Chain conditions and products

by

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Abstract. In unpublished work, R. Laver has shown that the continuum hypothesis implies that there are two partially ordered sets (or topological spaces) satisfying the countable chain condition, whose product does not satisfy the countable chain condition. We give a simple proof of Laver's result, and we prove some generalizations.

§ 1. Introduction. For convenience in defining infinite products, a partially ordered set is always assumed to have a least element, denoted by 0. If P_1 and P_2 are partially ordered sets, the Cartesian product $P_1 \times P_2$ is partially ordered by the rule

$$(x_1, x_2) \leqslant (y_1, y_2) \Leftrightarrow x_1 \leqslant y_1 \wedge x_2 \leqslant y_2 .$$

More generally, given partially ordered sets P_i $(i \in I)$, the (weak) product $\prod_{i \in I} P_i$ consists of all functions $f \in \bigtimes_{i \in I} P_i$ such that $f(i) = 0$ for all but finitely many $i \in I$, partially ordered by the rule

$$f \leqslant g \Leftrightarrow \forall i \in I \; f(i) \leqslant g(i) .$$

Let P be a partially ordered set, and let $\varkappa$ be a cardinal. Two elements $x, y \in P$ are *compatible* if there is an element $z \in P$ such that $x \leqslant z$ and $y \leqslant z$; otherwise they are *incompatible*. P satisfies the *$\varkappa$-chain condition* ($\varkappa$-c.c.) if there is no set of $\varkappa$ pairwise incompatible elements in P. The $\aleph_1$-chain condition is also called the *countable chain condition* (c.c.c.). A topological space X is said to satisfy the $\varkappa$-c.c. if every family of pairwise disjoint nonempty open sets of X has cardinality $<\varkappa$; in other words, if the $\varkappa$-c.c. is satisfied by the partially ordered set consisting of the nonempty open sets of X, ordered by reverse inclusion.

MA (Martin's axiom) and $\mathrm{MA}_{\aleph_1}$ are the axioms A and $\mathrm{A}_{\aleph_1}$ of Martin and Solovay [14, pp. 149–150]. Recall that MA is a weakened form of the continuum hypothesis, while $\mathrm{MA}_{\aleph_1}$ contradicts the continuum hypothesis.

If $\varkappa$ is a cardinal, $C(\varkappa)$ is the statement: if P and Q are any partially ordered sets satisfying the $\varkappa$-c.c., then their product $P \times Q$ also satisfies the $\varkappa$-c.c. ($C(\varkappa)$ is

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equivalent to the statement: if X and Y are any topological spaces satisfying the $\varkappa$-c.c., then their product $X\times Y$ satisfies the $\varkappa$-c.c. Moreover, if $\varkappa$ is an uncountable regular cardinal, then $C(\varkappa)$ implies that an arbitrary product of partially ordered sets, or of topological spaces with the Tychonoff product topology, will satisfy the $\varkappa$-c.c. provided each factor does. These points will be discussed in § 2.)

The question, whether $C(\aleph_1)$ is true, was raised by Marczewski [13, p. 141], and again by Kurepa [12, p. 108]. Kurepa showed that $C(\aleph_1)$ implies Souslin's hypothesis; in fact, if P is a Souslin tree, then $P\times P$ does not satisfy the c.c.c., although P of course does. K. Kunen, F. Rowbottom, and R. M. Solovay (independently) showed that $MA_{\aleph_1}$ implies $C(\aleph_1)$; see [2, p. 75] and [17, p. 17]. R. Laver showed (private communication) that the continuum hypothesis implies the negation of $C(\aleph_1)$. Laver's proof uses the partially ordered sets constructed by Baumgartner in [1], and is said to be somewhat complicated; in this paper we give a simple direct proof of Laver's result (¹). In view of Jensen's proof of the consistency of ZFC+GCH+SH [3, p. 113], Laver's result shows that Souslin's hypothesis is definitely weaker than $C(\aleph_1)$; this answers a question of Kunen [17, p. 63] and Tall [19, p. 333], [20, p. 614]. Finally, Roitman [16] has shown that $C(\aleph_1)$ is false in certain Cohen models of set theory.

In § 2 we summarize the well-known facts connecting partially ordered sets with (extremally disconnected compact Hausdorff) topological spaces. In § 3 we prove Laver's theorem in its generalized form: if $2^\varkappa = \varkappa^+$, then $C(\varkappa^+)$ is false. In § 4 we construct more general examples; e.g., assuming $2^\varkappa = \varkappa^+$, for every $n<\omega$ there is an extremally disconnected compact Hausdorff space X such that X^n satisfies the $\varkappa^+$-c.c. but X^{n+1} does not. In § 5 we show that the partition relation $\varkappa \to [\varkappa]^2_3$ implies $C(\varkappa)$.

§ 2. Topological spaces, partially ordered sets, and products. Most of the results in this paper can be stated either as theorems about partially ordered sets or as theorems about topological spaces. Using the (well-known) facts collected in this section, one can easily derive the topological formulations from the partial order formulations, and vice versa. The reader who is only interested in the partial order formulations should skip this section, except for Corollary 2.10 and Lemma 2.11.

By a *topological space* we mean simply a (nonempty) topological space, with no separation axioms assumed. If X is a topological space, then $\tau(X)$ is the collection of all open sets of X, and $\tau^+(X) = \tau(X)\setminus\{\emptyset\}$. A family $\mathcal{S}$ of sets has the *ν-intersection property* if $\bigcap \mathcal{S}' \neq \emptyset$ for every $\mathcal{S}'\subset\mathcal{S}$ with $|\mathcal{S}'|<\nu$; thus, the $\aleph_0$-intersection property is the finite intersection property.

Let $\varkappa, \lambda, \mu, \nu$ be cardinals. A topological space X satisfies the condition $\boldsymbol{P}(\varkappa, \nu)$ if $\tau^+(X)$ is the union of $\varkappa$ families, each having the ν-intersection property. The

(¹) Professor Laver has asked me to be more precise about the history of this result. Laver first proved that, assuming CH, the c.c.c. is weaker than property $\boldsymbol{K}$ (defined in § 2); I found a simpler proof of Laver's result; later I noticed that my method could be used to get a counterexample to $C(\aleph_1)$, whereupon Laver pointed out that his construction also gives this.

predensity of X is the least $\varkappa$ for which X satisfies $\boldsymbol{P}(\varkappa, \aleph_0)$; the *density* of X, denoted by $d(X)$, is the least $\varkappa$ such that $\tau^+(X)$ is the union of $\varkappa$ families, each having a nonempty intersection. X satisfies the condition $\boldsymbol{Q}(\varkappa, \lambda, \mu, \nu)$ if there is a partition $\tau^+(X) = \bigcup_{\alpha<\varkappa} \tau_\alpha$ such that, for each ordinal $\alpha<\varkappa$, there is a cardinal $\lambda_\alpha<\lambda$ such that, for any sets $U_\xi\in\tau_\alpha$ $(\xi<\lambda_\alpha)$, there exists $A\subset\lambda_\alpha$ such that $|A| = \mu$ and $\{U_\xi\colon \xi\in A\}$ has the ν-intersection property. Thus, $\boldsymbol{Q}(1, \varkappa^+, 2, 3)$ is the *$\varkappa$-chain condition*; $\boldsymbol{Q}(\varkappa, \lambda^+, 2, 3)$ is the *$(\varkappa, \lambda)$-chain condition*; $\boldsymbol{Q}(\varkappa, \lambda, 2, 3)$ is the *$(\varkappa, <\lambda)$-chain condition*; $\boldsymbol{Q}(1, \aleph_2, \aleph_1, 3)$ is *property $\boldsymbol{K}$*; a regular infinite cardinal $\varkappa$ is a *precaliber* for X if X satisfies $\boldsymbol{Q}(1, \varkappa^+, \varkappa, \aleph_0)$, and a *caliber* if X satisfies $\boldsymbol{Q}(1, \varkappa^+, \varkappa, \varkappa^+)$.

In fact, this paper is only concerned with the $\varkappa$-chain condition; the other notions defined in the preceding paragraph will not be mentioned outside of this section. However, I want to state the results of this section in some generality, so we can refer to them in [7].

2.1. Lemma. *Let X be a topological space, let μ be a cardinal, and suppose there is a family $\mathcal{S}$ of subsets of X such that every member of $\mathcal{S}$ contains a member of $\tau^+(X)$, every member of $\tau^+(X)$ contains a member of $\mathcal{S}$, and every subfamily of $\mathcal{S}$ that has the finite intersection property also has the μ^+-intersection property. Then*:

(1) *if X satisfies $\boldsymbol{P}(\varkappa, \aleph_0)$ and $|X|\leqslant\mu$, then $d(X)\leqslant\varkappa$;*

(2) *if X satisfies $\boldsymbol{Q}(\varkappa, \lambda, \mu, \aleph_0)$, then X satisfies $\boldsymbol{Q}(\varkappa, \lambda, \mu, \mu^+)$.*

2.2. Corollary. *Let X be a compact Hausdorff space.*

(1) *If X satisfies $\boldsymbol{P}(\varkappa, \aleph_0)$, then $d(X)\leqslant\varkappa$.*

(2) *If X satisfies $\boldsymbol{Q}(\varkappa, \lambda, \mu, \aleph_0)$, then X satisfies $\boldsymbol{Q}(\varkappa, \lambda, \mu, \mu^+)$.*

Let X and Y be topological spaces. We write $X\leqslant Y$ if there is a mapping $\varphi\colon \tau^+(X)\to\tau^+(Y)$ such that, for any $n<\omega$ and any $U_1, \ldots, U_n\in\tau^+(X)$,

$$U_1\cap\ldots\cap U_n = \emptyset \Rightarrow \varphi(U_1)\cap\ldots\cap\varphi(U_n) = \emptyset .$$

We write $X\equiv Y$ if $X\leqslant Y$ and $Y\leqslant X$. Clearly $\leqslant$ is a quasi-ordering and $\equiv$ is an equivalence relation.

