Table des matières du tome CII, fascicule 2

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The equiconsistency of two large cardinal axioms

by

E. M. Kleinberg (Cambridge, Mass.)

Abstract. In this paper, we prove that the large cardinal axioms "there exists a Rowbottom cardinal" and "there exists a Jonsson cardinal" are equiconsistent over the theory ZFC.

Perhaps the two most well-known of the large cardinal axioms whose inspiration arises from model theory are those which assert the existence of Jonsson cardinals and Rowbottom cardinals. Let us be more specific: a cardinal $\varkappa$ is said to be a Jonsson cardinal if every structure [1] of power $\varkappa$ has a proper elementary substructure of power $\varkappa$. Clearly ω is not a Jonsson cardinal — the structure whose domain is ω and which has each finite ordinal distinguished attests to this. Not so easily, but equally the case, is the result of Erdös and Hajnal that no $\aleph_n$ is a Jonsson cardinal — indeed, there is their general result that if a cardinal is not Jonsson, then neither is the least cardinal greater than it ([2]). The definition of a cardinal's being Rowbottom stems from the notion of two cardinal structure: assume that $\mathfrak{A}$ is a structure over a similarity type having a designated one-place relation. Then $\mathfrak{A}$ is said to have type $\langle\varkappa, \lambda\rangle$ where $\varkappa$ is the cardinality of the domain of $\mathfrak{A}$ and λ is the cardinality of the extension in $\mathfrak{A}$ of the designated one-place relation. We now define a cardinal $\varkappa$ to be Rowbottom if for every cardinal $\lambda<\varkappa$, every structure of type $\langle\varkappa, \lambda\rangle$ has a proper elementary substructure of type $\langle\varkappa, \omega\rangle$. It is immediate that every Rowbottom cardinal is Jonsson — indeed if $\varkappa$ is Rowbottom not only do structures of power $\varkappa$ have proper elementary substructures of power $\varkappa$, they have elementary substructures of power $\varkappa$ in which a distinguished relation, which, in the original structure, may have had any cardinality less than $\varkappa$, is now only countable.

The literature concerned with Jonsson and Rowbottom cardinals is fairly developed. Both axioms have been extensively studied and shown to have remarkable consequences in model theoretic as well as pure (combinatorial) areas of set theory. For background information consult the references to this paper.

What we shall do here is to prove these two axioms equiconsistent over Zermelo–Fraenkel set theory plus the axiom of choice (ZFC). Here is our result:

THEOREM. *The following two theories are equiconsistent*: ZFC+"*there exists a Jonsson cardinal*", ZFC+"*there exists a Rowbottom cardinal.*"

[1] Throughout, all structures are of countable length.

Proof. Since any Rowbottom cardinal is Jonsson we trivially have Con(ZFC+"there exists a Rowbottom cardinal) implies Con(ZFC ⊢ "there exists a Jonsson cardinal").

The converse is more difficult. We begin [1] by giving pure set theoretic definitions of Jonsson and Rowbottom cardinals. For convenience, if X is any set, let $[X]^{<\omega}$ denote the collection of finite subsets of X. Let $\varkappa$ and γ be cardinals. Then $\varkappa\to[\varkappa]^{<\omega}_{\gamma}$ denotes the assertion *for every function F from $[\varkappa]^{<\omega}$ into γ there exists a subset C of $\varkappa$ of cardinality $\varkappa$ such that the range of F on $[C]^{<\omega}$ is a proper subset of γ*. If $\varkappa$, γ, and δ are cardinals, then $\varkappa\to[\varkappa]^{<\omega}_{\gamma,\delta}$ ($\varkappa\to[\varkappa]^{<\omega}_{\gamma,<\delta}$) denotes the assertion *for every function F from $[\varkappa]^{<\omega}$ into γ there exists a subset C of $\varkappa$ of cardinality $\varkappa$ such that the range of F on $[C]^{<\omega}$ has cardinality $\leqslant\delta$ ($<\delta$)*. The connection, now, with the notions of Jonsson and Rowbottom is simply this: a cardinal $\varkappa$ is Jonsson iff $\varkappa\to[\varkappa]^{<\omega}_{\varkappa}$ — it is Rowbottom iff for every $\lambda<\varkappa$, $\varkappa\to[\varkappa]^{<\omega}_{\lambda,\omega}$ [2].

