A fixed point theorem for multivalued mappings in topological vector spaces

by

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Abstract. In this paper we shall prove, using a result of Kasahara [4], a fixed point theorem for multivalued mappings in topological vector space which is a generalization of Matusov's fixed point theorem from [5].

Recently Matusov proved the following fixed point theorem [5] which is a generalization of the well known Tihonov fixed point theorem.

THEOREM 1. *Let X be a Hausdorff topological vector space, K be a nonempty, convex and compact subset of X and $f\colon K \to K$ be a continuous mapping. Then there exists at least one fixed point of the mapping f.*

Let X be a topological vector space, $K \subset X$, 2^K the set off all nonempty subsets of K and 2_c^K the set of all convex subsets of K.

Further, let $\mathcal{U}$ be the fundamental system of open, balanced neighbourhoods of zero in X.

DEFINITION. We say that the mapping $f\colon A \to 2^X$ $(A \subset X)$ is *u-continuous* iff for every $V_1 \in \mathcal{U}$ there exists $V_2 \in \mathcal{U}$ such that the following implication holds:

For every $x_1, x_2 \in A$ such that $x_1 - x_2 \in V_2$ and every $y_1 \in f(x_1)$ there exists $y_2 \in f(x_2)$ such that $y_1 - y_2 \in V_1$.

Now, we shall prove for u-continuous multivalued mappings a fixed point theorem which is a generalization of Matusov's fixed point theorem. In the proof we shall obtain also a result on almost continuous selection property for u-continuous multivalued mappings.

THEOREM 2. *Let X be a Hausdorff topological vector space, K be a nonempty, convex and compact subset of Φ-type of* [6] *of X and $f\colon K \to 2_c^K$ be a closed u-continuous mapping. Then there exists at least one element $x \in K$ such that $x \in f(x)$.*

Proof. First, we shall show that the mapping f has the almost continuous selection property, i.e., that for every $V \in \mathcal{U}$ there exists a continuous mapping $g_V\colon K \to K$ such that:

$$g_V(x) \in f(x) + V, \quad \text{for every } x \in K. \tag{1}$$

Let $V\in\mathscr{U}$. In [4] is proved that X is a paranormed space $(X, ||\cdot||, \Phi)$ over a topological semifield R_Δ, where:

R_Δ is the set of all mappings from Δ into R with the Tihonov product topology and the operations addition and scalar multiplication as usual,

$||\cdot||\colon X\to P_\Delta$, where P_Δ is the cone of nonnegative elements in R_Δ,

$\Phi\colon R_\Delta\to R_\Delta$ is a positive, continuous, linear operator such that the following conditions are satisfied:

(i) $||x||\geqslant 0$ for every $x\in X$ and $||x||=0\Leftrightarrow x=0$,

(ii) $||\lambda x||=|\lambda|\cdot||x||$, for every $x\in X$ and for every λ in the scalar field of X,

(iii) $||x+y||\leqslant\Phi(||x||+||y||)$ for every $x, y\in X$.

Further, if U is a neighbourhood of zero in R_Δ then the set $\{x|\ ||x||\in U\}$ is a neighbourhood of zero in X. Suppose that $\varepsilon>0$ and $\mu=\{t_1, t_2, \ldots, t_n\}\subset\Delta$ so that

$$||x-y||\in U_{\mu,\varepsilon}\ \Rightarrow\ x-y\in V$$

where

$$U_{\mu,\varepsilon}=\{h|\ h\in R_\Delta,\ |h(t_j)|<\varepsilon \text{ for every } t_j\in\mu\}.$$

Since te mapping $\Phi\colon R_\Delta\to R_\Delta$ is continuous, positive, linear mapping there exists a neighbourhood $V_1(\mu,\varepsilon)$ of zero in R_Δ such that

$$||x-y||\in V_1(\mu,\varepsilon)\ \Rightarrow\ \Phi(||x-y||)\in U_{\mu,\varepsilon}.$$

Further, let $V_2\in\mathscr{U}$ be such that the following implication holds:

For every $x_1, x_2\in K$ such that $x_1-x_2\in V_2$ and $y_1\in f(x_1)$ there exists $y_2\in f(x_2)$ such that $||y_1-y_2||\in V_1(\mu,\varepsilon)$.

Since K is compact subset of X there exists a finite set $\{z_1, z_2, \ldots, z_n\}\subset K$ such that

$$K\subset\bigcup_{i=1}^{n}\{z_i+V_2\}.$$

Let $\{w_i\}_{i=1}^n$ be a partition of unity associated to the finite open covering $\{z_i+V_2\}_{i=1}^n$. Then $w_i(x)>0$ implies that $x-z_i\in V_2$. Further, let $y_i\in f(z_i)$ $(i=1, 2, \ldots, n)$ and

$$g_V(x)=\sum_{i=1}^{n}w_i(x)y_i.$$

Let us prove that (1) is satisfied. Namely, we shall prove that for every $x\in K$ there exists $z(x)\in f(x)$ such that

$$||g_V(x)-z(x)||\in U_{\mu,\varepsilon} \tag{2}$$

and so

$$g_V(x)-z(x)\in V.$$

Let $x\in K$ and $i\in\{1, 2, \ldots, n\}$ be such that $w_i(x)>0$. Then $x-z_i\in V_2$ and since $y_i\in f(z_i)$ it follows that there exists $u_i(x)\in f(x)$ such that $||y_i-u_i(x)||\in V_1(\mu,\varepsilon)$.