2.3. Lemma. *Let X and Y be topological spaces such that $X\leqslant Y$, and let $\varkappa, \lambda, \mu, \nu$ be cardinals, $\nu\leqslant\aleph_0$.*

(1) *If Y satisfies $\boldsymbol{P}(\varkappa, \nu)$, so does X.*

(2) *If Y satisfies $\boldsymbol{Q}(\varkappa, \lambda, \mu, \nu)$, so does X.*

The *Gleason space* [9] of a topological space X, denoted by $\boldsymbol{G}X$, is the Stone space of the Boolean algebra of regular open sets of X; alternatively, it can be defined as follows. The points of $\boldsymbol{G}X$ are the maximal subfamilies of $\tau(X)$ with the finite intersection property; the basic open sets are of the form $\{p\in\boldsymbol{G}X\colon U\in p\}$ where

$U \in \tau(X)$. The *pseudoweight* of X, denoted by $\pi(X)$, is the minimum cardinality of a family $\mathscr{B} \subset \tau^+(X)$ such that every member of $\tau^+(X)$ contains a member of X. X is *semiregular* if the regular open sets of X form a base for the topology.

2.4. Lemma. *For every topological space X:*

(1) *GX is an extremally disconnected compact Hausdorff space*;,

(2) $GX \equiv X$;

(3) *$\pi(GX) \leqslant \pi(X)$, with equality if X is semiregular.*

The *forcing topology* on a partially ordered set P has basic open sets of the form $\{x \in P\colon x \geqslant a\}$, $a \in P$. When a partially ordered set P is regarded as a topological space with the forcing topology, we denote it by X_P. A partially ordered set P is said to satisfy the condition $P(\varkappa, \nu)$ or $Q(\varkappa, \lambda, \mu, \nu)$ if the topological space X_P satisfies it. In particular, P satisfies the $\varkappa$-chain condition if and only if X_P does.

A family $\mathscr{S}$ of sets is *hereditary* if $A \subset B \in \mathscr{S} \Rightarrow A \in \mathscr{S}$.

2.5. Lemma. *For every topological space X, there is a partially ordered set P such that $X_P \equiv X$; moreover, we can take for P a hereditary family of finite sets ordered by inclusion.*

Proof. Let $P = \{p \subset \tau(X)\colon |p| < \aleph_0$ and $\bigcap p \neq \emptyset\}$. The mapping $\varphi\colon \tau^+(X) \to \tau^+(X_P)$, defined by $\varphi(U) = \{p \in P\colon U \in p\}$, establishes $X \leqslant X_P$. For each $U \in \tau^+(X_P)$, choose $p \in U$ and put $\psi(U) = \bigcap p$; the mapping $\psi\colon \tau^+(X_P) \to \tau^+(X)$ establishes $X_P \leqslant X$.

The product of partially ordered sets is the weak direct product defined in § 1; the product of topological spaces is the usual Tychonoff product.

2.6. Lemma. *Let P_i $(i \in I)$ be partially ordered sets, $P = \prod_{i\in I} P_i$; then $X_P \equiv \prod_{i \in I} X_{P_i}$.*

2.7. Lemma. *Let X_i, Y_i $(i \in I)$ be topological spaces.*

(1) *If $X_i \leqslant Y_i$ for all $i \in I$, then $\prod_{i\in I} X_i \leqslant \prod_{i \in I} Y_i$.*

(2) *If $X_i \equiv Y_i$ for all $i \in I$, then $\prod_{i\in I} X_i \equiv \prod_{i\in I} Y_i$.*

2.8. Theorem. *For every cardinal $\varkappa$, the following statements are equivalent:*

(1) *there are two partially ordered sets satisfying the $\varkappa$-c.c., whose product does not satisfy the $\varkappa$-c.c.*;

(2) *there are two topological spaces satisfying the $\varkappa$-c.c., whose product does not satisfy the $\varkappa$-c.c.*;

(3) *there are two extremally disconnected compact Hausdorff spaces satisfying the $\varkappa$-c.c., whose product does not satisfy the $\varkappa$-c.c.*

Proof. Suppose P and Q are partially ordered sets satisfying the $\varkappa$-c.c., while $P \times Q$ does not satisfy the $\varkappa$-c.c. I.e., X_P and X_Q satisfy the $\varkappa$-c.c., but $X_{P\times Q}$ does not. By Lemma 2.4, GX_P and GX_Q are extremally disconnected compact Hausdorff spaces, $GX_P \equiv X_P$ and $GX_Q \equiv X_Q$. By Lemma 2.3, since X_P and X_Q satisfy the $\varkappa$-c.c., so do GX_P and GX_Q. By Lemmas 2.7 and 2.6, we have $GX_P \times GX_Q \equiv X_P \times X_Q \equiv X_{P\times Q}$ (in fact, $X_P \times X_Q = X_{P\times Q}$); by Lemma 2.3, since $X_{P\times Q}$ does not satisfy the $\varkappa$-c.c., neither does $GX_P \times GX_Q$. This proves (1)⇒(3). Obviously (3)⇒(2). To prove (2)⇒(1), suppose X and Y are topological spaces such that X and Y satisfy the $\varkappa$-c.c., but $X \times Y$ does not. By Lemma 2.5, there are partially ordered sets P and Q such that $X_P \equiv X$ and $X_Q \equiv Y$, and then $X_{P\times Q} \equiv X_P \times X_Q \equiv X \times Y$ by Lemmas 2.6 and 2.7. By Lemma 2.3, X_P and X_Q satisfy the $\varkappa$-c.c., but $X_{P\times Q}$ does not; i.e., P and Q satisfy the $\varkappa$-c.c., but $P\times Q$ does not.

The next lemma is a special case of a theorem of Noble and Ulmer [15, Theorem 1.3, p. 331; Corollary 1.4(i), p. 332].

2.9. Lemma. *Let $\varkappa$ be an uncountable regular cardinal, and let X_i $(i \in I)$ be topological spaces; then $\prod_{i\in I} X_i$ satisfies the $\varkappa$-c.c. if and only if $\prod_{i\in F} X_i$ satisfies the $\varkappa$-c.c. for every finite $F \subset I$.*

2.10. Corollary. *Let $\varkappa$ be an uncountable regular cardinal, and let P_i $(i \in I)$ be partially ordered sets; then $\prod_{i\in I} P_i$ satisfies the $\varkappa$-c.c. if and only if $\prod_{i\in F} P_i$ satisfies the $\varkappa$-c.c. for every finite $F \subset I$.*

Thus, for uncountable regular $\varkappa$, if the $\varkappa$-c.c. is preserved by products of two factors, then it is preserved by arbitrary products. The proof of Lemma 2.9 is based on the following fact, which we will also need for our constructions.

2.11. Lemma. *Let $\varkappa$ be an uncountable regular cardinal, and let F_α $(\alpha < \varkappa)$ be finite sets. Then there exist $A \subset \varkappa$ and F such that $|A| = \varkappa$ and*

$$\alpha, \beta \in A, \alpha \neq \beta \Rightarrow F_\alpha \cap F_\beta = F.$$

For a proof of this lemma (in a more general form due to Erdös and Rado) and a discussion of its history, see [2, Theorem 3.2, p. 62] and [2, pp. 79–80].

§ 3. The basic construction.

In this section, assuming $2^\varkappa = \varkappa^+$, we construct partially ordered sets P_1 and P_2 such that P_1 and P_2 satisfy the $\varkappa^+$-c.c., but $P_1 \times P_2$ does not.

If A and B are sets and $\varkappa$ is a cardinal, we write $A \otimes B = \{\{a, b\}\colon a \in A, b \in B\}$, $[A]^\varkappa = \{X \subset A\colon |X| = \varkappa\}$, $[A]^{<\varkappa} = \{X \subset A\colon |X| < \varkappa\}$. Given a cardinal λ and a set $K \subset [\lambda]^2$, we define $P(\lambda, K) = \{F \in [\lambda]^{<\aleph_0}\colon [F]^2 \subset K\}$; $P(\lambda, K)$ is partially ordered by inclusion.

