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33–100	1488	158	45	—	44	1735
1–100	1617	767	214	8	44	2650

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As we mentioned before, Professor Mazurkiewicz (d. 1945) and Professor Sierpiński (d. 1969) were the editors of "Fundamenta Mathematicae" from the moment of the appearance of its first volume. Since 1952 the publishing of the journal is in the hands of the authors of the present note — Kazimierz Kuratowski, Editor in Chief, and Karol Borsuk, Deputy Editor.

Parametric inductive definitions and recursive operators over the continuum *

by

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Abstract. In this paper we consider the possible closure ordinals $|\Gamma|$ and sets $\mathrm{Cl}(\Gamma)$ of non-monotone recursive inductive operators Γ which define subsets of the continuum N^N. For an upper bound, we show that any Δ_1^1 operator has Δ_2^1 closure; it is well-known that $|\Gamma|\leqslant\aleph_1$ even for Σ_1^1 operators. On the other hand, we construct a recursive operator Γ with $|\Gamma| = \aleph_1$ and show that any Π_1^1 or Σ_1^1 set is reducible to the closure of some recursive operator. Using the notion of a *parametric* operator (essentially a class of operators, one for each real parameter, over the natural numbers N), we extend this last result in several ways.

Introduction. A great deal of research has been done in recent years on the general subject of inductive definability. The volume *Generalized Recursion Theory* cited in [3] is an excellent source for background in this area.

Briefly, an inductive operator Γ over a set X is a map from $P(X)$ to $P(X)$ such that for all A, $A\subseteq\Gamma(A)$. Γ determines a transfinite sequence $\{\Gamma^\sigma\colon \sigma \text{ an ordinal}\}$, where for all σ, $\Gamma^\sigma = \bigcup\{\Gamma(\Gamma^\tau)\colon \tau<\sigma\}$. The closure ordinal $|\Gamma|$ of Γ is the least ordinal σ such that $\Gamma^{\sigma+1} = \Gamma^\sigma$; clearly $|\Gamma|$ always has cardinality less than or equal to $\mathrm{Card}(X)$. The closure $\mathrm{Cl}(\Gamma)$ is $\Gamma^{|\Gamma|}$, the set inductively defined by Γ.

For a class C of operators, the closure ordinal $|C|$ is the supremum of the $|\Gamma|$ for Γ in C and the inductive closure $\mathrm{Cl}(C)$ is the class of subsets A of X which are reducible to $\mathrm{Cl}(\Gamma)$ for some Γ in C. (The precise notion of reducibility depending on X.)

The general problem in the field of inductive definitions is to characterize, for a given class C of operators, the ordinal $|C|$ and the class $\mathrm{Cl}(C)$.

A common restriction placed on an inductive operator Γ is that it be *monotone*, that is, for any A and B, $A\subseteq B$ implies $\Gamma(A)\subseteq\Gamma(B)$. This is a strong condition and makes monotone inductive operators easier to deal with than their non-monotone counterparts. Monotone inductive definitions over the continuum were studied in some detail in [2]. (We follow the usual convention of identifying the continuum with the Baire space ${}^\omega\omega$.)

For operators over the natural numbers, monotone and non-monotone Π_1^0 inductive definitions led to the same closure ordinal (ω_1, the first non-recursive ordinal)

* Some of the results of this paper were announced in [1].

and the same inductive closure (Π_1^1). It was therefore quite surprising when Richter [4] demonstrated the great differences that arose at the Π_2^0 level. Richter's results actually generalize at the Π_1^1 level for operators over the continuum.

But in fact, interesting things start happening right at the recursive level. It was shown in [2] that the class of positive Σ_1^0 operators has closure ordinal ω and inductive closure Σ_1^0 itself. As the following principal theorem indicates, the class of non-monotone recursive operators over the continuum is much richer.

THEOREM. (a) $|\Delta_1^0| = \aleph_1$; (b) $\Pi_1^1 \cup \Sigma_1^1 \subsetneq \mathrm{Cl}(\Delta_1^0) \subseteq \Delta_2^1$.

One direction of (a) is a corollary to a simple cardinality argument showing $|\Sigma_1^1| \leqslant \aleph_1$. The other direction of part (a) and the first half of (b) will be proven in § 1. The latter result will be strengthened in § 2 with the introduction of the concept of parametric inductive definitions. Finally, § 3 will conclude the proof of the Theorem; we show in fact that $\mathrm{Cl}(\Delta_1^1) \subset \Delta_2^1$.