The proofs of these equivalences are not difficult at all. To see that the set theoretic properties imply the model theoretic ones one simply looks at Skolem functions. In the Jonsson case $\varkappa\to[\varkappa]^{<\omega}_{\varkappa}$ yields a size $\varkappa$ set whose Skolem closure is not all of $\varkappa$, and in the Rowbottom case $\varkappa\to[\varkappa]^{<\omega}_{\lambda,\omega}$ yields a size $\varkappa$ set whose Skolem closure threw at most ω members of $\varkappa$ into the designated relation. For the converse of the equivalences, one need only find appropriate structures to code the function defined on $[\varkappa]^{<\omega}$. The domain of the appropriate substructure will be the desired set C.

Now suppose that $\varkappa$ is a Jonsson cardinal, that is, that $\varkappa\to[\varkappa]^{<\omega}_{\varkappa}$. Then we claim that for some $\gamma<\varkappa$, $\varkappa\to[\varkappa]^{<\omega}_{\gamma}$. For if not, let, for each $\gamma<\varkappa$, F_γ attest to $\varkappa\not\to[\varkappa]^{<\omega}_{\gamma}$. Then if $F\colon [\varkappa]^{<\omega}\to\varkappa$ is given by $F(\{\alpha_1,\dots,\alpha_n\}) = F_{\alpha_1}(\{\alpha_2,\dots,\alpha_n\})$ (where $\alpha_1<\alpha_2<\dots<\alpha_n$), F is a counterexample to $\varkappa\to[\varkappa]^{<\omega}_{\varkappa}$, contradiction. Let, then, δ be the least cardinal γ such that $\varkappa\to[\varkappa]^{<\omega}_{\gamma}$. Then we claim that δ is regular and that $\varkappa\to[\varkappa]^{<\omega}_{\delta,<\delta}$. For if not let g be an increasing map from $\mathrm{cf}(\delta)$ onto an unbounded subset of δ, let G attest to $\varkappa\not\to[\varkappa]^{<\omega}_{\mathrm{cf}(\delta),<\mathrm{cf}(\delta)}$, let the F_γ (for $\gamma<\delta$) be as above, and let $F\colon [\varkappa]^{<\omega}\to\delta$ be given by

$$F(\{\alpha_1,\dots,\alpha_2 i_3 j\}) = F_{g(G(\{\alpha_1,\dots,\alpha_i\}))}(\{\alpha_{i+1},\dots,\alpha_{i+j}\})$$

where $\alpha_1<\alpha_2<\dots<\alpha_2 i_3 j$. Then F is a counterexample to $\varkappa\to[\varkappa]^{<\omega}_{\delta}$-contradiction. Now suppose that for some γ, $\gamma\not\to[\gamma]^{<\omega}_{\gamma}$ and $\varkappa\to[\varkappa]^{<\omega}_{\gamma}$. Then we claim that $\varkappa\to[\varkappa]^{<\omega}_{\gamma,<\gamma}$. For suppose not. Let $G\colon [\varkappa]^{<\omega}\to\gamma$ attest to $\varkappa\not\to[\varkappa]^{<\omega}_{\gamma,<\gamma}$, let $H\colon [\gamma]^{<\omega}\to\gamma$ attest to $\gamma\not\to[\gamma]^{<\omega}$, and let us code via positive integers all possible ways to extract from any finite ordered set a collection of subsets. For any n let $g(n)$ be the cardinality of the finite set associated with the code n and let us assume matters to be arranged in such a way that $g(n)$ is always at most n. Then if $F\colon [\varkappa]^{<\omega}\to\gamma$ is given by $F(\{\alpha_1,\dots,\alpha_n\}) = H(\{G(\Delta_1), G(\Delta_2),\dots,G(\Delta_i)\})$ (where $\alpha_1<\dots<\alpha_n$ and $\Delta_1,\dots,\Delta_i$ are the subsets of $\{\alpha_1,\dots,\alpha_{g(n)}\}$ n tells one to extract), F is a counterexample to $\varkappa\to[\varkappa]^{<\omega}_{\gamma}$ — contradiction. These results easily combine to prove the following theorem (Kleinberg [3]):

Assume that $\varkappa$ is the least Jonsson cardinal and that δ is the least cardinal such that $\varkappa\to[\varkappa]^{<\omega}_{\delta}$. Then δ is a regular cardinal less than $\varkappa$ and is such that for every cardinal λ, $\delta<\lambda<\varkappa$, $\varkappa\to[\varkappa]^{<\omega}_{\lambda,<\delta}$.

