Proof of Theorem 0.1. We consider the diagram of the previous proposition. Since $\mathrm{Td}_k \in I^{2k}_{2k+1}$, by the very description of th_k that we gave, the image of th_k in this diagram consists of integral elements. Since $\mu\colon \mathrm{MU}^0(Y) \to \mathrm{KU}^0(Y)$ is onto it is clear that the image of ch_k, in the diagram, consists of integral elements, which is exactly what we want to prove.

References

[1] M. F. Atiyah and F. Hirzebruch, *Vector bundles and homogeneous spaces*, Proc. of Symposia in Pure and App. Math, Vol. III, Diff. Geom. American Math. Soc. 1961.

[2] E. H. Brown and F. P. Peterson, *Relations among characteristic classes*, Topology 3 (1964), pp. 39–52.

[3] P. E. Conner and E. E. Floyd, *The Relation of Cobordism to K-Theories*, Springer–Verlag, Lecture Notes in Math. 28 (1966).

[4] E. Dyer, *Chern character of certain complexes*, Math. Zeitschr. 80 (1963), pp. 363–373.

[5] A. Hattori, *Integral characteristic numbers for weakly almost complex manifolds*, Topology 5 (1966), pp. 259–280.

[6] P. J. Hilton and E. H. Spanier, *On the embeddability of certain complexes in Euclidean spaces*, Proc. Amer. Math. Soc. 11 (1960), pp. 523–526.

[7] F. Hirzebruch, *Topological methods in algebraic geometry*, Springer–Verlag, New York 1966.

[8] J. Milnor and J. D. Stasheff, *Characteristic classes*, Princeton University Press, Princeton N. J., 1974.

[9] F. P. Peterson, *Some non-embedding problems*, Bol. Soc. Mat. Mexicana 2 (2) (1957), pp. 9–15.

[10] — N. Stein, *The dual of a secondary cohomology operations*, III, J. of Mathematics 4 (1960), pp. 397–404.

[11] L. Smith, *The Todd character and the integrability Theorem for the Chern character*, III, J. of Mathematics 17 (1973), pp. 301–310.

[12] R. E. Stong, *Relations among characteristic numbers I*, Topology 4 (1965), pp. 267–281.

[13] — *Notes on cobordism theory*, Princeton Univ. Press 1968.

[14] G. W. Whitehead, *Recent advances in homotopy theory*, Reg. Conference Series in Mathematics 5, Amer. Math. Soc. 1970.

UNIVERSITY OF ATHENS
MATHEMATICAL INSTITUTE
Athens

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Retraceable homogeneous sets *

by

T. G. McLaughlin (Lubbock, Tex.)

Abstract. We show that every recursive partition $\langle P_1, P_2\rangle$ of $[N]^2$ admits a retraceable infinite homogeneous set X whose Turing jump is $\leqslant 0''$; this refines a result of Jockusch [1] by adding the condition of retraceability (in fact, retraceability by a finite-one retracing function).

1. Introduction. This article is intended as a contribution to the "fine analysis" of Ramsey's Theorem in terms of recursion-theoretic notions. Previous analyses of this sort ([3], [4]) led to Jockusch's paper [1] which from one point of view can be regarded as the last word on the subject. Further words can be said, however, if we consider other "descriptive" notions in place of or in addition to *classification within the Kleene hierarchy*. One such notion is that of retraceability: recall that an infinite sequence $a_0, a_1, a_2, \ldots$ of natural numbers is said to be *retraceable* if $(\forall n)[a_n < a_{n+1}]$ & there is a partial recursive function p such that $(\forall n)[p(a_{n+1})$ is defined and $= a_n]$. We shall replace Π^0_2 *classification*, in [1, Theorem 4.2], by the combination of retraceability and a fairly strong condition on jumps of Turing degrees. It remains an open question (as far as we know) whether retraceability can be *added to* Π^0_2 representability, in Theorem 4.2 of [1]; a brief discussion of this question is included at the end of the paper. Our terminology and notation, where not explicitly defined or entirely standard, is in line with that of [1].