1. The operator Θ. In this section we define an operator Θ over the continuum with the following properties:

(1) $$\Theta^\omega = \{\alpha\colon (\exists p)\alpha(p) = 0\}.$$

(2) There is a recursive relation P such that $P(\Theta^\omega)$ but for all $n<\omega$, $\neg R(\Theta^n)$.

The existence of such an operator Θ leads to the following general result which says that the class of recursive operators effectively includes all arithmetic operators.

PROPOSITION 1.1. (a) $|\Delta_1^0| \geqslant |\Delta_\omega^0|$; (b) $\mathrm{Cl}(\Delta_1^0) \supseteq \mathrm{Cl}(\Delta_\omega^0)$.

Combining this with the result from [2] that $|\Pi_1^0\text{-monotone}| = \aleph_1$ and $\mathrm{Cl}(\Pi_1^0\text{-monotone}) = \Pi_1^1$, we have the main conclusion of this section.

PROPOSITION 1.2. (a) $|\Delta_1^0| = \aleph_1$; (b) $\mathrm{Cl}(\Delta_1^0) \supseteq \Pi_1^1$.

The definition of the operator Θ is quite simple.

DEFINITION 1.3. $\alpha \in \Theta(A) \leftrightarrow \alpha(0) = 0 \vee (\lambda p)\alpha(p+1) \in A$.

The following lemma is easily verified.

LEMMA 1.4. *For all* $n \leqslant \omega$, $\Theta^n = \{\alpha\colon (\exists p<n)\alpha(p) = 0\}$. ■

Before defining the desired relation P of (2) we need to consider the sequence of reals $\{\gamma_n\colon n \leqslant \omega\}$, given by

$$\gamma_n(p) = \begin{cases} 0, & \text{if} \quad p \geqslant n; \\ 1, & \text{if} \quad p < n. \end{cases}$$

These make up a canonical set of reals for differentiating the levels Θ^n in the following sense:

(3) For all $m, n \leqslant \omega$, $\gamma_m \in \Theta^n$ iff $m<n$.

Now let $F\colon \omega \times P({}^\omega\omega) \to \{0, 1\}$ be given by $F(m, A) = \chi_A(\gamma_{m+1})$ and let $f_n(m) = F(m, \Theta^n)$ for $n>0$. Then $f_n(m) = 1$ iff $\gamma_{m+1} \in \Theta^n$ iff $m+1<n$ iff $m<n-1$ (for $n<\omega$). Thus for $0<n<\omega, f_n = \gamma_{n-1}$ and therefore $f_n \in \Theta^n$, whereas $f_\omega = \gamma_\omega$ and therefore $f_\omega \notin \Theta^\omega$.

But this provides the desired relation P. Just let $P(A) \leftrightarrow \gamma_0 \in A \wedge (\lambda m)F(m, A) \notin A$. It is immediate from the above discussion that $P(\Theta^\omega)$ but for all $n<\omega$, $\neg P(\Theta^n)$. (Note: it is instructive in considering the above argument to look at the "membership table" for the $\{\gamma_m\}$ and $\{\Theta^n\}$. This will consist of $\omega+1$ rows and columns with 0's (for $\notin$) on the main diagonal and below and 1's above. The relation $\neg P(\Theta^n)$ can be seen along the super-diagonal.)

Having now established (1) and (2), we give the following as an example of their use.

PROPOSITION 1.5. $|\Delta_1^0| > \omega$.

Proof. Just let Γ be defined by the following:

$$\alpha \in \Gamma(A) \leftrightarrow \alpha \in \Theta(A) \vee P(A).$$

It is clear that $|\Gamma| = \omega+1$ and $\mathrm{Cl}(\Gamma) = {}^\omega\omega$. ■

We now present the proof of Proposition 1.1 that any arithmetic operator can be duplicated by a Δ_1^0 operator.