If $i\in\{1, 2, \ldots, n\}$ is such that $w_i(x)=0$ then $u_i(x)$ is an arbitrary element from $f(x)$. Then

$$z(x)\overset{\text{def}}{=}\sum_{i=1}^{n}w_i(x)u_i(x)$$

and it is obvious that $z(x)\in f(x)$ since the set $f(x)$ is convex for every $x\in K$.

Further, we have similarly as in [5] (K is of Φ-type):

$$\begin{aligned}||g_V(x)-z(x)||(t_j)&=||\sum_{i=1}^{n}w_i(x)(y_i-u_i(x))||(t_j)\\&\leqslant\sum_{i=1}^{n}w_i(x)\Phi(||y_i-u_i(x)||)(t_j)\\&<\sum_{i=1}^{n}w_i(x)\varepsilon=\varepsilon\end{aligned}$$

for every $t_j\in\mu$ which means that (2) is satisfied. So, it follows that the relation (1) is satisfied. Further, the mapping $g_V\colon K\to K$ is continuous and from Matusov's fixed point theorem it follows that for every $V\in\mathscr{U}$ there exists $x_V\in K$ such that

$$x_V=g_V(x_V)\in f(x_V)+V.$$

Since K is compact there exists a convergent subset x_W such that $\lim_W x_W=x_0$.

The mapping f is closed and so we have that

$$x_0\in f(x_0),$$

which completes the proof.

In Corollaries 1, 2, 3 K is of Φ-type.

COROLLARY 1 [5]. *Let X be a Hausdorff topological vector space, K a nonempty, convex and compact subset of X and $f\colon K\to K$ a continuous mapping. Then there exists at least one fixed point of the mapping f.*

Proof. Since K is compact and f is continuous it follows that f is also a u-continuous mapping and a closed mapping and from Theorem just proved it follows that there exists at least one fixed point of the mapping f.

COROLLARY 2. *Let X be a Hausdorff topological vector space, W be a closed neighbourhood of $b\in X$, K be a compact, convex subset of X so that $b\in K$. Let $F\colon W\cap K\to 2_c^K$ be a closed u-continuous mapping such that the following implication holds:*

$$x\in\partial W\cap K\wedge\beta\in(1,\infty)\ \Rightarrow\ \beta x+(1-\beta)b\notin F(x).$$

Then there exists $x_0\in W\cap K$ so that $x_0\in F(x_0)$.

Proof. As in [3]:

$$X_0=\{x|\ x\in W\cap K,\ \exists t\in[0,1],\ x\in tF(x)+(1-t)b\}.$$

It is easy to see that X_0 is closed subset of K and since K is compact it follows that X_0 is compact also. Further, $\partial W\cap K$ is closed and

$$(\partial W\cap K)\cap X_0=\varnothing.$$

Since X is complete regular topological space, there exists a continuous mapping $\lambda\colon X \to [0, 1]$ such that:

$$\lambda(x) = \begin{cases} 0 & \text{for } x \in X_0, \\ 1 & \text{for } x \in \partial W \cap X. \end{cases}$$

As in [3] let

$$G(x) = \begin{cases} [1-\lambda(x)]F(x)+\lambda(x)b & \text{for } x \in W \cap K, \\ b & \text{for } x \in K \setminus W. \end{cases}$$

It is easy to see that the mapping G satisfies all the conditions of Theorem 2 and so $\operatorname{Fix}(G) \neq \emptyset$. If $x_0 \in G(x_0)$ then $x_0 \in X_0$ and so $\lambda(x_0) = 0$ which implies that $G(x_0) = F(x_0)$. This means that $x_0 \in F(x_0)$.

COROLLARY 3. *Let E be a Hausdorff topological vector space, K be a nonempty, closed and convex subset of E and $T\colon K \to 2_c^K$ a closed u-continuous mapping on every compact subset of K so that the following conditions are satisfied:*

(i) *There exists $C \subset K$ such that $C \subset \overline{\operatorname{co}}\, T(C)$.*

(ii) *For every closed and convex subset Q such that $\overline{\operatorname{co}}\, T(Q) = Q$ it follows that Q is compact.*

Then there exists $x_0 \in K$ such that $x_0 \in T(x_0)$.

Proof. The proof is similar to the proof of Theorem 1 from [1]. Let $\mathscr{F} = \{Q \mid Q \supset C,\ Q = \overline{\operatorname{co}}\, Q,\ T(Q) \subset Q\}$. The family $\mathscr{F}$ has the following property:

$$(*) \qquad Q \in \mathscr{F} \Rightarrow \overline{\operatorname{co}}\, T(Q) \in \mathscr{F}.$$

Let $K_0 = \bigcap_{Q \in \mathscr{F}} Q$. As in [1] it follows that $K_0 \neq \emptyset$ and from the implication $(*)$ we conclude that:

$$K_0 = \overline{\operatorname{co}}\, T(K_0).$$

From (ii) it follows that K_0 is compact. Since $T|K_0\colon K_0 \to 2_c^{K_0}$ is u-continuous mapping using Theorem 2 we conclude that for every $V \in \mathscr{F}$ there exists a continuous singlevalued mapping $\varphi_V\colon K_0 \to K_0$ such that $\varphi_V(x) \in T(x)+V$, $\forall x \in K_0$. The rest of the proof is equal as in Theorem 2.

Remark. This corollary is a generalization of Theorem 3 from [1] for singlevalued mapping.

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