

Our last axiom expresses the "canonicity" of the representations (cf. [12, Theorem 5.3] and [7, Theorem 6.13]) and may be stated as follows: starting from a $\mathrm{Sh}_{\mathrm{fc}}(B)$-model $\overline{M}$ of T, we obtain (by the universal property of spectra) a morphism

$$(\mathrm{Sh}_{\mathrm{fc}}(B), \overline{M}) \xrightarrow{(r_{\overline{M}}, f_{\overline{M}})} \mathrm{Sh}_{\mathrm{fc}}(B_{\Gamma\overline{M}}, \widetilde{\Gamma\overline{M}})$$

since (due to assumption (∗)) $\Gamma\overline{M}$ is a model of T_0.

Axiom (I). For every $\mathrm{Sh}_{\mathrm{fc}}(B)$-model $\overline{M}$ of T, the canonical morphism $(r_{\overline{M}}, f_{\overline{M}})$ is an isomorphism. In particular $B \simeq B_{\Gamma\overline{M}}$ and $r_M^*\widetilde{\Gamma\overline{M}} \simeq \overline{M}$.

Definition. Let $T_0 \to T$ be a quotient of geometric theories with an admissible class A of morphisms of models of T and satisfying (∗). We call the corresponding spectral theory a *Boolean representation theory* if axioms (B), (SR), and (I) are satisfied.

For quotients $T_0 \to T$ with an admissible class A as above, satisfying (∗), and whose spectral theory is a Boolean representation theory, we may separate the two ingredients of $\mathrm{Mod}_{\mathrm{Sets}}(T_0)$ quite neatly (as mentioned in the Introduction). Indeed, if $\mathscr{B}^*$ is a subcategory of the category of Boolean algebras $\mathscr{B}$ and $T \to T^*$ is a quotient (with T^* geometric), we let $(\mathscr{B}^*, T^*)$ be the category whose objects are couples (B, M) with $B \in \mathscr{B}^*$ and M a $\mathrm{Sh}_{\mathrm{fc}}(B)$-model of T^* and whose morphisms are couples $(B, M) \xrightarrow{(g,\alpha)} (B', M')$ such that $B \xrightarrow{g} B' \in \mathscr{B}^*$ and the canonical morphism $g^*M \xrightarrow{\alpha} M' \in \mathrm{Mod}_{\mathrm{Sh}_{\mathrm{fc}}(B')}(T^*)$ (composition of morphisms is defined in the obvious way).

PROPOSITION. *The functor* Φ: $\mathrm{Mod}_{\mathrm{Sets}}(T_0) \to (\mathscr{B}, T)$ *defined by* $\Phi(M) = (B_M, \tilde{M})$ *and* $\Phi(M \xrightarrow{\alpha} N) = (g_\alpha, h_\alpha)$ *is an equivalence of categories.*

Proof. Axiom (SR) and the properties of the factorization given by extremal-admissible maps (cf. [1], [4]) together imply that Φ is fully faithful, whereas the essential surjectivity (or denseness) of Φ follows from Axiom (I).

Remark. If $M \xrightarrow{\alpha} N \in \mathrm{Mod}_{\mathrm{Sets}}(T_0)$ is an extension and axiom (C) is satisfied, then in the couple $\Phi(\alpha) = (g_\alpha, h_\alpha)$, g_α is a monomorphism.

§ 2. **Macintyre's theorem and related results.** To state our main theorem, we recall some definitions (cf. [6], [9], [13]). A coherent theory T is *model complete* (resp. *positively model complete*) if for any existential (resp. coherent) formula φ there is an existential (resp. coherent) formula φ' such that $T \vdash \uparrow \Rightarrow \varphi \vee \varphi'$ and $T \vdash \varphi \wedge \varphi' \Rightarrow \downarrow$. (The pretopos associated to T is Boolean in the sense of [6], precisely when T is positively model complete.)

Remark. It follows immediately from the above definition, that a coherent theory T is positively model complete if and only if for every formula φ there exists a coherent formula φ' which is (classically) T-equivalent to φ. The following lemma, due to Michel Coste, is a strengthening of this remark.

LEMMA. *Let T be a positively model complete coherent theory. Then, for any formula φ there exists a coherent formula φ' which is intuitionistically T-equivalent to φ. In particular, for any topos $\mathscr{E}$ and morphism α: $M \to N$ of T-models in $\mathscr{E}$, given any formula φ, the diagram*

$$\begin{tikzcd} M^n \arrow[rr, "\alpha^n"] \arrow[dr, "\|\varphi\|_M"'] & & N^n \arrow[dl, "\|\varphi\|_N"] \\ & \Omega & \end{tikzcd}$$

commutes in $\mathscr{E}$.