Proof of Proposition 1.1. For simplicity's sake we consider only Π_1^0 operators; the proof generalizes easily to arbitrary arithmetic operators. Let T be given by $\alpha \in T(A) \leftrightarrow (\forall p)F(p, \alpha, A) = 1$, where F is a recursive mapping. For any set A and any $n<\omega$, let $(A)_n = \{\alpha\colon \langle n, \alpha\rangle \in A\}$. Define an operator Γ by putting

$$\langle 0, \alpha\rangle \in \Gamma(A) \leftrightarrow \alpha \in \Theta\big((A)_0\big) \quad \text{and} \quad \langle 1, \alpha\rangle \in \Gamma(A) \leftrightarrow P\big((A)_0\big) \wedge \alpha \in T\big((A)_1\big).$$

It is clear that for all ordinals σ, $T^\sigma = (\Gamma^{\omega+\sigma})_1$, so that $|\Gamma| = \omega+|T|$ and $\mathrm{Cl}(T)$ is reducible to $\mathrm{Cl}(\Gamma)$. But Γ is a Δ_1^0 operator, since under the assumption $P\big((A)_0\big)$ we have $\alpha \in T\big((A)_1\big)$ iff $(\forall p)F(p, \alpha, A) \neq 0$ iff $(\lambda p)F\big(p, \alpha, (A)_1\big) \notin (A)_0$. ■

Proposition 1.2 now follows, so we see that $\mathrm{Cl}(\Delta_1^0)$ is a fairly large. This fact can be demonstrated further after the introduction of the concept of parametric inductive definitions in the next section.

2. Parametric inductive definitions. The simplest type of inductive definition over the reals would be one in which the real variable appeared only as a parameter. For example, if R is a Π_1^0 relation such that for all α, Γ_α, defined by $n \in \Gamma_\alpha(A) \leftrightarrow R(n, \alpha, A)$, is an inductive operator over ω, then there is an obvious Π_1^0 operator Γ such that for all n and α, $n \in \mathrm{Cl}(\Gamma_\alpha)$ iff $\langle n, \alpha\rangle \in \mathrm{Cl}(\Gamma)$. A similar fact holds if Π_1^0 is replaced by any standard definability class.

It turns out that for Π_1^0-monotone this process can be reversed: If Γ is a Π_1^0 monotone operator over ${}^\omega\omega$, then there is a Π_1^0 relation R such that for all α, Γ_α, defined by $n \in \Gamma_\alpha(A) \leftrightarrow R(n, \alpha, A)$, is a monotone operator over ω and $\alpha \in \mathrm{Cl}(\Gamma)$ iff $1 \in \mathrm{Cl}(\Gamma_\alpha)$. We say that the Γ_α *parametrize* Γ.

This follows from the previously mentioned result that any Π_1^0 monotone operator over the continuum has Π_1^1 closure and from the following lemma (see [3], p. 233).

LEMMA 2.1. *If Q is a Π_1^1 relation on ${}^\omega\omega$, then there is a uniformly Π_1^0 class of operators Γ_α over ω such that for all α, $Q(\alpha)$ iff $1 \in \mathrm{Cl}(\Gamma_\alpha)$; furthermore, $0 \notin$ any $\mathrm{Cl}(\Gamma_\alpha)$.* ∎

In general, we say that $A \subset {}^\omega\omega$ can be *inductively parametrized* in C if there is a $\{\Gamma_\alpha : \alpha \in {}^\omega\omega\}$ in C and a recursive F such that for all α, $\alpha \in A$ iff $F(\alpha) \in \mathrm{Cl}(\Gamma_\alpha)$. For any definability class C, let $\mathrm{PCl}(C)$ be the class of subsets of ${}^\omega\omega$ which can be inductively parametrized in C.

By the above remarks, $\mathrm{PCl}(\Pi_1^0) = \Pi_1^1 = \mathrm{Cl}(\Pi_1^0\text{-monotone})$. Lemma 2.1 can also be applied to obtain Σ_1^1 sets.

Given a Π_1^1 set Q and a class of operators Γ_α as in Lemma 2.1, it is important to notice that for any fixed α, the operator Γ_α has a countable closure ordinal. Thus we can obtain the typical Σ_1^1 set ${}^\omega\omega - Q$ with the following operator T, defined by:

$$\langle m, \alpha\rangle \in T(A) \leftrightarrow m \in \Gamma_\alpha((A)_\alpha) \vee [m = 0 \wedge \Gamma_\alpha((A)_\alpha) = (A)_\alpha \wedge \langle 1, \alpha\rangle \notin A],$$

where $(A)_\alpha = \{p : \langle p, \alpha\rangle \in A\}$.

