An embedding theorem of infinite-dimensional manifold pairs in the model space *

by

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Abstract. Let $E = l^2$, Σ or σ. In this paper we obtain a sufficient condition under which an E-manifold pair (M, N) with N a Z-set in M can be embedded in E such that M is closed in E and N is a topological boundary of M in E. This condition is: N contains a certain deformation retract of M. This gives a partial answer to Anderson's problem raised in [1]. Our condition cannot be replaced by a weaker one: N is a retract of M.

0. Introduction. An infinite-dimensional manifold is a paracompact Hausdorff space admitting a cover by open sets homeomorphic to open subsets of a given infinite-dimensional (homogeneous) space called the model. An *E-manifold* is a manifold modelled on a space E.

A closed subset K of a space X is a *Z-set* in X if for each non-empty homotopically trivial open set U, $U \setminus K$ is non-empty and homotopically trivial. E.g., each collared closed set in X is a Z-set in X (collared in the sense of M. Brown [6]).

Hilbert space, l^2, is the space of all square-summable sequences of reals with the norm topology, i.e., $l^2 = \{(x_i) \mid \sum_{i=1}^{\infty} x_i^2 < \infty\}$. Let Σ be the linear span of the Hilbert cube $\prod_{i=1}^{\infty} [-1/i, 1/i]$ embedded in l^2 and let σ be the linear span of the usual orthonormal basis, i.e., $\sigma = \{(x_i) \in l^2 \mid x_i = 0$ except for finitely many ith-coordinates$\}$. Then Σ and σ are dense sigma-compact linear subspaces of l^2.

For such an infinite-dimensional space E, it is known that, in the E-manifold pair (M, N), N is a Z-set in M iff N is a collared closed subset of M (1). Then (M, N) may be considered as a manifold-with-boundary, N being the boundary. Thus the study of infinite-dimensional manifolds-with-boundary becomes the study of such manifold pairs.

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(1) E.g., this can be seen as following: Suppose that N is a Z-set in M. By Corollaries 2.7 and 3.5 in [12], there is an open embedding $j\colon M \to E \times [0, 1)$ such that $j(N) = j(M) \cap E \times \{0\}$. Let $k\colon N \to (0, 1]$ be a continuous map such that $\{pj(x)\} \times [0, k(x)) \subset j(M)$ for each $x \in N$, where $p\colon E \times [0, 1) \to E$ is the projection. Then we can get an open embedding $g\colon N \times [0, 1) \to M$ defined by $g(x, t) = j^{-1}\big(pj(x), k(x)t\big)$ for each $(x, t) \in N \times [0, 1)$. Therefore N is collared in M.

R. D. Anderson raised the problem in [1]: *Under what condition can M be embedded in l^2 such that N is the topological boundary under the embedding*? In [13], we obtained a partial answer of this Anderson's problem. In this paper, we shall find more mild condition under which M can be embedded in l^2 such a way, and moreover, it is equally true in the case of Σ- and σ-manifold pairs:

THEOREM. *Let $E = l^2$, Σ or σ and let (M, N) be a separable E-manifold pair with N a Z-set in M. If N contains some deformation retract of M, there exists a closed embedding $h\colon M \to E$ such that $h(N) = \mathrm{bd}(h(M))$ and moreover $h(N)$ is bicollared in E.*

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1. Preliminaries. Let X and Y be spaces and let α be open cover of Y. Maps $f, g\colon X \to Y$ are said to be *α-near* (or *f is α-near to g*) provided that for each $x \in X$ there exists some $U \in \alpha$ such that $f(x), g(x) \in U$. When $X = Y$ and $g = \mathrm{id}$, f is said to be *α-limited.* A homotopy (or an isotopy) $\psi\colon X \times I \to Y$ (where $I = [0, 1]$) is said to be an *α-homotopy* (or an *α-isotopy*) provided that for each $x \in X$ there exists some $U \in \alpha$ such that $\psi(\{x\} \times I) \subset U$. For $\psi\colon X \times I \to Y$, let $\psi_t\colon X \to Y (t \in I)$ be defined by $\psi_t(x) = \psi(x, t)$ for each $x \in X$. Then $f = \psi_0$, $g = \psi_1$ are said to be *α-homotopic* (or *f is α-homotopic to g*). For each $B \subset Y$, we write

$$\mathrm{st}(B; \alpha) = \bigcup \{U \in \alpha \mid B \cap U \neq \emptyset\}$$

and then $\mathrm{st}^0(\alpha) = \alpha$ and, inductively,

$$\mathrm{st}^n(\alpha) = \{\mathrm{st}(V; \alpha) \mid V \in \mathrm{st}^{n-1}(\alpha)\} \quad \text{for} \quad n = 1, 2, \ldots$$

In following lemmas, $E = l^2$, Σ or σ and E-manifolds are separable.