2. Recursive partitions and a theorem of Jockusch. If N = the natural numbers and $X \subseteq N$ is infinite, then $[X]^2$ denotes the set $\{\{x, y\} \mid x \in X \,\&\, y \in X \,\&\, x \neq y\}$. The classical theorem of Ramsey asserts that if $[N]^2$ is divided into complementary subsets P_1 and P_2 then there is an infinite set $X \subseteq N$ such that either $[X]^2 \subseteq P_1$ or $[X]^2 \subseteq P_2$; such an X is called a *homogeneous set* (or a *set of indiscernibles*) for the partition $\langle P_1, P_2\rangle$. One can ask about the degree of constructivity possible for X in case P_1 and P_2 are *recursive* sets of pairs. This question, first dealt with by Specker [3], has been answered in a definitive way relative to the Kleene hierarchy in Jockusch [1] (which paper, in addition, contains instructive commentary on what is probably the most elegant possible proof (due independently to various mathematicians) of the

* A version of the central result of this paper, Theorem 3, was independently proved by Gordon Phillips, a student of Jockusch. Our inquiries have led us to conclude that Phillips does not presently intend to publish his proof.

classical Ramsey Theorem). In this section, we shall give a proof of the "tree-theoretic" form of [1, Proposition 4.6]; the tree construction used for this purpose will also yield our principal theorem in § 3. Our proof will be longer and messier than the corresponding argument in [1], for three reasons: (1) our style is inherently longer and messier than that of Jockusch; (2) we wish to place detailed emphasis on tree structure; and (3) an explicit verification of the classical Ramsey Theorem (for recursive partitions) is included in our proof. By a *retracing tree*, we mean a set $\mathcal{T}$ of ordered pairs of natural numbers constituting a *function*, such that (i) $\varrho\mathcal{T}\subseteq\delta\mathcal{T}$ and (ii) $(\forall x)[x\in\delta\mathcal{T}\Rightarrow\mathcal{T}(x)\leqslant x]$; here "$\varrho$" denotes range while "$\delta$" denotes domain. A retracing tree $\mathcal{T}$ is called *recursive* (*recursively enumerable*) if $\mathcal{T}$ is recursive (recursively enumerable) as a set of ordered pairs; $\mathcal{T}$ is *strongly recursive* if it is recursive and $\delta\mathcal{T}$ is a recursive subset of N. $\mathcal{T}$ is *non-trivial* if it has at least one infinite branch, where by a *branch* is meant a maximal subset A of $\delta\mathcal{T}$ such that

$$(\forall x)(\forall y)[(x\in A \,\&\, y\in A) \Rightarrow (x\in\hat{\mathcal{T}}(y)\vee y\in\hat{\mathcal{T}}(x))]\,;$$

here $\hat{\mathcal{T}}(x)$, for $x\in\delta\mathcal{T}$, is defined as $\{x,\mathcal{T}(x),\mathcal{T}^2(x),\ldots,\mathcal{T}^{\mathcal{T}^*(x)}(x)\}$ where $\mathcal{T}^*(x)\underset{\mathrm{df}}{=}$ the least n such that $\mathcal{T}^{n+1}(x)=\mathcal{T}^n(x)$. As usual, we denote by $\underset{\sim}{\alpha}$ the Turing degree of a set α of natural numbers and by $\underset{\sim}{f}$ the Turing degree of a (total) number-theoretic function f; $\underset{\sim}{\alpha}'$ ($\underset{\sim}{f}'$) denotes the *jump* of $\underset{\sim}{\alpha}$ (of $\underset{\sim}{f}$). If β is an infinite set of numbers, p_β denotes the function which enumerates β in increasing order.

THEOREM 1 (Jockusch). *Let* $\langle P_1,P_2\rangle$ *be a recursive partition of* $[N]^2$. *Then there exists a non-trivial recursive retracing tree* $\mathcal{T}$ *such that* $(\forall\beta[(\beta=$ *an infinite branch of* $\mathcal{T})\Rightarrow(\exists X)[X\subseteq\beta\,\&\,\underset{\sim}{X}\leqslant\underset{\sim}{\beta}\,\&\,X=$ an infinite homogeneous set for $\langle P_1,P_2\rangle]]$.

Proof. We shall build a proof of the basic Ramsey Theorem into our proof of Theorem 1. Let $\langle P_1,P_2\rangle$ be a given recursive partition of $[N]^2$. The tree $\mathcal{T}$ will be binary-branching, and will be obtained as the union, $\bigcup_s\mathcal{T}_s$, of a recursive sequence $\langle\mathcal{T}_s\rangle_{s=0}^{\infty}$ of finite subtrees. The details of the construction are as follows.

Stage 0. Set $\mathcal{T}_0=\{\langle 0,0\rangle\}$.