For each α, T constructs $\mathrm{Cl}(\Gamma_\alpha)$ in code with α and then puts in $\langle 0, \alpha\rangle$ if and only if $1 \notin \mathrm{Cl}(\Gamma_\alpha)$, i.e., iff $\alpha \notin Q$. By Proposition 1.1 T can essentially be duplicated by a Δ_1^0 operator. We have thus proven the following.

PROPOSITION 2.2. $\Sigma_1^1 \subseteq \mathrm{Cl}(\Delta_1^0)$. ∎

Hopefully the proof of Proposition 2.2 provides some insight into the rather more complicated proof of the following central result of this section.

THEOREM 2.3. $\mathrm{PCl}(\Pi_1^1) \cup \mathrm{PCl}(\Sigma_1^1) \subseteq \mathrm{Cl}(\Delta_1^0)$.

Proof. We are given a set D of reals and uniformly Π_1^1 class of operators $\{\Delta_\alpha : \alpha \in {}^\omega\omega\}$ such that for any α, $\alpha \in D \leftrightarrow 1 \in \mathrm{Cl}(\Delta_\alpha)$. Let Δ_α be given by the Π_1^1 relation Q, so that $m \in \Delta_\alpha(B) \leftrightarrow Q(m, \alpha, B)$ for all $m < \omega$, $\alpha \in {}^\omega\omega$ and $B \subseteq \omega$. Generalizing Lemma 2.1, there is a uniformly Π_1^0 set of operators $\Gamma_{\alpha B}$ such that for all m, α, B:

$$m \in \Delta_\alpha(B) \leftrightarrow Q(m, \alpha, B) \leftrightarrow \langle m, 1\rangle \in \mathrm{Cl}(\Gamma_{\alpha B}). \tag{1}$$

We want to construct an operator T satisfying

$$\alpha \in D \leftrightarrow 1 \in \mathrm{Cl}(\Delta_\alpha) \leftrightarrow \langle 1, \alpha\rangle \in \mathrm{Cl}(T). \tag{2}$$

Recall that $(A)_\alpha = \{m : \langle m, \alpha\rangle \in A\}$; now fix α and let $A_0 = (A)_\alpha$ and and $A_1 = \{n : \langle n, \alpha, A_0\rangle \in A\}$. The desired operator T is given by the following:

$$\langle n, \alpha, A_0\rangle \in T(A) \leftrightarrow n \in \Gamma_{\alpha A_0}(A_1)$$

and

$$\langle m, \alpha\rangle \in T(A) \leftrightarrow \Gamma_{\alpha A_0}(A_1) = A_1 \wedge \langle m, 1\rangle \in A_1.$$

The operator T builds successively the levels $\Delta_\alpha^{\sigma+1}$ in code with α and Δ_α^σ by applying the operator $\Gamma_{\alpha\Delta^\sigma}$ repeatedly. It is not difficult to show that

$$\mathrm{Cl}(T) = \{\langle n, \alpha, \Delta^\sigma\rangle : n \in \Delta^{\sigma+1}\} \cup \{\langle m, \alpha\rangle : m \in \mathrm{Cl}(\Delta_\alpha)\}. \tag{3}$$

From this it follows immediately that T satisfies (2).

For a set of Σ_1^1 operators Δ_α, the definition of T should be modified slightly to read $\langle m, 1\rangle \notin A_1$ at the end instead of $\langle m, 1\rangle \in A_1$.

This theorem can be extended to include compositions of Π_1^1 and Σ_1^1 operators as described in [3], p. 243.

PROPOSITION 2.4. *For any n,* $\mathrm{PCl}((\Pi_1^1)^n) \cup \mathrm{PCl}((\Sigma_1^1)^n) \subseteq \mathrm{Cl}(\Delta_1^0)$. ∎

These results give some idea of the extent of $\mathrm{Cl}(\Delta_1^0)$, although we do not as yet have an exact characterization of the entire class. In the final section of this paper we establish an upper bound on $\mathrm{Cl}(\Delta_1^0)$.

3. All Δ_1^0 operators have Δ_2^1 closure.

In this section we prove the following theorem.

THEOREM 3.1. $\mathrm{Cl}(\Delta_1^0) \subseteq \Delta_2^1$.