3.1. Lemma. *Let λ be a cardinal. If $K_1, K_2 \subset [\lambda]^2$ and $K_1 \cap K_2 = \emptyset$, then $P(\lambda, K_1) \times P(\lambda, K_2)$ does not satisfy the λ-c.c.*

Proof. Clearly, $\{(\{\alpha\}, \{\alpha\})\colon \alpha < \lambda\}$ is a set of λ pairwise incompatible elements of $P(\lambda, K_1) \times P(\lambda, K_2)$.

3.2. Lemma. *Let $\varkappa$ be an infinite cardinal and let A be a set. Suppose that, for each $\varrho < \varkappa$, we have a set I_ϱ with $|I_\varrho| = \varkappa$ and finite sets $E^\xi_\varrho \subset A$ $(\xi \in I_\varrho)$ such that, for* *every* $a \in A$, $|\{\xi \in I_\varrho\colon a \in E_\varrho^\xi\}| < \aleph_0$. *Then there are pairwise disjoint sets* $A_\nu \subset A$ $(\nu < \varkappa)$ *such that*

$$\forall \nu < \varkappa \ \forall \varrho < \varkappa \ |\{\xi \in I_\varrho\colon E_\varrho^\xi \subset A_\nu\}| = \varkappa .$$

The proof of Lemma 3.2 is a straightforward induction of length $\varkappa$.

3.3. THEOREM. *If $\varkappa$ is an infinite cardinal such that $2^\varkappa = \varkappa^+$, then there are partially ordered sets P_1 and P_2 such that P_1 and P_2 satisfy the $\varkappa^+$*-c.c., *but $P_1 \times P_2$ does not.*

Proof. By Lemma 3.1, it will suffice to construct disjoint sets $K_1, K_2 \subset [\varkappa^+]^2$ such that $P(\varkappa^+, K_1)$ and $P(\varkappa^+, K_2)$ satisfy the $\varkappa^+$-c.c. In fact, we are going to define disjoint sets $K_1(\alpha)$, $K_2(\alpha) \subset \alpha$ for every $\alpha < \varkappa^+$; we then put

$$K_i = \{\{\beta, \alpha\}\colon \alpha < \varkappa^+, \beta \in K_i(\alpha)\} .$$

Let $(\mathscr{F}_\mu\colon \mu < \varkappa^+)$ enumerate all $\varkappa$-sequences of pairwise disjoint finite subsets of $\varkappa^+$; $\mathscr{F}_\mu = (F_\mu^\xi\colon \xi < \varkappa)$. The sets $K_1(\alpha)$ and $K_2(\alpha)$ will be constructed simultaneously, by induction on α, so as to satisfy the condition:

(*) if $i \in \{1, 2\}$, $\mu < \alpha$, $\bigcup\limits_{\xi<\varkappa} F_\mu^\xi \subset \alpha$, $X \in [\alpha]^{<\aleph_0}$, and $|\{\xi\colon F_\mu^\xi \otimes X \subset K_i\}| = \varkappa$, then $|\{\xi\colon F_\mu^\xi \otimes X \subset K_i$ and $F_\mu^\xi \subset K_i(\alpha)\}| = \varkappa$, i.e., $|\{\xi\colon F_\mu^\xi \otimes (X \cup \{\alpha\}) \subset K_i\}| = \varkappa$.

Suppose that $K_1(\beta)$ and $K_2(\beta)$ have already been defined for all $\beta < \alpha$. Note that, if $\bigcup\limits_{\xi<\varkappa} F_\mu^\xi \subset \alpha$ and $X \in [\alpha]^{<\aleph_0}$, then $\{\xi\colon F_\mu^\xi \otimes X \subset K_i\}$ is well-defined, since it depends only on $K_i \cap [\alpha]^2$, which is already determined. Let $((i_\varrho, \mu_\varrho, X_\varrho)\colon \varrho < \varkappa)$ enumerate all triples (i, μ, X) satisfying the hypotheses of (*) for our fixed α. Using Lemma 3.2, with $A = \alpha$, $I_\varrho = \{\xi\colon F_{\mu_\varrho}^\xi \otimes X_\varrho \subset K_{i_\varrho}\}$, and $E_\varrho^\xi = F_{\mu_\varrho}^\xi$, we obtain disjoint sets $K_1(\alpha)$, $K_2(\alpha) \subset \alpha$ satisfying the requirements of (*).

We have defined disjoint sets $K_1, K_2 \subset [\varkappa^+]^2$ satisfying (*); now we have to show that the partially ordered sets $P(\varkappa^+, K_1)$ and $P(\varkappa^+, K_2)$ satisfy the $\varkappa^+$-c.c. Let $i \in \{1, 2\}$ and suppose $E^\xi \in P(\varkappa^+, K_i)$ for $\xi < \varkappa^+$; we have to show that, for some ordinals $\xi < \eta < \varkappa^+$, we have $E^\xi \cup E^\eta \in P(\varkappa^+, K_i)$, i.e., $[E^\xi \cup E^\eta]^2 \subset K_i$. By Lemma 2.11, we can assume that there is a set E such that $E^\xi \cap E^\eta = E$ for $\xi < \eta < \varkappa^+$. Then the sets $F^\xi = E^\xi \setminus E$ $(\xi < \varkappa^+)$ are pairwise disjoint. Moreover, since $[E^\xi \cup E^\eta]^2 = [E^\xi]^2 \cup [E^\eta]^2 \cup (F^\xi \otimes F^\eta)$ for $\xi < \eta < \varkappa^+$, and since $[E^\xi]^2 \cup [E^\eta]^2 \subset K_i$, it will suffice to find ordinals $\xi < \eta < \varkappa^+$ such that $F^\xi \otimes F^\eta \subset K_i$. Choose $\mu < \varkappa^+$ so that $F_\mu^\xi = F^\xi$ for all $\xi < \varkappa$.

Choose $\beta < \varkappa^+$ so that $\mu < \beta$ and $\bigcup\limits_{\xi<\varkappa} F_\mu^\xi \subset \beta$. Since the sets F^ξ are pairwise disjoint, we can choose an ordinal η so that $\varkappa \leqslant \eta < \varkappa^+$ and $F^\eta \cap \beta = \emptyset$; hence $\mu < \alpha$ and $\bigcup\limits_{\xi<\varkappa} F_\mu^\xi \subset \alpha$ for all $\alpha \in F^\eta$. Let $F^\eta = \{\alpha_1, \ldots, \alpha_n\}$, $\alpha_1 < \ldots < \alpha_n$.

Now, by applying (*) n times, with $\alpha_1, \alpha_2, \ldots, \alpha_n$ successively playing the role of α and $\emptyset$, $\{\alpha_1\}, \ldots, \{\alpha_1, \ldots, \alpha_{n-1}\}$ the role of X, we see that

$$|\{\xi < \varkappa\colon F_\mu^\xi \otimes F^\eta \subset K_i\}| = \varkappa .$$

In particular, there is some $\xi < \varkappa \leqslant \eta$ such that $F^\xi \otimes F^\eta \subset K_i$; this completes the proof.

3.4. COROLLARY. *If $\varkappa$ is an infinite cardinal such that $2^\varkappa = \varkappa^+$, then there are extremally disconnected compact Hausdorff spaces X_1 and X_2 such that X_1 and X_2 satisfy the $\varkappa^+$-c.c., but $X_1 \times X_2$ does not.*

Proof. Theorems 3.3 and 2.8.

The *character* of a topological space X at a point $x \in X$, denoted by $\chi(X, x)$, is the minimum cardinality of a neighborhood base at x; the *character* of the space X is $\chi(X) = \sup\{\chi(X, x)\colon x \in X\}$. Thus, X is *first countable* if $\chi(X) \leqslant \aleph_0$. The following refinement of Corollary 3.4 was pointed out by E. van Douwen (private communication) and is included here with his permission.

3.5. THEOREM. *If $\varkappa$ is an infinite cardinal such that $2^\varkappa = \varkappa^+$, then there are zero-dimensional Hausdorff spaces X_1 and X_2, of character $\leqslant \varkappa$, such that X_1 and X_2 satisfy the $\varkappa^+$-c.c., but $X_1 \times X_2$ does not.*

Theorem 3.5 may be proved by using the sets K_1 and K_2 constructed in the proof of Theorem 3.3. The points of X_i are the maximal sets $A \subset \varkappa^+$ with $[A]^2 \subset K_i$; the basic open sets are of the form $\{A \in X_i\colon F \subset A\}$, $F \in [\varkappa^+]^{<\aleph_0}$. The fact that $\chi(X_i) \leqslant \varkappa$ follows from the fact that $|A| \leqslant \varkappa$ for all $A \in X_i$. Alternatively, Theorem 3.5 follows from Corollary 3.4 and the following lemma, which was also pointed out by Van Douwen.