Stage $s+1$. Assume that $\mathcal{T}_s$ is a finite retracing tree whose exact contents are known to us (i.e., we have a complete list of the pairs belonging to $\mathcal{T}_s$). Let $n_1,\ldots,n_q$ be a listing of all those numbers n in $\delta\mathcal{T}_s$ such that $n\notin\varrho\mathcal{T}_s$. Further, let $m_1,\ldots,m_r$ be those elements m (if any) of $\varrho\mathcal{T}_s$ such that $\mathrm{card}(\mathcal{T}_s^{-1}(m))=1$. (Unlike $\{n_1,\ldots,n_q\}$, it is possible that $\{m_1,\ldots,m_r\}=\emptyset$.) Now, for any element w of $\delta\mathcal{T}_s$ and any number y, we define:

$$y \text{ is } s\text{-}acceptable\ for\ w \underset{\mathrm{df}}{\Leftrightarrow} y>w\,\&\,(\forall k)(\forall j)[(k\in\hat{\mathcal{T}}_s(w)\,\&\,j\in\hat{\mathcal{T}}_s(w)\,\&$$
$$\&\,k\neq j\,\&\,k=\mathcal{T}_s(j))\Rightarrow(\{y,k\}\in P_1\Leftrightarrow\{j,k\}\in P_1)]\,.$$

For $1\leqslant i\leqslant q$, we define:

$$B_1(n_i)=\begin{cases}\emptyset, \text{ if } \neg(\exists y)[y\leqslant s+1\,\&\,y\notin\delta\mathcal{T}_s\,\&\,\{y,n_i\}\in P_1\,\&\,y \text{ is } s\text{-acceptable for } n_i];\\ \{(\mu y)[y\leqslant s+1\,\&\,y\notin\delta\mathcal{T}_s\,\&\,\{y,n_i\}\in P_1\,\&\,y \text{ is } s\text{-acceptable for } n_i]\}, \text{ otherwise.}\end{cases}$$

and

$$B_2(n_i)=\begin{cases}\emptyset, \text{ if } \neg(\exists y)[y\leqslant s+1\,\&\,y\notin\delta\mathcal{T}_s\,\&\,\{y,n_i\}\in P_2\,\&\,y \text{ is } s\text{-acceptable for } n_i];\\ \{(\mu y)[y\leqslant s+1\,\&\,y\notin\delta\mathcal{T}_s\,\&\,\{y,n_i\}\in P_2\,\&\,y \text{ is } s\text{-acceptable for } n_i]\}, \text{ otherwise.}\end{cases}$$

Similarly, for $1\leqslant i\leqslant r$ we define:

$$B_1(m_i)=\begin{cases}\emptyset, \text{ if either } (\exists y)[y>m_i\,\&\,\langle y,m_i\rangle\in\mathcal{T}_s\,\&\,\{y,m_i\}\in P_1] \text{ or}\\ \quad\neg(\exists y)[y\leqslant s+1\,\&\,y\notin\delta\mathcal{T}_s\,\&\,\{y,m_i\}\in P_1\,\&\,y \text{ is } s\text{-acceptable for } m_i];\\ \{(\mu y)[y\leqslant s+1\,\&\,y\notin\delta\mathcal{T}_s\,\&\,\{y,m_i\}\in P_1\,\&\,y \text{ is } s\text{-acceptable for } m_i]\}, \text{ otherwise}\end{cases}$$

and

$$B_2(m_i)=\begin{cases}\emptyset, \text{ if either } (\exists y)[y>m_i\,\&\,\langle y,m_i\rangle\in\mathcal{T}_s\,\&\,\{y,m_i\}\in P_2] \text{ or}\\ \quad\neg(\exists y)[y\leqslant s+1\,\&\,y\notin\delta\mathcal{T}_s\,\&\,\{y,m_i\}\in P_2\,\&\,y \text{ is } s\text{-acceptable for } m_i];\\ \{(\mu y)[y\leqslant s+1\,\&\,y\notin\delta\mathcal{T}_s\,\&\,\{y,m_i\}\in P_2\,\&\,y \text{ is } s\text{-acceptable for } m_i]\}, \text{ otherwise.}\end{cases}$$

Now let $u_1,\ldots,u_{q+r}$ be the listing of $\{n_1,\ldots,n_q\}\cup\{m_1,\ldots,m_r\}$ in *increasing order* (i.e., $1\leqslant i<j\leqslant q+r\Rightarrow u_i<u_j$). If $B_1(u_i)=B_2(u_i)=\emptyset$ for all i such that $1\leqslant i\leqslant q+r$, we set $\mathcal{T}_{s+1}=\mathcal{T}_s$ and proceed to Stage $s+1$. Otherwise, let $j_0=(\mu j)[1\leqslant j\leqslant q+r\,\&\,\&\,B_1(u_j)\cup B_2(u_j)\neq\emptyset]$ and consider cases as follows.