Proof. By induction on φ.

Comer's version of the Feferman–Vaught's theorem may be stated in two parts (cf. [2], [14]).

PROPOSITION 1. *Let T be a positively model complete coherent theory and let B be a Boolean algebra.*

(i) *If M is a $\mathrm{Sh}_{\mathrm{fc}}(B)$-model of T and φ is any formula, then there is a factorization*

$$\begin{tikzcd} M^n \arrow[rr, "\|\varphi\|_M"] \arrow[dr] & & \Omega \\ & B \arrow[ur, hook] & \end{tikzcd}$$

in the topos $\mathrm{Sh}_{\mathrm{fc}}(B)$, where B also stands for the subsheaf of Ω in $\mathrm{Sh}_{\mathrm{fc}}(B)$ whose value at an element b of the Boolean algebra B is given by $\{a \wedge b \mid a \in B\}$.

(ii) *If* $M \xrightarrow{\alpha} N \in \mathrm{Mod}_{\mathrm{Sh}_{\mathrm{cf}}(B)}(T)$ *and φ is any formula, then the diagram (given by* (i))

$$\begin{tikzcd} M^n \arrow[rr, "\alpha^n"] \arrow[dr, "\|\varphi\|_M"'] & & N^n \arrow[dl, "\|\varphi\|_N"] \\ & B & \end{tikzcd}$$

commutes.

Proof. (i) is proved by first remarking that for coherent φ, the subobject of M^n classified by $\|\varphi\|_M$ is complemented, and then using the lemma, whereas (ii) follows from (i) and the lemma.

PROPOSITION 2. *Let T be a positively model complete coherent theory and B a Boolean algebra. If φ is any formula in the language of T there is a formula φ^* of the language of the theory of Boolean algebras as well as a finite sequence $\psi_1, \ldots, \psi_m$ of formulas in the language of T with the same number of variables as φ such that: given any $\mathrm{Sh}_{\mathrm{fc}}(B)$-model $\overline{M}$ of T and any global section $1 \xrightarrow{\vec{a}} \overline{M}^n$*

$$\Gamma\overline{M} \models \varphi[\vec{a}] \quad \text{iff} \quad B \models \varphi^*[\|\psi_1(\vec{a})\|_{\overline{M}}, \ldots, \|\psi_m(\vec{a})\|_{\overline{M}}].$$

Furthermore, if φ is existential, φ^* may be chosen existential.

Finally, let us say that a full subcategory $\mathscr{M}$ of the category $\mathrm{Mod}(T)$ of models of a coherent theory T is *model complete* (resp. *positively model complete*) if for every extension (resp. every morphism) $M \xrightarrow{\alpha} N \in \mathrm{Mod}(T)$, α (preserves and) reflects existential formulas, i.e., for any existential formula φ and a sequence $\vec{a}$ of elements of M,

$$M \models \varphi[a_1, \ldots, a_n] \quad \text{iff} \quad N \models \varphi[\alpha(a_1), \ldots, \alpha(a_n)].$$

(This definition replaces an earlier one that was incorrect as pointed out by C. Mulvey and E. Nelson.)

Assume from now on that we have a quotient $T_0 \to T$ with a class A as in the definition satisfying Axiom (C).

THEOREM. *Let $\mathscr{B}^*$ be a model complete full subcategory of $\mathscr{B}$ and let $T \to T^*$ be a quotient with T^* a coherent positively model complete theory. Then $(\mathscr{B}^*, T^*)$ is model complete (considered as a full subcategory of $\mathrm{Mod}_{\mathrm{Sets}}(T_0) \simeq (\mathscr{B}, T)$).*

Proof. Let $M \xrightarrow{\alpha} N \in (\mathscr{B}^*, T^*)$ be an extension (which we may assume to be an inclusion) and let φ be a coherent formula such that $N \models \varphi[\bar{a}]$, where $\bar{a}$ is a sequence of elements of M. We apply Proposition 2 (identifying N with $\Gamma\tilde{N}$) to conclude

$$B_N \models \varphi^*[||\psi_1(\bar{a})||_{\tilde{N}}, \ldots, ||\psi_m(\bar{a})||_{\tilde{N}}].$$

On the other hand, Proposition 1(ii) gives

$$p_\alpha^*||\psi_i(\bar{a})||_{\tilde{M}} = ||\psi_i(\bar{a})||_{p_\alpha^*\tilde{M}} = ||\psi_i(\bar{a})||_{\tilde{N}}.$$

Since the Boolean homomorphism $B_M \xrightarrow{g_\alpha} B_N \in \mathscr{B}^*$ may be chosen as the inclusion (cf. the remark at the end of § 1) we conclude that

$$B_M \models \varphi^*[||\psi_1(\bar{a})||_{\tilde{M}}, \ldots, ||\psi_m(\bar{a})||_{\tilde{M}}], \quad \text{i.e.} \quad M \models \varphi[\bar{a}],$$

by using Proposition 2 once again.