3.6. LEMMA. *Let λ be an infinite cardinal. Suppose there are topological spaces X_1 and X_2 such that X_1 and X_2 satisfy the λ-c.c., but $X_1 \times X_2$ does not. Then there are zero-dimensional Hausdorff spaces Y_1 and Y_2, with $\chi(Y_i, y) < \lambda$ for all $y \in Y_i$, such that Y_1 and Y_2 satisfy the λ-c.c., but $Y_1 \times Y_2$ does not.*

Proof. Choose $U_1^\alpha \in \tau^+(X_1)$ and $U_2^\alpha \in \tau^+(X_2)$ $(\alpha < \lambda)$ so that the rectangles $U_1^\alpha \times U_2^\alpha$ are pairwise disjoint. For $i \in \{1, 2\}$, the points of Y_i are the maximal sets $A \subset \lambda$ such that $\{U_i^\alpha\colon \alpha \in A\}$ has the finite intersection property; the basic open sets are of the form $\mathscr{B}_i^F = \{A \in Y_i\colon F \subset A\}$, $F \in [\lambda]^{<\aleph_0}$. It is easy to check that Y_1 and Y_2 are zero-dimensional Hausdorff spaces and satisfy the λ-c.c. To see that $Y_1 \times Y_2$ does not satisfy the λ-c.c., observe that the rectangles $\mathscr{B}_1^{\{\alpha\}} \times \mathscr{B}_2^{\{\alpha\}}$ $(\alpha < \lambda)$ are pairwise disjoint nonempty open sets of $Y_1 \times Y_2$. Now consider any $A \in Y_1$. If $\alpha, \beta \in A$, $\alpha \neq \beta$, then $U_1^\alpha \cap U_1^\beta \neq \emptyset$ while $(U_1^\alpha \times U_2^\alpha) \cap (U_1^\beta \times U_2^\beta) = \emptyset$; hence $U_2^\alpha \cap U_2^\beta = \emptyset$. I.e., the sets U_2^α $(\alpha \in A)$ are pairwise disjoint. Since X_2 satisfies the λ-c.c., we must have $|A| < \lambda$. Since $\{\mathscr{B}_1^F\colon F \in [A]^{<\aleph_0}\}$ is a neighborhood base at A, it follows that $\chi(Y_1, A) < \lambda$. Similarly, $\chi(Y_2, A) < \lambda$ for $A \in Y_2$.

Van Douwen's arguments also apply, to a certain extent, to the more general examples of § 4. For example, as Van Douwen has remarked, assuming $2^\varkappa = \varkappa^+$, for every $n < \omega$ there is a zero-dimensional Hausdorff space X of character $\leqslant \varkappa$ such that X^n satisfies the $\varkappa^+$-c.c. but X^{n+1} does not.

§ 4. More general examples. The main result in this section says that, assuming $2^\varkappa = \varkappa^+$, we can construct families of partially ordered sets or topological spaces so that prescribed products fail to satisfy the $\varkappa^+$-c.c.; for example, we can construct spaces X, Y, Z such that $X\times Y^2\times Z^3$ does not satisfy the $\varkappa^+$-c.c., while $Y^\omega\times Z^\omega$, $X^\omega\times Y\times Z^\omega$, and $X^\omega\times Y^\omega\times Z^2$ all satisfy it. Still assuming $2^\varkappa=\varkappa^+$, we construct a partially ordered set P such that P satisfies the $\varkappa^+$-c.c., but the set P^* of all pairwise compatible finite subsets of P, ordered by inclusion, does not; in this case, some (finite) power of P must fail to satisfy the $\varkappa^+$-c.c. We obtain similar results for the $2^{\aleph_0}$-c.c. assuming MA.

Let λ be a cardinal. A set $K\subset[\lambda]^2$ will be called *big* if, given any $n<\omega$ and any $H_1,\ldots,H_n\subset[\lambda]^2$, if $K\subset H_1\cap\ldots\cap H_n$, then $P(\lambda,H_1)\times\ldots\times P(\lambda,H_n)$ satisfies the λ-c.c. The following lemma is a strengthening of the result of Erdös, Hajnal, and Rado [4, Theorem 17, p. 145] that $2^\varkappa=\varkappa^+$ implies $\varkappa^+\nrightarrow[\varkappa^+]^2_{\varkappa^+}$.

4.1. Lemma. *If $\varkappa$ is an infinite cardinal such that $2^\varkappa=\varkappa^+$, then there are $\varkappa^+$ pairwise disjoint big subsets of $[\varkappa^+]^2$.*

Proof. Let $(\mathscr{F}_\mu\colon \mu<\varkappa^+)$ enumerate all $\varkappa$-sequences $(F^\xi\colon \xi<\varkappa)$ of finite subsets of $\varkappa^+$ such that, for every $\alpha\in\varkappa^+$, $|\{\xi\colon \alpha\in F^\xi\}|<\aleph_0$; $\mathscr{F}_\mu=(F^\xi_\mu\colon \xi<\varkappa)$. As in the proof of Theorem 3.3, we can use Lemma 3.2 to construct pairwise disjoint sets $K_\nu\subset[\varkappa^+]^2$ $(\nu<\varkappa^+)$ satisfying the following condition for every $\alpha<\varkappa^+$:

(*) if $\nu<\alpha$, $\mu<\alpha$, $\bigcup\limits_{\xi<\varkappa}F^\xi_\mu\subset\alpha$, $X\in[\alpha]^{<\aleph_0}$, and $|\{\xi\colon F^\xi_\mu\otimes X\subset K_\nu\}|=\varkappa$, then

$$|\{\xi\colon F^\xi_\mu\otimes(X\cup\{\alpha\})\subset K_\nu\}|=\varkappa\,.$$

Now consider any $\nu<\varkappa^+$; we have to show that K_ν is big. Let $n<\omega$ and suppose $H_1,\ldots,H_n\subset[\varkappa^+]^2$, $K_\nu\subset H_1\cap\ldots\cap H_n$; we have to show that

$$P(\varkappa^+,H_1)\times\ldots\times P(\varkappa^+,H_n)$$

satisfies the $\varkappa^+$-c.c. Suppose $(E^\xi_1,\ldots,E^\xi_n)\in P(\varkappa^+,H_1)\times\ldots\times P(\varkappa^+,H_n)$ for $\xi<\varkappa^+$; we have to find ordinals $\xi<\eta<\varkappa^+$ such that

$$(E^\xi_1\cup E^\eta_1,\ldots,E^\xi_n\cup E^\eta_n)\in P(\varkappa^+,H_1)\times\ldots\times P(\varkappa^+,H_n)\,,$$

i.e., $[E^\xi_i\cup E^\eta_i]^2\subset H_i$ for $1\leqslant i\leqslant n$. By Lemma 2.11, we can assume that there are sets $E_1,\ldots,E_n$ such that $E^\xi_i\cap E^\eta_i=E_i$ for $1\leqslant i\leqslant n$ and $\xi<\eta<\varkappa^+$. Let $F^\xi_i=E^\xi_i\setminus E_i$; since $[E^\xi_i\cup E^\eta_i]^2=[E^\xi_i]^2\cup[E^\eta_i]^2\cup(F^\xi_i\otimes F^\eta_i)$ for $\xi\neq\eta$, and since $[E^\xi_i]^2\cup[E^\eta_i]^2\subset H_i$, it will suffice to find ordinals $\xi<\eta<\varkappa^+$ such that $F^\xi_i\otimes F^\eta_i\subset H_i$ for $1\leqslant i\leqslant n$. Let $F^\xi=F^\xi_1\cup\ldots\cup F^\xi_n$; since $K_\nu\subset H_1\cap\ldots\cap H_n$, it will suffice to find ordinals $\xi<\eta<\varkappa^+$ such that $F^\xi\otimes F^\eta\subset K_\nu$. Since the sets F^ξ_i for fixed i are pairwise disjoint, we have $|\{\xi\colon \alpha\in F^\xi\}|\leqslant n$ for every $\alpha\in\varkappa^+$; hence there is an ordinal $\mu<\varkappa^+$ such that $F^\xi_\mu=F^\xi$ for all $\xi<\varkappa$. Choose $\beta<\varkappa^+$ so that $\mu<\beta$, $\nu<\beta$, and $\bigcup\limits_{\xi<\varkappa}F^\xi_\mu\subset\beta$; choose η so that $\varkappa\leqslant\eta<\varkappa^+$ and $F^\eta\cap\beta=\emptyset$. Now, as in the proof of Theorem 3.3, we can use the condition (*) to show that $F^\xi\otimes F^\eta\subset K_\nu$ for some $\xi<\varkappa$. This completes the proof of Lemma 4.1.

4.2. Lemma. *Assume* MA *and let $\varkappa<2^{\aleph_0}$. Suppose we are given a set A and finite sets $E^\xi_\varrho\subset A$ $(\xi<\omega,\ \varrho<\varkappa)$ such that, for every $\varrho<\varkappa$ and every $a\in A$, $|\{\xi\colon a\in E^\xi_\varrho\}|<\aleph_0$. Then there are pairwise disjoint sets $A_n\subset A$ $(n<\omega)$ such that*

$$\forall n<\omega\ \forall\varrho<\varkappa\ |\{\xi\colon E^\xi_\varrho\subset A_n\}|=\aleph_0\,.$$

The proof of Lemma 4.2 is a straightforward application of Martin's axiom.