This ends our brief digression.

We now return to our desired relative consistency result. What we wish to show is that Con(ZFC+"there exists a Jonsson cardinal") implies Con(ZFC+"there exists a Rowbottom cardinal"). So let M be a countable standard transitive model for ZFC+"there exists a Jonsson cardinal" [1]. We shall put together a countable standard transitive model for ZFC+"there exists a Rowbottom cardinal".

The basic idea of our proof is quite simple. Let $\varkappa$ be the least Jonsson cardinal of M. Then we know from our digression that $\varkappa$ itself is almost Rowbottom, namely that for some δ less than $\varkappa$, $\varkappa\to[\varkappa]^{<\omega}_{\lambda,<\delta}$ for every λ such that $\delta\leqslant\lambda<\varkappa$. We shall simply form a Cohen extension $M[G]$ of M by collapsing δ to $\aleph_1$. Then assuming that in forming the extension we have not ruined $\forall\lambda(\delta\leqslant\lambda<\varkappa\to(\varkappa\to[\varkappa]^{<\omega}_{\lambda,<\delta}))$, $\varkappa$ will be Rowbottom in $M[G]$.

Perhaps a few words are in order concerning our proof. There is a well-known collection of results and techniques concerned with so called "preservation of large cardinals under mild Cohen extensions". A result of this type would be of the form "if M is a model in which $\varkappa$ is a large cardinal of such and such a type and $M[G]$ is a Cohen extension of M got by forcing over a partial ordering of power less than $\varkappa$, then $\varkappa$ remains a such and such type of large cardinal in $M[G]$". This form of result is true for measurable cardinals, Ramsey cardinals, and many others. However, what is strongly used in proofs of such results is that the cardinal in question is a strong limit cardinal or at least exceeds the cardinality of the power set of the partial ordering. Now assuming GCH, Jonsson cardinals are strong limit cardinals and hence the usual techniques would permit one to collapse the δ and preserve $\forall\lambda(\delta\leqslant\lambda<\varkappa\to(\varkappa\to[\varkappa]^{<\omega}_{\lambda,<\delta}))$. The result one would then get is Con(ZF+GCH+"there exists a Jonsson cardinal") iff Con(ZF+GCH+"there exists a Rowbottom cardinal"). This is precisely what Devlin observed upon seeing our final theorem of the digression. However, when discussing Jonsson cardinals, the assumption of the GCH is an extreme and unreasonable extravagance. Indeed, as Devlin has shown, if it is consistent for there to exist a Jonsson cardinal at all, it is consistent for there to be one $\leqslant 2^{\aleph_0}$.

Keeping this in mind we see that something quite different is needed to push through our desired result, for if our Jonsson cardinal in M is at most $2^{\aleph_0}$ none of the usual tricks suffice.

So let us return to the proof. M is a countable standard transitive model for ZFC+"there exists a Jonsson cardinal", $\varkappa$ is the least Jonsson cardinal of M, δ is

(1) We are about to enter a brief digression. We will give results whose proofs we only sketch. For a complete and detailed discussion of these results see Kleinberg [3].

(2) We exclude ω and $\aleph_1$ from being Rowbottom. These characterizations are due to Rowbottom.

(1) We assume the existence of such an M for convenience. As usual our proof can be recast so that the relative consistency result is provable in elementary number theory.

the least cardinal γ such that $\varkappa\to[\varkappa]^{<\omega}_{\gamma}$, and so δ is an uncountable regular cardinal less than $\varkappa$ satisfying $\varkappa\to[\varkappa]^{<\omega}_{\lambda,<\delta}$ for every λ such that $\delta\leqslant\lambda<\varkappa$. We begin by gently collapsing δ to $\aleph_1$. Namely, let $\mathscr{I}$ (in M) be the set of functions f from a finite subset of $\omega\times\delta$ into δ such that $\langle n,\alpha\rangle\in\operatorname{dmn}(f)$ implies $f(\langle n,\alpha\rangle)<\alpha$, and let us assume $\mathscr{I}$ to be partially ordered by inclusion.