Let Γ be an arbitrary Δ_1^1 operator, given by:

$$\alpha \in \Gamma(A) \leftrightarrow (\forall\gamma)(\exists n)\{a_0\}(n, \gamma, \alpha, \chi_A) \cong 0$$

and

$$\alpha \notin \Gamma(A) \leftrightarrow (\forall\gamma)(\exists n)\{a_1\}(n, \gamma, \alpha, \chi_A) \cong 0.$$

Although Γ itself is not necessarily monotone, we will define a Π_1^1 monotone operator Γ_Π and a Σ_1^1 monotone operator Γ_Σ both of which can be used to obtain the action of Γ. For any $B \subseteq \{0, 1\} \times {}^\omega\omega$, let F_B be the (multiple-valued) partial function having B for its graph. Now define

$$\Gamma_\Pi(B) = \{\langle i, \alpha\rangle : (\forall\gamma)(\exists n)\{a_i\}(n, \gamma, \alpha, F_B) \cong 0\};$$

Γ_Σ is obtained similarly from Σ_1^1 definitions of Γ.

It is important to note that these operators are not inclusive (that is, $\Gamma_\Pi(B)$ does not necessarily include B) but *are* monotone since making B larger simply increases the probability that the functions $\{a_i\}$ will converge to 0 when applied to the indicated arguments (and similarly for Γ_Σ).

The following lemma is easily verified.

LEMMA 3.2. *For any A and B, if $B = \{1\} \times A \cup \{0\} \times ({}^\omega\omega - A)$, then $\Gamma_\Pi(B) = \Gamma_\Sigma(B) = \{1\} \times \Gamma(A) \cup \{0\}({}^\omega\omega - \Gamma(A))$.* ∎

We want to use the operator Γ_Π to define another Π_1^1 monotone operator T (this one to be inclusive) which will keep track of all of the levels of Γ simultaneously. To do this, we need to code ordinals as reals in the usual fashion. Let W be the set of reals δ which are the characteristic functions of well-orderings of subsets of ω, and let $|\delta|$ be the length of the ordering given by such a δ. For any $p < \omega$, let $\delta \upharpoonright p$ give the ordering of δ restricted to numbers preceding p in the ordering.

PROPOSITION 3.3. *The relation E, defined by $E(\alpha, \delta) \leftrightarrow W(\delta) \wedge \alpha \in \Gamma^{|\delta|}$, is Π_1^1.*

Proof. In fact, if we let $E^* = \{\langle \alpha, \delta, 1\rangle : W(\delta) \wedge a \in \Gamma^{|\delta|}\} \cup \{\langle \alpha, \delta, 0\rangle : \alpha \notin \Gamma^{|\delta|}\}$, then E^* can be given by a Π_1^1 monotone inductive definition T and is therefore Π_1^1 by a basic result of [2]. Define the required operator T by

$$T(B) = \{\langle \alpha, \delta, i\rangle : (\exists p)[\langle i, \alpha\rangle \in \Gamma_\Pi(\{\langle j, \beta\rangle : \langle \beta, \delta \upharpoonright p, j\rangle \in B\})] \wedge W(\delta)\}.$$

Notice that the $(\exists p)$ is just an effective version of the $\bigcup$ in our original convention that $\Gamma^\sigma = \bigcup\{\Gamma(\Gamma^\tau)\colon \tau<\sigma\}$. It now follows rather easily from Lemma 3.2 that $\mathrm{Cl}(T) = E^*$, completing the proof of the proposition. ■

COROLLARY 3.4. *$\mathrm{Cl}(\Gamma)$ is Σ^1_2.*

Proof. $\alpha \in \mathrm{Cl}(\Gamma) \leftrightarrow (\exists\delta) E(\alpha, \delta)$.

It remains to be shown that $\mathrm{Cl}(\Gamma)$ is also Π^1_2. We make use of the fact that since Γ_Σ is a Σ^1_1 operator, checking for a particular β and B whether β is in $\Gamma_\Sigma(B)$ will depend only on some countable subset of B. Specifically, we have the following improvement of Lemma 3.2. ■

LEMMA 3.5. *For any real α and any A, $\alpha \notin \Gamma(A)$ iff there exist countable $B_1 \subseteq A$ and $B_0 \subseteq {}^\omega\omega - A$ such that $\langle 0, \alpha\rangle \in \Gamma_\Sigma(\{1\}\times B_1 \cup \{0\}\times B_0)$.* ■

Call $B \subseteq \{0,1\}\times{}^\omega\omega$ *self-sustaining* (abbreviated $S(B)$) if for all $\alpha \in B_0$, $\langle 0, \alpha\rangle \in \Gamma_\Sigma(B)$. The relation S is clearly Σ^1_1. Its significance is given by the following lemma:

LEMMA 3.6. *For any ordinal σ and any set B with $B_1 \subseteq \Gamma^\sigma$ and $B_0 \subseteq {}^\omega\omega - \Gamma^\sigma$, if $S(B)$ then $B_0 \subseteq {}^\omega\omega - \mathrm{Cl}(\Gamma)$.*

Proof. We actually show by induction on τ that for all $\tau \geqslant \sigma$, $B_0 \subseteq {}^\omega\omega - \Gamma^\tau$. This is true for $\tau = \sigma$ by assumption. Now suppose that for all ξ with $\sigma \leqslant \xi < \tau$, $B_0 \subseteq {}^\omega\omega - \Gamma^\xi$; for all such ξ we also have $B_1 \subseteq \Gamma^\sigma \subseteq \Gamma^\tau$. Since $S(B)$, for any $\alpha \in B_0$ we have $\langle 0, \alpha\rangle \in \Gamma_\Sigma(B)$; but Γ_Σ is monotone, so by the preceding sentence

$$\langle 0, \alpha\rangle \in \Gamma_\Sigma(\{1\}\times\Gamma^\xi \cup \{0\}\times{}^\omega\omega - \Gamma^\xi).$$

Now by Lemma 3.2 it follows that for all such ξ $(\sigma \leqslant \xi < \tau)$, $\alpha \notin \Gamma(\Gamma^\xi)$ and therefore that $\alpha \notin \Gamma^\tau$. As this is true for any $\alpha \in B_0$, we now have $B_0 \subseteq {}^\omega\omega - \Gamma^\tau$ as promised. ■

PROPOSITION 3.7. *For any real α, $\alpha \notin \mathrm{Cl}(\Gamma)$ iff*

$$(\exists\sigma<\aleph_1)\ (\exists \text{ countable } B)\ [B_1 \subseteq \Gamma^\sigma \wedge B_0 \subseteq {}^\omega\omega - \Gamma^\sigma \wedge S(B) \wedge \alpha \in B_0].$$

Proof. The direction $(\leftarrow)$ is immediate from Lemma 3.6. Now let $\mathrm{Cl}(\Gamma) = C = \Gamma(C)$ and suppose that $\alpha \notin C$. For any real $\beta \notin C$ and $i = 0$ or 1, let $B_i(\beta)$ be the countable set whose existence is given by Lemma 3.5. Let $D_0 = B_0(\alpha)$ and for any n let $D_{n+1} = \bigcup\{B_0(\beta)\colon \beta \in D_n\}$. It is clear that for any n, $\{0\}\times D_n \subseteq \Gamma_\Sigma(\{0\}\times D_{n+1} \cup \{1\}\times C)$. If we now let $B_0 = \bigcup\{D_n\colon n<\omega\}$ and let $B_1 = \bigcup\{B_1(\beta)\colon \beta \in B_0\}$, then $B = (\{0\}\times B_0) \cup (\{1\}\times B_1)$ is countable and it follows by the monotonicity of Γ_Σ that $\{0\}\times B \subseteq \Gamma_\Sigma(B)$, i.e., $S(B)$. It is clear that $B_1 \subseteq C$, $B_0 \subseteq {}^\omega\omega - C$ and $\alpha \in B_0$. Finally, pick σ large enough so that $B_1 \subseteq \Gamma^\sigma$; since $\Gamma^\sigma \subseteq C$, it is automatic that $B_0 \subseteq {}^\omega\omega - C \subseteq {}^\omega\omega - \Gamma^\sigma$. ■

COROLLARY 3.8. *$\mathrm{Cl}(\Gamma)$ is Π^1_2.*

Sketch of proof. The ordinal σ can be coded as a real from W, the countable set B can be coded as a single real, and the relation E of (3.3) can be used to represent "$B_1 \subseteq \Gamma^\sigma$" in Π^1_1 form and $B_0 \subseteq {}^\omega\omega - \Gamma^\sigma$ in Π^1_1 form. (A similar proof is given in detail for Corollary 4.5 of [2].) ■

This completes the proof of Theorem 3.1. It is an interesting open problem whether $\mathrm{Cl}(\Delta^1_1)$ is a strict subset of Δ^1_2. It would also be nice to have a similar result for the class of semirecursive operators; the techniques of this section do not seem to lend themselves to this class.

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