Remark. If $\mathscr{B}^*$ is an elementary class, then so is $(\mathscr{B}^*, T^*)$ and its theory is model-complete.

In particular, let $\mathscr{B}^*$ = the class of atomless Boolean algebras and let T^* be the theory of algebraically closed fields. A simple argument (cf. [9]) gives that $(\mathscr{B}^*, T^*)$ is the theory T_0^* of integrally closed non-trivial regular rings with 1 and no minimal idempotents. Similarly for T^* = the theory of real closed fields, we obtain that $(\mathscr{B}^*, T^*)$ is the theory T_0^* of non-trivial regular f-rings with 1 and no minimal idempotents such that $x \wedge 0 = 0 \Rightarrow \exists y(x = y^2)$ and all monic polynomials of odd degree have roots. Hence, both of these theories are model complete.

The following results complement our main theorem and pave the way for the applications.

We shall say that a full subcategory $\mathscr{M}$ of the category $\mathrm{Mod}_{\mathrm{Sets}}(T)$ of models of a geometric theory T is *cofinal* if for every $M \in \mathrm{Mod}_{\mathrm{Sets}}(T)$ there is an extension $M \overset{\alpha}{\rightarrowtail} N \in \mathrm{Mod}_{\mathrm{Sets}}(T)$ with $N \in \mathscr{M}$.

We shall now assume (for our next theorems) that the quotient $T_0 \to T$ satisfies (besides our assumptions) the following conditions:

(*)′ T_0 has an axiomatization of the form $\uparrow \Rightarrow \varphi$ or $\varphi \Rightarrow \downarrow$, where φ is a conjunction of atomic formulas

(**) If $M \in \mathrm{Mod}_{\mathrm{Sets}}(T)$, then every $M \to M_0 \in \mathrm{Mod}_{\mathrm{Sets}}(T_0)$ is an extension.

Notice that both conditions are satisfied in our applications.

THEOREM. *Let $\mathscr{B}^* \hookrightarrow \mathscr{B}$ be cofinal and let $T \to T^*$ be a quotient such that $\mathrm{Mod}\,T^* \hookrightarrow \mathrm{Mod}\,T$ is cofinal (more precisely, we assume that the statement "every model of T can be extended to one of T^*" is a theorem of* ZFC *set theory). Assume, furthermore, that any $B \in \mathscr{B}^*$ can be extended to a complete Boolean algebra in $\mathscr{B}^*$. Then, $(\mathscr{B}^*, T^*) \hookrightarrow \mathrm{Mod}_{\mathrm{Sets}}(T_0)$ is cofinal.*

LEMMA. *Let T be a coherent theory, B a complete Boolean algebra and let $\mathrm{Sh}_\infty(B) \underset{i}{\overset{\alpha}{\leftrightarrows}} \mathrm{Sh}_{\mathrm{fc}}(B)$ be the subtopos of $\mathrm{Sh}_{\mathrm{fc}}(B)$ given by the $\neg\neg$ topology. If $M \xrightarrow{\alpha} N \in \mathrm{Mod}_{\mathrm{Sh}_\infty(B)}(T)$, then $iM \xrightarrow{i\alpha} iN \in \mathrm{Mod}_{\mathrm{Sh}_{\mathrm{fc}}(B)}(T)$.*

Proof. Since $\mathrm{Sh}_\infty(B)$ satisfies the axiom of choice (i.e. epis split), i preserves images and it is enough to show that i preserves $\vee$ (i preserves $\wedge$, $\uparrow$ automatically since it has a left adjoint). But this follows, by using characteristic morphisms, from the fact that $B \hookrightarrow \Omega \in \mathrm{Sh}_{\mathrm{fc}}(B)$ preserves $\vee$, wherever B is complete. (In topological terms: in an extremally disconnected Stone space, the regular open sets coincide with the clopen sets and the supremum of two clopen is just their union).