Case 1. $B_1(u_{j_0})\neq\emptyset$. Let $n_0=(\mu n)[u_{j_0}<n\leqslant s+1\,\&\,n\notin\delta\mathcal{T}_s\,\&\,n$ is s-acceptable for $u_{j_0}\,\&\,\{n,u_{j_0}\}\in P_1]$. Define $\mathcal{T}_{s+1}=\mathcal{T}_s\cup\{\langle n_0,u_{j_0}\rangle\}$; then go on to Stage $s+2$.

Case 2. $B_1(u_{j_0})=\emptyset\,\&\,B_2(u_{j_0})\neq\emptyset$. Let $m_0=(\mu m)[u_{j_0}<m\leqslant s+1\,\&\,m\notin\delta\mathcal{T}_s\,\&\,m$ is s-acceptable for $u_{j_0}\,\&\,\{n,u_{j_0}\}\in P_2]$. Define $\mathcal{T}_{s+1}=\mathcal{T}_s\cup\{\langle m_0,u_{j_0}\rangle\}$; then proceed to Stage $s+2$.

That completes our description of Stage $s+1$, and with it our construction of the sequence $\langle\mathcal{T}_s\rangle_{s=0}^{\infty}$; it is clear from the construction that $\langle\mathcal{T}_s\rangle_{s=0}^{\infty}$ is a fully effective sequence of finite retracing trees such that $(\forall s)[\mathcal{T}_s\subseteq\mathcal{T}_{s+1}]$. We define: $\mathcal{T}=\bigcup_s\mathcal{T}_s$. In order to verify that $\mathcal{T}$, so defined, is a recursive tree, we need only show (since $\mathcal{T}$ is r.e.) the existence of a recursive function τ with the property: $(\forall x)[x\in\delta\mathcal{T}\Leftrightarrow x\in\delta\mathcal{T}_{\tau(x)}]$. (The existence of such a function τ establishes, indeed, that $\mathcal{T}$ is *strongly* recursive.) We establish the existence of τ by induction. Clearly, we may take $\tau(0)=0$. Suppose $\tau(x)$ has been defined for all $x\leqslant n$, in such a way that

$$x\leqslant n\Rightarrow(x\in\delta\mathcal{T}\Leftrightarrow x\in\delta\mathcal{T}_{\tau(x)})\,.$$

Let $s_0=(\mu s)[s>\max\{n+1,\max\{\tau(x)|\ x\leqslant n\}\}]$. Now, at the beginning of Stage s_0+1 we can tell effectively whether $n+1$ is s_0-acceptable for some number $y<n+1$ such that $\mathrm{card}(\mathcal{T}_{s_0}^{-1}(y))\leqslant 1$. If no such y exists then, in view of our choice of s_0, $n+1$ *will never enter* $\delta\mathcal{T}$ *unless it is already in* $\delta\mathcal{T}_{s_0}$. Suppose, on the other hand, that such numbers $y<n+1$ *do* exist, and that $n+1\notin\delta\mathcal{T}_{s_0}$; let y_1 be the *least* such y. By our