LEMMA. *$\mathscr{I}$ satisfies the δ-antichain condition, that is, any set of pairwise incompatible elements of $\mathscr{I}$ has cardinality less than δ.*

Proof of the lemma. Assume the lemma to be false and so let S be a set of pairwise incompatible members of $\mathscr{I}$ such that $\overline{\overline{S}}=\delta$. Since δ is an uncountable regular cardinal and since each member of S has a finite domain we may assume that there exists an integer n_0 such that $f\in S$ implies $\overline{\overline{\operatorname{dmn}(f)}}=n_0$. Now the only way two elements of S can be incompatible is if they differ in value at some point common to their two domains. Thus there is a point $\langle n,\alpha\rangle$ in $\omega\times\delta$ such that $\langle n,\alpha\rangle$ is in the domain of δ many members of S. But if $\langle n,\alpha\rangle\in\operatorname{dmn}(f)$ and $f\in\mathscr{I}$ then $f(\langle n,\alpha\rangle)<\alpha<\delta$ and so there must be δ many members of S, S_1, which all agree at $\langle n,\alpha\rangle$. (Keep in mind that δ is a regular cardinal.) But since S_1 is pairwise incompatible and of size δ we can repeat the above argument to find $S_2\subseteq S_1$ such that S_2 is pairwise incompatible, is of size δ, and is such that for some $\langle n_1,\alpha_1\rangle\in\omega\times\delta$, $\langle n_1,\alpha_1\rangle\neq\langle n,\alpha\rangle$, each member of S_2 has the same value at $\langle n_1,\alpha_1\rangle$. If we now repeat this argument n_0-2 more times we would be left with a size δ set of identical members of S, an absurdity. ■

Let, now, G be an M-generic filter on $\mathscr{I}$. Then $M[G]$ is a model for ZFC in which δ is the cardinal $\aleph_1$. For each α less than δ, $\bigcup G\restriction\omega\times\{\alpha\}$ maps ω onto α and yet as $\mathscr{I}$ has the δ antichain condition δ remains a cardinal in $M[G]$. All we need show to complete our proof, then, is that in $M[G]$ $\varkappa\to[\varkappa]^{<\omega}_{\lambda,<\delta}$ for every λ such that $\delta\leqslant\lambda<\varkappa$. So let us assume that $\delta\leqslant\lambda<\varkappa$ and that F: $[\varkappa]^{<\omega}\to\lambda$, $F\in M[G]$. Let us work in M for a while: we first choose, for each subset Q of $\mathscr{I}$ of cardinality less than δ, a function F_Q from $[\varkappa]^{<\omega}$ into Q such that for any subset C of $\varkappa$ of cardinality $\varkappa$, $F_Q''[C]^{<\omega}=Q$. We can do this since δ is the *least* cardinal γ such that $\varkappa\to[\varkappa]^{<\omega}_{\gamma}$. Also, for each x in $[\varkappa]^{<\omega}$ let us choose a maximal incompatible set Q_x of conditions which force a value for $F(x)$, that is, for each x in $[\varkappa]^{<\omega}$ we choose a maximal incompatible subset Q_x of $\{p\in\mathscr{I}|$ for some $\beta<\lambda$, $p\Vdash$ "$\underline{F(x)}=\underline{\beta}$"$\}$. We are now in a position to define (still in M) the following function H from $[\varkappa]^{<\omega}$ into λ: if $\{\alpha_1,\ldots,\alpha_{2i3j}\}\in[\varkappa]^{<\omega}$ (where $\alpha_1<\ldots<\alpha_{2i3j}$) then $H(\{\alpha_1,\ldots,\alpha_{2i3j}\})=\beta$, where β is the value for $\underline{F(\{\alpha_1,\ldots,\alpha_i\})}$ forced by the condition in $Q_{\{\alpha_1,\ldots,\alpha_i\}}$ chosen by $F_{Q_{\{\alpha_1,\ldots,\alpha_i\}}}$ at $\{\alpha_{i+1},\ldots,\alpha_{i+j}\}$, i.e., β is the unique ordinal such that

$$F_{Q_{\{\alpha_1,\ldots,\alpha_i\}}}(\{\alpha_{i+1},\ldots,\alpha_{i+j}\})\Vdash \text{“}\underline{F(\{\alpha_1,\ldots,\alpha_i\})}=\underline{\beta}\text{”}.$$