Remark. In [8], Loullis proves that i preserves all finitary logic (i.e. $\forall$, $\to$, $\neg$). Indeed, i preserves all higher order logic because of the following (easily checked by adjointness).

PROPOSITION. *If $\mathscr{E} \underset{i}{\overset{a}{\leftrightarrows}} \mathscr{F}$ is a subtopos of $\mathscr{F}$ (i.e., a is left-exact, $a \dashv i$ and $ai = \mathrm{Id}$), then i preserves Π_f.*

Proof (of the theorem). Let $M \in \mathrm{Mod}_{\mathrm{Sets}}(T_0)$. By assumption, there is $B_M \overset{g}{\rightarrowtail} B^* \in \mathscr{B}$ with B^* a complete Boolean algebra in $\mathscr{B}^*$.

We obtain geometric morphisms

$$\mathrm{Sh}_{\mathrm{fc}}(B_M) \underset{g_*}{\overset{g^*}{\rightleftarrows}} \mathrm{Sh}_{\mathrm{fc}}(B^*) \underset{i}{\overset{a}{\rightleftarrows}} \mathrm{Sh}_\infty(B^*).$$

Since $ag^*\tilde{M}$ is a model of T in $\mathrm{Sh}_\infty(B^*)$ and this topos is a model of ZFC set theory, there is

$$ag^*\tilde{M} \rightarrowtail M^* \in \mathrm{Mod}_{\mathrm{Sh}_\infty(B^*)}(T)$$

with M^* a $\mathrm{Sh}_\infty(B^*)$-model of T^*.

By adjointness, we obtain a map

$$\tilde{M} \to g_* iM^* \in \mathrm{Mod}_{\mathrm{Sh}_{\mathrm{fc}}(B_M)}(T_0).$$

(By the lemma, iM^* is a model of T^* and by the assumption on T_0, $g_* iM^*$ is a model of T_0, since g_* preserves $\downarrow$, as can be easily checked). By the assumption (**), this map is an extension (since this topos has enough points, it is sufficient to check it in Sets).

Taking global sections, we obtain an extension

$$M \rightarrowtail \Gamma(iM^*), \quad \text{with} \quad \Gamma(iM^*) \in (\mathscr{B}^*, T^*).$$

Using the same techniques, we can prove (under the same hypotheses)

THEOREM. *Assume that* $\mathscr{B}^* \hookrightarrow \mathscr{B}$ *has the amalgamation property, and let* $T \to T^*$ *be a quotient such that* $\mathrm{Mod}(T^*)$ *has the amalgamation property (more precisely, we assume that this statement is a theorem of* ZFC). *If every* $B \in \mathscr{B}^*$ *can be extended to a complete Boolean algebra in* $\mathscr{B}^*$, *then* $(\mathscr{B}^*, T^*)$ *has the amalgamation property.*

Proof. Given $M \begin{array}{l} \nearrow M_1 \\ \searrow M_2 \end{array}$ in $(\mathscr{B}^*, T^*) \hookrightarrow \mathrm{Mod}_{\mathrm{Sets}}(T_0)$ we obtain (by (C)), Boolean monomorphisms $B_M \begin{array}{l} \nearrow B_{M_1} \\ \searrow B_{M_2} \end{array}$. We now amalgamate with a complete Boolean algebra in $\mathscr{B}^*$

$$\begin{tikzcd} & B_{M_1} \arrow[dr] & \\ B_M \arrow[ur] \arrow[dr] & & B^* \\ & B_{M_2} \arrow[ur] & \end{tikzcd}$$

and we proceed as before. Details are lengthy, but straightforward.

Remark. In particular, for $\mathscr{B}^* = \mathscr{B}$ and $T^* = T$, we obtain that $(\mathscr{B}, T) \simeq \mathrm{Mod}_{\mathrm{Sets}}(T_0)$ has the amalgamation property, provided that $\mathrm{Mod}_{\mathrm{Sets}}(T)$ has it. From this, we obtain immediately the results of Carson and of Weispfenning mentioned in [9] and [10], whereby the theory of non-trivial commutative regular rings, respectively the theory of non-trivial commutative regular f-rings, both have the amalgamation property. We conclude (for the theorems in this section) that in each case referred to in the remark after the main theorem, T_0^* is the model completion of T_0. In addition to these examples, the first author has obtained (J. Algebra 68 (1981), pp. 79–96) similar applications to certain theories of differential rings. In the same paper, transfer theorems on the existence of prime model extensions are given also in the context of spectra and localizations.

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