choice of s_0 and by the minimality constraints on the choices of j_0 and n_0 (or m_0) at positive stages of the construction, we see that there must be a first stage $t \geqslant s_0+1$, say t_0, such that at stage t we have $j_0 = y_1$ & $n+1 = n_0$ or m_0 according as Case 1 or Case 2 applies. It follows that we can define either $\tau(n+1) = s_0+1$ or $\tau(n+1) = t_0$; moreover, we can effectively decide which of these two definitions is needed. Hence, by induction, τ exists as required, and so $\mathscr{T}$ is a recursive tree. To show that $\mathscr{T}$ is non-trivial, it suffices (by König's Lemma, since $\mathscr{T}$ is binary-branching) to verify that for every positive integer k there exists a $\mathscr{T}$-segment of length k. (As will be seen, this verification amounts, in effect, to a proof of the basic Ramsey Theorem for $\langle P_1, P_2\rangle$.) Obviously there is a $\mathscr{T}$-segment of length 1 (namely, $\mathscr{T}_0$). Suppose, for the sake of a proof by contradiction, that $\mathscr{T}$-segments of length k fail to exist for some $k>1$. Then $\mathscr{T} = \mathscr{T}_s$ holds for some s, since $\mathscr{T}$ is binary-branching. Let s_0 = the least such s; and let k_0 = the maximum length of a branch of $\mathscr{T}_{s_0}$. Now, either $\{x \mid x>0 \,\&\, \{x, 0\} \in P_1\}$ is infinite or $\{x \mid x>0 \,\&\, \{x, 0\} \in P_2\}$ is infinite; hence there exists an *infinite* set S_0 of numbers x such that $x>0 \,\&\, (\forall t)[t>s_0 \Rightarrow x$ is t-acceptable for 0]. It follows easily, in view of our definition of the sequence $\langle \mathscr{T}_s\rangle_{s=0}^{\infty}$, that $\mathscr{T}_{s_0}^{-1}(0)$ contains at least one number $x>0$. Let x_0 = the least positive member of $\mathscr{T}_{s_0}^{-1}(0)$. Suppose $\{x_0, 0\} \in P_1$ and that there do not exist infinitely many $x>x_0$ such that x is t-acceptable for x_0 whenever $t>\max\{x, s_0\}$. Then, for all but finitely many $x>x_0$, we have $\{x, 0\} \in P_2$. It therefore follows, from the construction of $\mathscr{T}$, that there is a number $x_1 \in \mathscr{T}_{s_0}^{-1}(0)$ for which $x_1>0 \,\&\, \{x_1, 0\} \in P_2$; moreover, infinitely many $x>x_1$ are t-acceptable for x_1 whenever $t>\max\{x, s_0\}$. Were we to suppose that $\{x_0, 0\} \in P_2$, a parallel argument would apply with interchange of the roles of P_1 and P_2. Our conclusion is: $\mathscr{T}_{s_0}^{-1}(0)$ contains at least one number z such that infinitely many $x>z$ are t-acceptable for z whenever $t>\max\{x, s_0\}$. Let $z' = (\mu z)[z \in \mathscr{T}_{s_0}^{-1}(0)$ & infinitely many $x>z$ are t-acceptable for z whenever $t>\max\{x, s_0\}]$. We can now repeat the entire foregoing argument with z' in place of 0, and obtain the conclusion: $\mathscr{T}_{s_0}^{-1}(z')$ contains at least one number z such that infinitely many $x>z$ are t-acceptable for z whenever $t>\max\{x, s_0\}$. Iteration yields: there exists an element z of $\delta\mathscr{T}_{s_0}$ such that $\mathscr{T}_{s_0}^{*}(z) = k_0-1 \,\&\, \mathscr{T}_{s_0}^{-1}(z) \neq \emptyset$. Since a number of $\mathscr{T}_{s_0}^{*}$-height k_0-1 tops a $\mathscr{T}_{s_0}$-segment of length k_0, we have a contradiction. We conclude that $\mathscr{T}$ is non-trivial. Let β be an infinite branch of $\mathscr{T}$. We need to find an infinite subset X of β, recursive in β, such that $X = a$ set of indiscernibles for $\langle P_1, P_2\rangle$. For an arbitrarily given number n, we designate $p^{\beta}(n)$ as a P_1-*number* if $\{p_\beta(n+1), p_\beta(n)\} \in P_1$; otherwise, $p_\beta(n)$ is designated a P_2-*number*. Let $X_1 \underset{\mathrm{df}}{=} \{p_\beta(n) \mid p_\beta(n)$ is a P_1-number$\}$, $X_2 \underset{\mathrm{df}}{=} \beta - X_1$. Since $\langle P_1, P_2\rangle$ is a recursive partition, both X_1 and X_2 are recursive in β. Moreover, it is evident from the construction of $\mathscr{T}$ that each of X_1, X_2 is a set of indiscernibles for $\langle P_1, P_2\rangle$. Since one of them is infinite, we are done. ∎

3. Retraceable homogeneous sets for recursive partitions. For our proof of Theorem 3, we shall require the following lemma, which is simply the relativized tree-form version of [2, Theorem 2.1]:

LEMMA 2 (Soare, Jockusch). *Let $\mathscr{T}$ be a non-trivial recursive retracing tree such that only finitely many branchings occur at each node (i.e., $\mathscr{T}^{-1}(x)$ is finite for each $x \in \varrho\mathscr{T}$). Then $\mathscr{T}$ has at least one infinite branch β such that $\underset{\sim}{\beta'} \leqslant \underset{\sim}{0''}$.*