4.3. Lemma. *Assuming* MA, *there are $\aleph_0$ pairwise disjoint big subsets of $[2^{\aleph_0}]^2$.*

Proof. Let $(\mathscr{F}_\mu\colon \mu<2^{\aleph_0})$ enumerate all ω-sequences $(F^\xi\colon \xi<\omega)$ of finite subsets of $2^{\aleph_0}$ such that, for every $\alpha\in 2^{\aleph_0}$, $|\{\xi\colon \alpha\in F^\xi\}|<\aleph_0$; $\mathscr{F}_\mu=(F^\xi_\mu\colon \xi<\omega)$. Imitating the proof of Lemma 4.1, with Lemma 4.2 taking the place of Lemma 3.2, we can construct pairwise disjoint sets $K_n\subset[2^{\aleph_0}]^2$ $(n<\omega)$ satisfying the following condition for every $\alpha<2^{\aleph_0}$:

(*) if $n<\omega$, $\mu<\alpha$, $\bigcup\limits_{\xi<\omega}F^\xi_\mu\subset\alpha$, $X\in[\alpha]^{<\aleph_0}$, and $|\{\xi\colon F^\xi_\mu\otimes X\subset K_n\}|=\aleph_0$, then

$$|\{\xi\colon F^\xi_\mu\otimes(X\cup\{\alpha\})\subset K_n\}|=\aleph_0\,.$$

Now, as in the proof of Theorem 3.3 or Lemma 4.1, we can use (*) to show that the sets K_n are big.

It should be noted that Lemma 4.3 is only useful in case $2^{\aleph_0}$ is weakly inaccessible. If MA holds and $2^{\aleph_0}$ is not weakly inaccessible, then we have $2^{\aleph_0}=\varkappa^+=2^\varkappa$ for some $\varkappa$ [14, Theorem 1, p. 164], and it follows from Lemma 4.1 that there are $2^{\aleph_0}$ pairwise disjoint big subsets of $[2^{\aleph_0}]^2$. The conclusion of Lemma 4.3 can not be improved without some additional assumption, since Martin and Solovay have proved that, if the existence of an uncountable measurable cardinal is consistent, then MA is consistent with the assumption that $2^{\aleph_0}$ carries a nontrivial $\aleph_1$-saturated $2^{\aleph_0}$-complete ideal [14, p. 175]. Now, by a theorem of Solovay [18, Theorem 5, p. 406], if $2^{\aleph_0}$ carries such an ideal, then $2^{\aleph_0}\to[2^{\aleph_0}]^2_{\aleph_1}$; in particular, there is no family of $\aleph_1$ pairwise disjoint big subsets of $[2^{\aleph_0}]^2$.

4.4. Lemma. *Let λ be an infinite cardinal, and let $\mathscr{S}$ be a family of pairwise disjoint nonempty finite subsets of 2^λ. Then there are sets $N_\alpha\subset\lambda$ $(\alpha<2^\lambda)$ such that, for every finite set $A\subset 2^\lambda$, we have $\bigcap\limits_{\alpha\in A}N_\alpha=\emptyset$ if and only if A contains a member of $\mathscr{S}$.*

Proof. By a theorem of Hausdorff (see [11] or [2, Theorem 3.16, p. 76]), there are functions $f_\nu\colon 2^\lambda\to\{0,1\}$ $(\nu<\lambda)$ such that, given any finite set $A\subset 2^\lambda$ and any function $g\colon A\to\{0,1\}$, there exists $\nu<\lambda$ such that $f_\nu(\alpha)=g(\alpha)$ for all $\alpha\in A$. For $\nu<\lambda$, define $B_\nu=\{\alpha\colon f_\nu(\alpha)=0\}$ and $C_\nu=B_\nu\setminus\bigcup\{S\in\mathscr{S}\colon S\subset B_\nu\}$. Finally, for $\alpha<2^\lambda$, let $N_\alpha=\{\nu<\lambda\colon \alpha\in C_\nu\}$. Consider any finite set $A\subset 2^\lambda$; we show that $\bigcap\limits_{\alpha\in A}N_\alpha\neq\emptyset$ if and only if A contains no member of $\mathscr{S}$. First, suppose $\nu\in\bigcap\limits_{\alpha\in A}N_\alpha$; then $A\subset C_\nu$, and it is clear from the definition of C_ν that C_ν contains no member of $\mathscr{S}$. Conversely, suppose that A contains no member of $\mathscr{S}$. Then we can choose $\nu<\lambda$ so that f_ν vanishes identically on A, but does not vanish identically on any of the (finitely many) sets $S\in\mathscr{S}$ such that $S\cap A\neq\emptyset$. Then $A\subset C_\nu$, i.e., $\nu\in\bigcap\limits_{\alpha\in A}N_\alpha$.

In fact, the assumption in Lemma 4.4, that the members of $\mathcal{S}$ are pairwise disjoint, could be weakened considerably; however, the lemma as stated is sufficiently general for our purposes.

4.5. Lemma. *Assume that either $\varkappa \geqslant \aleph_0$ and $\lambda = 2^\varkappa = \varkappa^+$, or else $\lambda = \varkappa = \aleph_0$ and MA holds. Let $\mathcal{S}$ be any family of pairwise disjoint nonempty finite subsets of 2^λ. Then there are partially ordered sets P_α $(\alpha < 2^\lambda)$ such that, for any indexed family $\alpha_i < 2^\lambda$ $(i \in I)$, the product $\prod_{i\in I} P_{\alpha_i}$ satisfies the $2^\varkappa$-c.c. if and only if the set $\{\alpha_i : i \in I\}$ contains no member of $\mathcal{S}$.*

Proof. By Lemmas 4.1 and 4.3, there are pairwise disjoint big sets $K_\nu \subset [2^\varkappa]^2$ $(\nu < \lambda)$. By Lemma 4.4, there are sets $N_\alpha \subset \lambda$ $(\alpha < 2^\lambda)$ such that, for every finite set $A \subset 2^\lambda$, we have $\bigcap_{\alpha\in A} N_\alpha = \emptyset$ if and only if A contains a member of $\mathcal{S}$. For $\alpha < 2^\lambda$ let $P_\alpha = P(2^\varkappa, H_\alpha)$, where $H_\alpha = \bigcup_{\nu\in N_\alpha} K_\nu$. Now suppose $\alpha_i < 2^\lambda$ for $i \in I$; we have to show that the product $\prod_{i\in I} P_{\alpha_i}$ satisfies the $2^\varkappa$-c.c. if and only if the set $\{\alpha_i : i \in I\}$ contains no member of $\mathcal{S}$.

Suppose $\{\alpha_i : i \in I\} \supset A \in \mathcal{S}$; then $A = \{\alpha_i : i \in I_0\}$ for some finite set $I_0 \subset I$. Since $A \in \mathcal{S}$, we have $\bigcap_{i\in I_0} N_{\alpha_i} = \emptyset$; moreover, since the sets K_ν are pairwise disjoint, we also have $\bigcap_{i\in I_0} H_{\alpha_i} = \emptyset$. For $\xi < 2^\varkappa$ define $f_\xi \in \prod_{i\in I} P_{\alpha_i}$ so that $f_\xi(i) = \{\xi\}$ for $i \in I_0$, $f_\xi(i) = \emptyset$ for $i \in I \setminus I_0$. Then $\{f_\xi : \xi < 2^\varkappa\}$ is a set of $2^\varkappa$ pairwise incompatible elements of $\prod_{i\in I} P_{\alpha_i}$.

Now suppose $\{\alpha_i : i \in I\}$ contains no member of $\mathcal{S}$; we have to show that $\prod_{i\in I} P_{\alpha_i}$ satisfies the $2^\varkappa$-c.c. By Corollary 2.10, it will suffice to show that $\prod_{i\in F} P_{\alpha_i}$ satisfies the $2^\varkappa$-c.c. for every finite set $F \subset I$. Since $\{\alpha_{i_1}, \ldots, \alpha_{i_n}\}$ contains no member of $\mathcal{S}$, we have $N_{\alpha_{i_1}} \cap \ldots \cap N_{\alpha_{i_n}} \neq \emptyset$. Choose $\nu \in N_{\alpha_{i_1}} \cap \ldots \cap N_{\alpha_{i_n}}$; then $K_\nu \subset H_{\alpha_{i_1}} \cap \ldots \cap H_{\alpha_{i_n}}$. Since K_ν is a big subset of $[2^\varkappa]^2$, it follows that $P(2^\varkappa, H_{\alpha_{i_1}}) \times \ldots \times P(2^\varkappa, H_{\alpha_{i_n}})$ satisfies the $2^\varkappa$-c.c., i.e., $\prod_{i\in F} P_{\alpha_i}$ satisfies the $2^\varkappa$-c.c. This completes the proof of Lemma 4.5.

The next lemma generalizes Lemma 4.5 by eliminating the requirement that the family $\mathcal{S}$ consist of pairwise disjoint sets.