Now since $H\in M$ and since M satisfies $\varkappa\to[\varkappa]^{<\omega}_{\lambda,<\delta}$, there exists a subset C of $\varkappa$ in M such that C has cardinality $\varkappa$ in M and such that $\overline{\overline{H''[C]^{<\omega}}}<\delta$. Of course C is also a member of $M[G]$, and since $\mathscr{I}$ has the δ antichain condition, C is of cardinality $\varkappa$ in $M[G]$. We would thus be finished upon establishing the following

LEMMA. *In $M[G]$, $F''[C]^{<\omega}\subseteq H''[C]^{<\omega}$.*

Proof of the lemma. Suppose $\{\alpha_1,\ldots,\alpha_i\}\in[C]^{<\omega}$ and

$$M[G]\models F(\{\alpha_1,\ldots,\alpha_i\})=\beta\,.$$

We seek an x in $[C]^{<\omega}$ such that $H(x)=\beta$. Our first goal is to show that there is a condition q in $Q_{\{\alpha_1,\ldots,\alpha_i\}}$ such that $q\Vdash\underline{F(\{\alpha_1,\ldots,\alpha_i\})}=\underline{\beta}$. Now since $M[G]\models$ "F is a function from $[\varkappa]^{<\omega}$ into λ and $F(\{\alpha_i,\ldots,\alpha_n\})=\beta$", there exists a condition p in G such that $p\Vdash$ "$\underline{F}$ is a function from $[\varkappa]^{<\omega}$ into $\underline{\lambda}$ and $\underline{F(\{\alpha_1,\ldots,\alpha_i\})}=\underline{\beta}$". As p thus forces a value for $\underline{F(\{\alpha_1,\ldots,\alpha_i\})}$, p is compatible with a condition q in $Q_{\{\alpha_1,\ldots,\alpha_i\}}$ (for $Q_{\{\alpha_1,\ldots,\alpha_i\}}$ is a maximal incompatible set of such conditions). So let r be an extension of both p and q. Now q forces a value for $\underline{F(\{\alpha_1,\ldots,\alpha_i\})}$. If a value it forces $\underline{F(\{\alpha_1,\ldots,\alpha_i\})}$ equal to is different from β, then r would force $\underline{F(\{\alpha_1,\ldots,\alpha_i\})}$ equal to two different values at $\{\alpha_1,\ldots,\alpha_i\}$ (r extends both p and q). But as p forces $\underline{F}$ to be a function, so does r, and hence r cannot force $\underline{F}$ to assume two different values at the same point. We are thus forced to conclude that q forces "$\underline{F(\{\alpha_1,\ldots,\alpha_i\})}=\underline{\beta}$". Now since C-$(\max\{\alpha_1,\ldots,\alpha_i\}+1)$ has power $\varkappa$, our choice of the function $F_{Q_{\{\alpha_1,\ldots,\alpha_i\}}}$ tells us that for some $\{\alpha_{i+1},\ldots,\alpha_{i+j}\}$ in $[C$-$(\max\{\alpha_1,\ldots,\alpha_i\}+1]^{<\omega}$,

$$F_{Q_{\{\alpha_1,\ldots,\alpha_i\}}}(\{\alpha_{i+1},\ldots,\alpha_{i+j}\})=q\,.$$

Let, now, $\{\alpha_{i+j+1},\ldots,\alpha_{2i3j}\}\in[C$-$(\max\{\alpha_{i+1},\ldots,\alpha_{i+j}\}+1)]^{<\omega}$ and let

$$x=\{\alpha_1,\ldots,\alpha_i,\ldots,\alpha_{i+j},\ldots,\alpha_{2i3j}\}\,.$$

Then it is routine to check that $x\in[C]^{<\omega}$ and $H(x)=\beta$. This completes the proof of our lemma. ■

The theorem is thus established. ■

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