Jockusch observed in [1] that if $\langle P_1, P_2\rangle$ is any recursive partition of $[N]^2$, then there exists an infinite homogeneous set X for $\langle P_1, P_2\rangle$ satisfying the relation $\underset{\sim}{X'} \leqslant \underset{\sim}{0''}$. In the next theorem, we shall strengthen this observation to read: X can be chosen so that $\underset{\sim}{X'} \leqslant \underset{\sim}{0''}$ *and* X is retraceable. The proof of Theorem 3 is based on an analysis of our foregoing proof of Theorem 1; here again, Jockusch's notion ([1]) of a P_1-*number* (P_2-*number*) is the key to the argument.

THEOREM 3. *Let $\langle P_1, P_2\rangle$ be a recursive partition of $[N]^2$. Then there exists an infinite homogeneous set X for $\langle P_1, P_2\rangle$ such that X is retraceable and $\underset{\sim}{X'} \leqslant \underset{\sim}{0''}$. Moreover, it can be required that X admit a finite-one retracing function f such that every infinite set retraced by f is a homogeneous set for $\langle P_1, P_2\rangle$.*

Proof. Let $\mathscr{T}$ be the retracing tree constructed, for $\langle P_1, P_2\rangle$, in the proof of Theorem 1. If $x \in \delta\mathscr{T}$ and $\mathscr{T}(x) \neq x$, we shall say that $\mathscr{T}(x)$ is a P_1-*number for* x or a P_2-*number for* x according as $\{x, \mathscr{T}(x)\} \in P_1$ or $\{x, \mathscr{T}(x)\} \in P_2$. Since $\mathscr{T}$ is recursive and finite-branching, it follows from Lemma 2 that $\mathscr{T}$ has at least one infinite branch β such that $\underset{\sim}{\beta'} \leqslant \underset{\sim}{0''}$. Let β_0 be a particular infinite branch of $\mathscr{T}$, chosen so that $\underset{\sim}{\beta_0'} \leqslant \underset{\sim}{0''}$. We shall assume, without loss of generality, that $\{p_{\beta_0}(n) \mid p_{\beta_0}(n)$ is a P_1-number for $p_{\beta_0}(n+1)\}$ is infinite. (Our entire procedure would suffer only a minor notational change, namely the replacement of P_1 by P_2, if such were not the case.) Let

$$b_0 = (\mu x)\big[x \in \beta_0 \,\&\, (\forall y)[(y \in \beta_0 \,\&\, y \neq x \,\&\, \mathscr{T}(y) = x) \Rightarrow \{y, x\} \in P_1]\big];$$

i.e., b_0 = the *least* of those infinitely many elements of β_0 which are P_1-numbers for their $\mathscr{T}$-preimages in β_0. We shall construct a new non-trivial retracing tree $\mathscr{T}'$ (not necessarily finite-branching) in such a way that (a) if τ is any infinite branch of $\mathscr{T}'$ then there exists an infinite branch β of $\mathscr{T}$ such that

$$b_0 \in \beta \,\&\, \tau = \{p_\beta(n) \mid b_0 \in \hat{\mathscr{T}}(p_\beta(n)) \,\&\, p_\beta(n)$$

is a P_1-number for $p_\beta(n+1)\}$ and (b) $\{p_{\beta_0}(n) \mid p_{\beta_0}(n)$ is a P_1-number for $p_{\beta_0}(n+1)\}$ is an infinite branch of $\mathscr{T}'$. The definition of $\mathscr{T}'$, with b_0 as its unique root, is as follows:

$$\mathscr{T}'_0 \underset{\mathrm{df}}{=} \{\langle b_0, b_0\rangle\};$$

$$\mathscr{T}'_{s+1} \underset{\mathrm{df}}{=} \mathscr{T}'_s \cup \{\langle x, y\rangle \mid x \leqslant s+1 \,\&\, x \notin \delta\mathscr{T}'_s \,\&\, y \in \delta\mathscr{T}'_s \cap (\hat{\mathscr{T}}(x) - \{x\}) \,\&$$
$$\& \{x, y\} \in P_1 \,\&\, (\forall z)[(x>z>y \,\&\, z \in \hat{\mathscr{T}}(x)) \Rightarrow \{x, z\} \in P_2]\};$$