4.6. Lemma. *Assume that either $\varkappa \geqslant \aleph_0$ and $\lambda = 2^\varkappa = \varkappa^+$, or else $\lambda = \varkappa = \aleph_0$ and MA holds. Let $\mathcal{S}$ be any family of nonempty finite subsets of 2^λ. Then there are partially ordered sets P_α $(\alpha < 2^\lambda)$ such that, for any indexed family $\alpha_i < 2^\lambda$ $(i \in I)$, the product $\prod_{i\in I} P_{\alpha_i}$ satisfies the $2^\varkappa$-c.c. if and only if the set $\{\alpha_i : i \in I\}$ contains no member of $\mathcal{S}$.*

Proof. Choose a 1-to-1 mapping $\varphi : \mathcal{S} \to 2^\lambda$, and define

$$\mathcal{S}' = \{S \times \{\varphi(S)\} : S \in \mathcal{S}\}.$$

Then $\mathcal{S}'$ is a family of pairwise disjoint nonempty finite subsets of $2^\lambda \times 2^\lambda$. By Lemma 4.5, there are partially ordered sets $P_{\alpha,\beta}$ $(\alpha, \beta < 2^\lambda)$ such that, for any indexed family $(\alpha_i, \beta_i) \in 2^\lambda \times 2^\lambda$ $(i \in I)$, the product $\prod_{i\in I} P_{\alpha_i,\beta_i}$ satisfies the $2^\varkappa$-c.c. if and only if the set $\{(\alpha_i, \beta_i) : i \in I\}$ contains no member of $\mathcal{S}'$. Now the partially ordered sets $P_\alpha = \prod_{\beta<2^\lambda} P_{\alpha,\beta}$ $(\alpha < 2^\lambda)$ have the desired properties.

Given partially ordered sets P_i $(i \in I)$, we define the sum

$$\sum_{i\in I} P_i = \{f \in \prod_{i\in I} P_i : |\{i : f(i) \neq 0\}| \leqslant 1\},$$

partially ordered by the rule

$$f \leqslant g \Leftrightarrow \forall i \in I\, f(i) \leqslant g(i).$$

In other words, $\sum_{i\in I} P_i$ is the union of copies of the P_i's which have the same zero element but are otherwise disjoint; for $i \neq j$, nonzero elements of P_i are incomparable with nonzero elements of P_j.

4.7. Theorem. *Assume that either $\varkappa \geqslant \aleph_0$ and $\lambda = 2^\varkappa = \varkappa^+$, or else $\lambda = \varkappa = \aleph_0$ and MA holds. Let $\mathcal{F}$ be any family of functions $f: 2^\lambda \to \omega$ such that*

$$0 < |\{\alpha : f(\alpha) \neq 0\}| < \aleph_0$$

for each $f \in \mathcal{F}$. Then there are partially ordered sets P_α $(\alpha < 2^\lambda)$ such that, for every cardinal-valued function g defined on 2^λ, the following statements are equivalent:

(1) $\prod_{\alpha<2^\lambda} P_\alpha^{g(\alpha)}$ *does not satisfy the $2^\varkappa$-c.c.;*

(2) $\exists f \in \mathcal{F}\ \forall \alpha < 2^\lambda\ f(\alpha) \leqslant g(\alpha)$.

Proof. For $f \in \mathcal{F}$, let $S_f = \{(\alpha, r) \in 2^\lambda \times \omega : r < f(\alpha)\}$; then $\mathcal{S} = \{S_f : f \in \mathcal{F}\}$ is a family of nonempty finite subsets of $2^\lambda \times \omega$. By Lemma 4.6, there are partially ordered sets $P_{\alpha,r}$ $(\alpha < 2^\lambda,\ r < \omega)$ such that, for any indexed family $(\alpha_i, r_i) \in 2^\lambda \times \omega$ $(i \in I)$, the product $\prod_{i\in I} P_{\alpha_i,r_i}$ satisfies the $2^\varkappa$-c.c. if and only if the set $\{(\alpha_i, r_i) : i \in I\}$ contains no member of $\mathcal{S}$. For $\alpha < 2^\lambda$ let $P_\alpha = \sum_{r<\omega} P_{\alpha,r}$.

We have to show that (1) and (2) are equivalent for every cardinal-valued function g defined on 2^λ.

Suppose (1) holds. Then, by Corollary 2.10, some finite partial product of $\prod_{\alpha<2^\lambda} P_\alpha^{g(\alpha)}$ (considered as a product of P_α's) fails to satisfy the 2^λ-c.c.; i.e., there exist $n < \omega$ and $\alpha_1, \ldots, \alpha_n < 2^\lambda$ such that each $\alpha < 2^\lambda$ occurs at most $g(\alpha)$ times in the sequence $\alpha_1, \ldots, \alpha_n$, and $P = P_{\alpha_1} \times \ldots \times P_{\alpha_n}$ does not satisfy the $2^\varkappa$-c.c. Now P is the union of countably many subsets, each isomorphic to $P_{\alpha_1,r_1} \times \ldots \times P_{\alpha_n,r_n}$ for some $r_1, \ldots, r_n < \omega$; hence there exist $r_1, \ldots, r_n < \omega$ such that $P_{\alpha_1,r_1} \times \ldots \times P_{\alpha_n,r_n}$ does not satisfy the $2^\varkappa$-c.c. Then we must have $S_f \subset \{(\alpha_1, r_1), \ldots, (\alpha_n, r_n)\}$ for some $f \in \mathcal{F}$. It follows that each $\alpha < 2^\lambda$ occurs at least $f(\alpha)$ times in the sequence $\alpha_1, \ldots, \alpha_n$. Hence $f(\alpha) \leqslant g(\alpha)$ for all $\alpha \leqslant 2^\lambda$; i.e., (2) holds.

To see that (2)⇒(1), we observe that the product $\prod_{(\alpha,r)\in S_f} P_{\alpha,r}$ does not satisfy the $2^{\varkappa}$-c.c., and is embedded in $\prod_{\alpha<2^{\lambda}} P_{\alpha}^{g(\alpha)}$ in such a way that incompatible elements of $\prod_{(\alpha,r)\in S_f} P_{\alpha,r}$ remain incompatible in $\prod_{\alpha<2^{\lambda}} P_{\alpha}^{g(\alpha)}$. This completes the proof of Theorem 4.7.

4.8. COROLLARY. *Assume that either $\varkappa\geqslant\aleph_0$ and $\lambda = 2^{\varkappa} = \varkappa^{+}$, or else $\lambda = \varkappa = \aleph_0$ and* MA *holds. Let $\mathscr{F}$ be any family of functions $f\colon 2^{\lambda}\to\omega$ such that*

$$0<|\{\alpha\colon f(\alpha)\neq 0\}|<\aleph_0$$

for each $f\in\mathscr{F}$. Then there are extremally disconnected compact Hausdorff spaces X_{α} $(\alpha<2^{\lambda})$ such that, for every cardinal-valued function g defined on 2^{λ}, the following statements are equivalent:

(1) $\prod_{\alpha<2^{\lambda}} X_{\alpha}^{g(\alpha)}$ *does not satisfy the $2^{\varkappa}$-c.c.;*

(2) $\exists f\in\mathscr{F}\ \forall\alpha<2^{\lambda}\ f(\alpha)\leqslant g(\alpha)$.

Proof. Let P_{α} $(\alpha<2^{\lambda})$ be the partially ordered sets given by Theorem 4.7 for the same $\varkappa$, λ, and $\mathscr{F}$. Put $X_{\alpha} = GX_{P_{\alpha}}$. By Lemma 2.4, each X_{α} is an extremally disconnected compact Hausdorff space, and $X_{\alpha}\equiv X_{P_{\alpha}}$. By Lemmas 2.7(2) and 2.6 we have $\prod_{\alpha<2^{\lambda}} X_{\alpha}^{g(\alpha)} \equiv \prod_{\alpha<2^{\lambda}} (X_{P_{\alpha}})^{g(\alpha)} \equiv X_P$ where $P = \prod_{\alpha<2^{\lambda}} P_{\alpha}$. It follows by Lemma 2.3(2) that $\prod_{\alpha<2^{\lambda}} X_{\alpha}^{g(\alpha)}$ satisfies the $2^{\varkappa}$-c.c. if and only if $\prod_{\alpha<2^{\lambda}} P_{\alpha}$ satisfies it, i.e., if and only if (2) holds.

4.9. COROLLARY. *Assume that either $\varkappa\geqslant\aleph_0$ and $2^{\varkappa} = \varkappa^{+}$, or else $\varkappa = \aleph_0$ and* MA *holds. Then, for every $n<\omega$, there is an extremally disconnected compact Hausdorff space X such that X^n satisfies the $2^{\varkappa}$-c.c. but X^{n+1} does not.*

Proof. In Corollary 4.8 take $\mathscr{F} = \{f\}$ where $f(0) = n+1$, $f(\alpha) = 0$ for $0<\alpha<2^{\lambda}$.

The next theorem shows that, in the case $\lambda = 2^{\varkappa} = \varkappa^{+}$, Theorem 4.7 is best possible, in the sense that we could not replace 2^{λ} by a larger cardinal.