$$\mathscr{T}' \underset{\mathrm{df}}{=} \bigcup_s \mathscr{T}'_s.$$

It is clear that each $\mathscr{T}'_s$ is a finite retracing tree, and that $(\forall s)[\mathscr{T}'_s \subseteq \mathscr{T}'_{s+1}]$. Furthermore, the sequence $\langle \mathscr{T}'_s\rangle_{s=0}^{\infty}$ is r.e., since both $\mathscr{T}$ and the partition $\langle P_1, P_2\rangle$ are recursive. Thus, $\mathscr{T}'$ is a retracing tree. To verify (a), let τ be an infinite branch of $\mathscr{T}'$. Clearly, $b_0 \in \hat{\mathscr{T}}'(p_\tau(n))$ holds for all n; moreover, it is plain from the definition of $\mathscr{T}'_{s+1}$ that $\tau \subseteq \beta$ for some (uniquely determined) branch β of $\mathscr{T}$. We first note that every number $p_\tau(n)$ has the property:

$$(\forall x)[(x \in \beta \;\&\; x \neq p_\tau(n) \;\&\; \mathscr{T}(x) = p_\tau(n)) \Rightarrow \{x, p_\tau(n)\} \in P_1];$$

i.e., $p_\tau(n)$ is a P_1-number for its $\mathscr{T}$-preimage in β. For, suppose n_0 were a counterexample: let $x_0 \in \beta$, $x_0 \neq p_\tau(n_0)$, $\mathscr{T}(x_0) = p_\tau(n_0)$, and $\{x_0, p_\tau(n_0)\} \in P_2$. Then, *by the construction of* $\mathscr{T}$, $\{p_\tau(n_0+1), p_\tau(n_0)\} \in P_2$ and so, *by the construction of* $\mathscr{T}'$, $\langle p_\tau(n_0+1), p_\tau(n_0)\rangle \notin \mathscr{T}'$: contradiction. To complete the verification of (a), it remains to show that $\varrho p_\tau =$ *all of* $\{x \in \beta|\ b_0 \in \hat{\mathscr{T}}(x) \;\&\; x$ is a P_1-number for its $\mathscr{T}$-preimage in $\beta\}$. Suppose x_0 were a counterexample; i.e., $x_0 \in \beta$, x_0 is a P_1-number for its $\mathscr{T}$-preimage in β, $b_0 \in \hat{\mathscr{T}}(x_0)$, and $x_0 \notin \tau$. Let n_0 be such that $p_\tau(n_0) < x_0 < p_\tau(n_0+1)$. Now, since x_0 is a P_1-number for its $\mathscr{T}$-preimage in β, it follows from the construction of $\mathscr{T}$ that $\{p_\tau(n_0+1), x_0\} \in P_1$; hence, by the definition of $\mathscr{T}'$, we have $\mathscr{T}'(p_\tau(n_0+1)) \geqslant x_0$: contradiction. (a) is thus proven. To establish (b), we proceed by induction. Let $\langle b_i\rangle_{i=0}^{\infty}$ be the enumeration, in strictly increasing order, of all $x \in \beta_0$ such that x is a P_1-number for its $\mathscr{T}$-preimage in β_0. From the definitions of $\mathscr{T}$ and $\mathscr{T}'$, we readily see that $\langle b_1, b_0\rangle \in \mathscr{T}'$; moreover, if $\langle b_{i+1}, b_i\rangle \in \mathscr{T}'$ then (again from the constructions of $\mathscr{T}$ and $\mathscr{T}'$) it must also be the case that $\langle b_{i+2}, b_{i+1}\rangle \in \mathscr{T}'$. Thus, $\{b_i|\ i \in N\}$ is an infinite branch of $\mathscr{T}'$, which proves (b). It is obvious that $\{b_i|\ i \in N\}$ is recursive in β_0 and therefore that $\{b_i|\ i \in N\}' \leqslant 0''$; moreover, the set $\{b_i|\ i \in N\}$ is (by the construction of $\mathscr{T}$) a homogeneous set for $\langle P_1, P_2\rangle$. If $\mathscr{T}'$ is finite-branching, we are done. Suppose $\mathscr{T}'$ is *not* finite-branching, and let j_0 be the *smallest* positive integer j such that the set J_0 defined by $J_0 = \{x \in \delta\mathscr{T}'|\ \mathscr{T}'^*(x) = j\}$ is infinite. There is no harm in assuming that $\mathscr{T}'(x) = \mathscr{T}'(y)$ for all $x, y \in J_0$, and for the sake of the rest of our argument, we so assume. We use J_0 to define a subtree of $\mathscr{T}$, as follows: $\mathscr{T}_1 \underset{\mathrm{df}}{=}$ the, union of the sets $\{\langle x, \mathscr{T}(x)\rangle, \langle \mathscr{T}(x), \mathscr{T}(\mathscr{T}(x))\rangle, \ldots, \langle \mathscr{T}^{\mathscr{T}^*(x)-1}(x), b_0\rangle, \langle b_0, b_0\rangle\}$ where $x \in J_0$. Plainly, $\mathscr{T}_1$ is a recursively enumerable retracing subtree of $\mathscr{T}$. Since $\mathscr{T}_1$ is a subtree of $\mathscr{T}$, it is finite-branching. It is an easy application of König's Lemma (in view of the definition of $\mathscr{T}_1$ and the fact that $\mathscr{T}$ is finite-branching) to show that $\mathscr{T}_1$ is non-trivial. If β is any infinite branch of $\mathscr{T}_1$, and if q_0 is the minimal $\mathscr{T}$-height of any element of $J_0 \cap \beta$, then $\{x \in \beta|\ \mathscr{T}^*(x) \geqslant q_0\}$ is homogeneous for $\langle P_1, P_2\rangle$; in fact, we readily see from the definitions of $\mathscr{T}$ and $\mathscr{T}'$ that