4.10. THEOREM. *If λ is any infinite cardinal, then there do not exist partially ordered sets P_{α} and Q_{α} $(\alpha<(2^{\lambda})^{+})$ such that $P_{\alpha}\times Q_{\alpha}$ fails to satisfy the λ-c.c. for $\alpha<(2^{\lambda})^{+}$, while $P_{\alpha}\times Q_{\beta}$ satisfies the λ-c.c. for $\alpha<\beta<(2^{\lambda})^{+}$.*

Proof. Suppose that $P_{\alpha}\times Q_{\alpha}$ fails to satisfy the λ-c.c. for $\alpha<(2^{\lambda})^{+}$. For each α, choose a subset $P'_{\alpha}\subset P_{\alpha}$ such that: $P'_{\alpha}\times Q_{\alpha}$ contains a set of λ pairwise incompatible elements; any two elements of P'_{α}, which are compatible in P_{α}, are also compatible in P'_{α}; and $|P_{\alpha}|\leqslant\lambda$. Since there are only 2^{λ} nonisomorphic partially ordered sets of cardinality λ, we can choose $\alpha<\beta<(2^{\lambda})^{+}$ so that P'_{α} and P'_{β} are isomorphic. Hence $P'_{\alpha}\times Q_{\beta}$ is isomorphic to $P'_{\beta}\times Q_{\beta}$. Since $P'_{\beta}\times Q_{\beta}$ does not satisfy the λ-c.c., neither does $P'_{\alpha}\times Q_{\beta}$. Since incompatible elements of $P'_{\alpha}\times Q_{\beta}$ remain incompatible in $P_{\alpha}\times Q_{\beta}$, it follows that $P_{\alpha}\times Q_{\beta}$ does not satisfy the λ-c.c.

Recall that, if P is a partially ordered set, then P^* consists of all pairwise compatible finite subsets of P, partially ordered by inclusion.

4.11. LEMMA. *Let λ be a cardinal with $\mathrm{cf}\,\lambda>\omega$, let $r<\omega$, and let $P_1, \ldots, P_r$ be partially ordered sets. If $P_1^n\times\ldots\times P_r^n$ satisfies the λ-c.c. for every $n<\omega$, then $P_1^*\times\ldots\times P_r^*$ satisfies the λ-c.c.*

Proof. Suppose $(F_1^{\xi}, \ldots, F_r^{\xi})\in P_1^*\times\ldots\times P_r^*$ for $\xi<\lambda$; we have to show that, for some ordinals $\xi<\eta<\lambda$, we have $(F_1^{\xi}\cup F_1^{\eta}, \ldots, F_r^{\xi}\cup F_r^{\eta})\in P_1^*\times\ldots\times P_r^*$. Since $\mathrm{cf}\,\lambda>\omega$, we can assume that, for each $i\in\{1, \ldots, r\}$, there is a number $m_i<\omega$ such that $|F_i^{\xi}| = m_i$ for all $\xi<\lambda$. Let $n_i = \binom{m_i+1}{2}$, and choose a 1-to-1 mapping

$$\varphi_i\colon \{\{j,k\}\colon 1\leqslant j\leqslant k\leqslant m_i\}\to\{1, \ldots, n_i\}.$$

Write $F_i^{\xi} = \{x_{i,1}^{\xi}, \ldots, x_{i,m_i}^{\xi}\}$. Since the elements of F_i^{ξ} are pairwise compatible in P_i, we can choose $y_i^{\xi} = (y_{i,1}^{\xi}, \ldots, y_{i,n_i}^{\xi})\in P_i^{n_i}$ so that $x_{i,j}^{\xi}, x_{i,k}^{\xi}\leqslant y_{i,\varphi_i(\{j,k\})}^{\xi}$ for $1\leqslant j\leqslant k\leqslant m_i$. Let $y^{\xi} = (y_1^{\xi}, \ldots, y_r^{\xi})\in P_1^{n_1}\times\ldots\times P_r^{n_r}$. By hypothesis, $P_1^{n_1}\times\ldots\times P_r^{n_r}$ satisfies the λ-c.c.; hence y^{ξ} and y^{η} are compatible for some $\xi<\eta<\lambda$. We have to show that $(F_1^{\xi}\cup F_1^{\eta}, \ldots, F_r^{\xi}\cup F_r^{\eta})\in P_1^*\times\ldots\times P_r^*$, i.e., that $x_{i,j}^{\xi}$ and $x_{i,k}^{\eta}$ are compatible in P_i for $1\leqslant i\leqslant r$, $1\leqslant j\leqslant m_i$, and $1\leqslant k\leqslant m_i$. In fact, let $h = \varphi_i(\{j,k\})$; then $x_{i,j}^{\xi}\leqslant y_{i,h}^{\xi}$ and $x_{i,k}^{\eta}\leqslant y_{i,h}^{\eta}$. Since $y_{i,h}^{\xi}$ and $y_{i,h}^{\eta}$ are compatible in P_i, so are $x_{i,j}^{\xi}$ and $x_{i,k}^{\eta}$.

4.12. LEMMA. *Consider the following statements for a fixed cardinal λ:*

(1) *there are partially ordered sets P_0, P_1, and P_2 such that $P_0^n\times P_1^n$, $P_0^n\times P_2^n$, and $P_1^n\times P_2^n$ satisfy the λ-c.c. for every $n<\omega$, but $P_0\times P_1\times P_2$ does not satisfy the λ-c.c.;*

(2) *there is a partially ordered set P such that P satisfies the λ-c.c., but P^* does not;*

(3) *there are partially ordered sets P and Q such that P and Q satisfy the λ-c.c., but $P\times Q$ does not.*

If $\mathrm{cf}\,\lambda>\omega$, then (1)⇒(2)⇒(3).

Proof. First suppose there are partially ordered sets P_0, P_1, P_2 as in (1). Define a subset $P = Q_0\cup Q_1\cup Q_2\subset P_0^*\times P_1^*\times P_2^*$, where

$$Q_i = \{(F_0, F_1, F_2)\in P_0^*\times P_1^*\times P_2^*\colon F_i = \varnothing\}.$$

Then Q_0, Q_1 and Q_2 are isomorphic, respectively, to $P_1^*\times P_2^*$, $P_0^*\times P_2^*$, and $P_0^*\times P_1^*$. It follows from Lemma 4.11 that Q_0, Q_1, and Q_2 satisfy the λ-c.c., and consequently P satisfies the λ-c.c. Choose pairwise incompatible elements $(x_0^{\xi}, x_1^{\xi}, x_2^{\xi})\in P_0\times P_1\times P_2$ $(\xi<\lambda)$, and define

$$E^{\xi} = \{(\{x_0^{\xi}\}, \varnothing, \varnothing), (\varnothing, \{x_1^{\xi}\}, \varnothing), (\varnothing, \varnothing, \{x_2^{\xi}\})\}.$$

Then $\{E^{\xi}\colon \xi<\lambda\}$ is a set of λ pairwise incompatible elements of P^*. Now suppose there is a partially ordered set P satisfying the λ-c.c., such that P^* does not satisfy the λ-c.c. Then, by Lemma 4.11, there is a number $n<\omega$ such that P^n does not satisfy the λ-c.c. Let n be the least such number, and let $Q = P^{n-1}$; then P and Q satisfy the λ-c.c., but $P\times Q$ does not.

4.13. THEOREM. *Assume that either $\varkappa \geq \aleph_0$ and $2^\varkappa = \varkappa^+$, or else $\varkappa = \aleph_0$ and MA holds. Then there is a partially ordered set P such that P satisfies the $2^\varkappa$-c.c., but P^* does not.*

Proof. By Theorem 4.7 (or Lemma 4.5), there are partially ordered sets P_0, P_1, and P_2 satisfying (1) of Lemma 4.12 for $\lambda = 2^\varkappa$.

§ 5. **A positive result.** Let us define a special case of the "square bracket" partition relation of Erdös, Hajnal, and Rado [4, p. 144]. For cardinals $\varkappa$ and λ, the symbol $\lambda \to [\lambda]^2_\varkappa$ denotes the following statement: for any partition $[A]^2 = \bigcup_{i \in I} K_i$ where $|A| = \lambda$ and $|I| = \varkappa$, there exist $B \subset A$ and $i_0 \in I$ such that $|B| = \lambda$ and $[B]^2 \subset \bigcup_{i \in I \setminus \{i_0\}} K_i$. For $n < \omega$, the relation $\lambda \to [\lambda]^2_{n+1}$ is easily seen to be equivalent to the following: for any partition $[A]^2 = \bigcup_{i \in I} K_i$ where $|A| = \lambda$ and $|I| < \aleph_0$, there exists $B \subset A$ and $J \subset I$ such that $|B| = \lambda$, $|J| \leq n$, and $[B]^2 \subset \bigcup_{i \in J} K_i$.

In this section we prove that, if two partially ordered sets (or topological spaces) satisfy the λ-c.c., where λ is a cardinal such that $\lambda \to [\lambda]^2_3$, then their product also satisfies the λ-c.c. The only nontrivial case is when λ is a regular cardinal such that $\lambda \to [\lambda]^2_3$ but $\lambda \nrightarrow [\lambda]^2_3$; unfortunately, 2 do not know if the existence of such cardinals is consistent with ZFC. It is known that $\lambda \to [\lambda]^2_3$ does not hold if λ is $\aleph_1$, $2^{\aleph_0}$, or cf $2^{\aleph_0}$; in fact $\mathrm{cf}\,2^{\aleph_0} \nrightarrow [\mathrm{cf}\,2^{\aleph_0}]^2_{\aleph_0}$ [8, Lemma 4, p. 170], $2^{\aleph_0} \nrightarrow [2^{\aleph_0}]^2_{\aleph_0}$ [8, Theorem 1, p. 170], and $\aleph_1 \nrightarrow [\aleph_1]^2_4$ [8, Theorem 2, p. 171].

5.1. LEMMA. *Let λ be a regular cardinal such that $\lambda \to [\lambda]^2_3$. If a partially ordered set P satisfies the λ-c.c., then P also satisfies the condition $Q(1, \lambda^+, \lambda, 3)$, i.e., every subset of P of cardinality λ contains a pairwise compatible subset of cardinality λ.*

Proof. Let P be a partially ordered set satisfying the λ-c.c., and let $A \subset P$, $|A| = \lambda$. The conclusion is trivial if $\lambda \to [\lambda]^2_2$, so we assume that $\lambda \nrightarrow [\lambda]^2_2$; then, by a theorem of Hanf [10], there is a linear ordering $<$ of A such that no subset of A of cardinality λ is well-ordered by $<$ or $>$. Choose a 1-to-1 mapping $\varphi\colon A \to \lambda$. Define

$$H_1 = \{\{x, y\} \in [A]^2\colon x < y \text{ and } \varphi(x) < \varphi(y)\};$$

$$H_2 = \{\{x, y\} \in [A]^2\colon x < y \text{ and } \varphi(x) > \varphi(y)\};$$

$$K_1 = \{\{x, y\} \in [A]^2\colon x \text{ and } y \text{ are compatible in } P\};$$

$$K_2 = \{\{x, y\} \in [A]^2\colon x \text{ and } y \text{ are incompatible in } P\}.$$

Then $[A]^2 = H_1 \cup H_2 = K_1 \cup K_2$, so we have

$$[A]^2 = (H_1 \cap K_1) \cup (H_1 \cap K_2) \cup (H_2 \cap K_1) \cup (H_2 \cap K_2).$$

By $\lambda \to [\lambda]^2_3$, there is a subset $B \subset A$ with $|B| = \lambda$, such that $[B]^2$ is covered by 2 of the 4 classes $H_1 \cap K_1$, $H_1 \cap K_2$, $H_2 \cap K_1$, $H_2 \cap K_2$. We can not have $[B]^2 \subset H_1$ or $[B]^2 \subset H_2$, as B would then be well-ordered by $<$ or $>$, respectively; therefore we have $[B]^2 \subset (H_1 \cap K_i) \cup (H_2 \cap K_j)$ for some $i, j \in \{1, 2\}$. We want to show that $[B]^2 \subset K_1$, i.e., that $i = j = 2$. Suppose, on the contrary, that $i = 2$ or $j = 2$. We must also have $i = 1$ or $j = 1$; otherwise we would have $[B]^2 \subset K_2$, contradicting the assumption that P satisfies the λ-c.c. Thus $\{i, j\} = \{1, 2\}$; by symmetry we can assume that $i = 1$ and $j = 2$. This means that, if $x, y \in B$, $x < y$, then x and y are compatible in P if and only if $\varphi(x) < \varphi(y)$. By a lemma of Erdös and Rado [5, Lemma 1, p. 446], there are sets $C, D \subset B$ such that $|C| = |D| = \lambda$ and $c < d$ for all $c \in C$, $d \in D$. Since λ is regular and φ is 1-to-1, we can choose $c_\xi \in C$ and $d_\xi \in D$ $(\xi < \lambda)$ so that $\varphi(c_\xi) < \varphi(d_\xi) < \varphi(c_\eta)$ for $\xi < \eta < \lambda$. For each $\xi < \lambda$, c_ξ and d_ξ are compatible in P, since $c_\xi < d_\xi$ and $\varphi(c_\xi) < \varphi(d_\xi)$; let $x_\xi \in P$ be a common upper bound for c_ξ and d_ξ in the partial ordering of P. Since P satisfies the λ-c.c., x_η and x_η must be compatible for some $\xi < \eta < \lambda$. Then c_η and d_ξ are compatible; but this is a contradiction, since $c_\eta < d_\xi$ and $\varphi(c_\eta) > \varphi(d_\xi)$.

5.2. THEOREM. *Let λ be a cardinal such that $\lambda \to [\lambda]^2_3$. If two partially ordered sets P_1 and P_2 satisfy the λ-c.c., then their product $P_1 \times P_2$ satisfies the λ-c.c.*

Proof. If λ is regular, the result follows easily from Lemma 5.1. We assume, then, that λ is singular, $\mathrm{cf}\lambda = \varkappa < \lambda$. Let P_1 and P_2 be partially ordered sets satisfying the λ-c.c., and suppose there are λ pairwise incompatible elements

$$(x_1^\xi, x_2^\xi) \in P_1 \times P_2 \quad (\xi < \lambda).$$

Choose a mapping $\varphi\colon \lambda \to \varkappa$ which is not constant on any set of cardinality λ. Define

$$H_1 = \{\{\xi, \eta\} \in [\lambda]^2\colon \varphi(\xi) = \varphi(\eta)\};$$

$$H_2 = \{\{\xi, \eta\} \in [\lambda]^2\colon \varphi(\xi) \neq \varphi(\eta)\};$$

$$K_1 = \{\{\xi, \eta\} \in [\lambda]^2\colon x_1^\xi \text{ and } x_1^\eta \text{ are incompatible}\};$$

$$K_2 = \{\{\xi, \eta\} \in [\lambda]^2\colon x_2^\xi \text{ and } x_2^\eta \text{ are incompatible}\}.$$

Then $[\lambda]^2 = H_1 \cup H_2 = K_1 \cup K_2$, so we have

$$[\lambda]^2 = (H_1 \cap K_1) \cup (H_1 \cap K_2) \cup (H_2 \cap K_1) \cup (H_2 \cap K_2).$$

By $\lambda \to [\lambda]^2_3$, there is a set $B \subset \lambda$ with $|B| = \lambda$, such that $[B]^2 \subset (H_1 \cap K_i) \cup (H_2 \cap K_j)$ for some $i, j \in \{1, 2\}$. By symmetry we can assume that $i = 1$. Then $[B]^2 \cap H_1 \subset K_1$; i.e., if $\xi, \eta \in B$, $\xi \neq \eta$, and $\varphi(\xi) = \varphi(\eta)$, then x_1^ξ and x_1^η are incompatible. Now, by a theorem of Erdös and Tarski [6, Theorem 1, p. 320], since P_1 satisfies the λ-c.c. and λ is singular, it follows that P_1 satisfies the λ_0-c.c. for some cardinal $\lambda_0 < \lambda$. Since $|B| = \lambda$, there is a subset $B_0 \subset B$ with $|B_0| = \lambda_0$, such that φ is constant on B_0. Then $\{x_1^\xi\colon \xi \in B_0\}$ is a set of λ_0 pairwise incompatible elements of P_1, contradicting the fact that P_1 satisfies the λ_0-c.c.

5.3. THEOREM. *Let λ be a cardinal such that $\lambda \to [\lambda]^2_3$. If two topological spaces X_1 and X_2 satisfy the λ-c.c., then their product $X_1 \times X_2$ satisfies the λ-c.c.*

Proof. Theorems 5.2 and 2.8.

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Continua whose hyperspace is a product

by

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Abstract. Let X be a nondegenerate metric continuum. By the *hyperspace of* X is meant $C(X) = \{A\colon A$ is a nonempty subcontinuum of $X\}$ with the Hausdorff metric. An investigation is made of when $C(X)$ is homeomorphic to a cartesian product of nondegenerate continua. Some examples are given using techniques in infinite-dimensional topology and some unsolved problems are stated.

1. Introduction. In [21] are some results concerning the structure of all finite-dimensional continua whose hyperspace and cone are homeomorphic. Among other results, I showed that there are *exactly eight* such hereditarily decomposable continua [21, (1.1)] and that for such continua which are indecomposable, each proper subcontinuum is an arc. In [25] we showed that any finite-dimensional continuum whose hyperspace and suspension are homeomorphic must be an arc. Using [15, 5.4], 9.7 of [10] may be restated as follows:

(1.1) THEOREM [10]. *Let X be a locally connected continuum. If $C(X)$ is a finite-dimensional cartesian product of* (*nondegenerate*) *continua, then X is an arc or a circle* (*and conversely*).

The above-mentioned results provide the principal motivation for the following question:

(Q) For what continua X is $C(X)$ homeomorphic to a cartesian product (of nondegenerate continua)?

In this paper I give some answers to (Q). The next section is devoted to giving some general results, and some complete answers to (Q) in some special cases. In Section 3, I consider the situation when X is locally connected and $C(X)$ is an infinite-dimensional cartesian product. The section contains several examples and Theorem (3.15) which hopefully [see (3.19)] will lead to a characterization completely answering (Q).

I adopt the following notation. The term *nondegenerate* means consisting of more than one point. The letters X, Y, and Z always denote continua (a *continuum* is a nondegenerate compact connected metric space). I refer the reader to [15] for preliminary information about the space $C(X)$. Whenever I say $C(X)$ *is a cartesian product*, I mean that $C(X)$ is homeomorphic to the cartesian product of