$$[\{x \in \beta|\ \mathscr{T}^*(x) \geqslant q_0\}]^2 \subseteq P_2.$$

It is now obvious how to use any such β, say β_1, to extract from $\mathscr{T}_1$ a non-trivial, recursively enumerable subtree $\mathscr{T}_2$, with unique root lying in β_1, which (in light of Lemma 2) meets all the requirements of the theorem. That completes the proof of Theorem 3. ■

QUESTIONS. (a) We know from [1, Theorem 4.2] or [4, Theorem 2] that if $\langle P_1, P_2\rangle$ is a recursive partition of $[N]^2$ then $\langle P_1, P_2\rangle$ admits an infinite Π_2^0 set X of indiscernibles. Does $\langle P_1, P_2\rangle$ necessarily admit an infinite *retraceable* set X of

indiscernibles for which $X \in \Pi_2^0$? (Jockusch's construction in [1, proof of Theorem 4.2] does not in any obvious way lead to such a set X, since in that construction one repeatedly appeals for information to an oracle of degree $0'$. Nor do we see a way to locate Π_2^0 infinite branches in the trees $\mathscr{T}'$ constructed for Theorem 3. It seems possible, but by no means obvious, that a positive answer is obtainable via the standard trick of approximating functions of degree $0'$ by recursive sequences of recursive functions.)

(b) While our proof of Theorem 3 does not produce *retracing* trees unless $\langle P_1, P_2\rangle$ is recursive, it does not seem at all unreasonable that less constructive partitions should also admit retraceable homogeneous sets (retraceable sets, after all, occur in all degrees of unsolvability). Do they? (It is straightforward to construct a partition $\langle P_1, P_2\rangle$ of $[N]^2$ which is recursive in $0''$ and admits no infinite homogeneous set retraceable by a *finite-to-one* retracing function. Moreover, by relativizing [1, Theorem 3.1] and applying Lemma 2, we see that there is a partition $\langle Q_1, Q_2\rangle$ of $[N]^2$, recursive in $0'$, such that no infinite homogeneous set for $\langle Q_1, Q_2\rangle$ is retraced by a *finite-to-one* retracing function which retraces *only* sequences which are homogeneous for $\langle Q_1, Q_2\rangle$. Beyond these easy observations, however, we are in ignorance.)

References

[1] C. G. Jockusch, Jr., *Ramsey's theorem and recursion theory*, J. Symb. Logic 37 (1972) pp. 268–280.

[2] — and R. I. Soare, *Π_1^0 classes and degrees of theories*, Trans. Amer. Math. Soc. 173 (1972), pp. 33–56. MR 47, 4775.

[3] E. Specker, *Ramsey's theorem does not hold in recursive set theory*, Studies in Logic and the Foundations of Mathematics, North-Holland Publ. Co., Amsterdam 1971, pp. 439–442. MR 43, 4667.

[4] C. E. M. Yates, *A note on arithmetical sets of indiscernibles*, Studies in Logic and the Foundations of Mathematics, North-Holland Publ. Co., Amsterdam 1971, pp. 443–451. MR 43, 4668.

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