LEMMA 1. *Let (M, N), (X, Y) be E-manifold pairs with N, Y Z-sets in M, X respectively.*

(i) *If for each open cover α of X, there exist homeomorphisms $f\colon M \to X$ and $f'\colon N \to Y$ such that $f \mid N$, f' are α-near, then there exists a homeomorphism $g\colon (M, N) \to (X, Y)$.*

(ii) *Let $h\colon M \to X$ be a continuous map. If for each open cover α of X, there exist homeomorphisms $f\colon M \to X$ and $f'\colon N \to Y$ such that f is α-near to h and $f \mid N$, f' are α-near, then for each open cover β of X, there exists a homeomorphism $g\colon (M, N) \to (X, Y)$ which is β-homotopic to h. Moreover, if h is a map of (M, N) into (X, Y), $g, h\colon (M, N) \to (X, Y)$ are relatively β-homotopic.*

Proof. Since X is an ANR, for each open cover γ of X, there exists an open refinement α of γ such that α-near maps are γ-homotopic (S.-T. Hu [10], Ch. IV, Theorem 1.1). By the Homeomorphism Extension Theorem ([4] or Theorems 1 and 2 of [7]; and Theorem 11.1 of [8]), there exists an $\mathrm{st}^n(\gamma)$-limited homeomorphism $\psi\colon X \to X$ such that $\psi f \mid N = f'$. Then $g = \psi f$ is a desired homeomorphism in (i). If f is α-near to h, then g is $\mathrm{st}^{n+1}(\gamma)$-near to h. Choose γ so fine that $\mathrm{st}^{n+1}(\gamma)$-near maps are β-homotopic since X is an ANR. Then g is β-homotopic to h. The last statement in (ii) follows from Corollaries 2.7 and 3.5 of [12] by the same argument as in the proof of Theorems 2.6′ and 3.4′ of [12]. (Here $n = 3$ when $E = l^2$; $n = 28$ when $E = \Sigma$ or $= \sigma$.) ■

LEMMA 2. *Let (M, N) be an E-manifold pair with N a Z-set in M. For each open cover α of M, there exists a homeomorphism $f\colon (M \times E, N \times E) \to (M, N)$ relatively α-homotopic to the projection.*

Proof. By Lemma 1 and the Stability Theorem ([5] or [15]), it is easily proved. ■

2. Proof of theorem. By the condition in the theorem, N contains a deformation retract M_0 of M. Let

$$L = M \times \{1\} \times \{0\} \cup N \times \{1\} \times [0, 1] \cup C(M_0) \times [0, 1]$$

where $C(M_0) = M \times (0, 1] \cup \{0\}$ denotes the closed metric cone over M_0 (see [16] for definition). Then L is homotopic to $M \times \{1\} \cup C(M_0)$. Since M_0 is a strong deformation retract of M (S.-T. Hu [10], Ch. VII, Theorem 2.1), $M \times \{1\} \cup C(M_0)$ is contractible, therefore so is L. Then L is an AR because L is an ANR. By the result of Toruńczyk, $L \times E \cong E$. Since

$$\mathrm{bd}_{L \times E} M \times \{1\} \times \{0\} \times E = N \times \{1\} \times \{0\} \times E$$

is a Z-set in each $M \times \{1\} \times \{0\} \times E$ (Lemma 2.3 in [18]) and

$$(N \times \{1\} \times [0, 1] \cup C(M_0) \times [0, 1)) \times E$$

i.e., bicollared in $L \times E$, and since

$$(M \times \{1\} \times \{0\} \times E, N \times \{1\} \times \{0\} \times E) \cong (M, N)$$

by Lemma 2, it is easy to construct a desired embedding. ■

Remark. Using the concept of *(f-d) cap-sets* (see [3] or [8]), we can also translate the theorem for l^2-manifold pairs into one for Σ- and σ-manifold pairs as the following way: By Proposition 3.1 in [12], there is an l^2-manifold pair (X, Y) with Y a Z-set in X such that M is an (f-d) cap-set in X and $N = M \cap Y$ is also an (f-d) cap-set in Y. Then one can assume that Y contains a deformation retract of X. This can be proved by applying the Extension Theorem ([11], p. 328) to a deformation of M into N, scrapping surplusses being Z_σ-sets ([2]) and constructing new deformation, skillfully. Then X may be considered a closed set in l^2 with a topological boundary Y. And then construct an (f-d) cap-set L of l^2 such that $L \cap X = M$. From the fundamental property of (f-d) cap-sets, one can construct a desired embedding.

From such a proof, we can obtain the following result for Σ- and σ-manifold pairs which is a slight improvement of the theorem:

Let (M, N) be a separable Σ-(or σ-)manifold pair with N a Z-set in M. If N contains some deformation retract of M, there exists a closed embedding $h\colon M \to \Sigma$

(or $h\colon M\to\sigma$) such that $\mathrm{bd}(h(M)) = h(N)$ *is bicollared in Σ (or in σ) and moreover, $(X, Y) = (\mathrm{cl}_{l^2}h(M), \mathrm{cl}_{l^2}h(N))$ is an l^2-manifold pair with Y a Z-set in X and $Y = \mathrm{bd}_{l^2}X$ is bicollared in l^2 and M and N are $(f\text{-}d)$ cap-sets for X and Y, respectively.*

The following example shows that we cannot replace the condition *N contains a deformation retract of M* by the condition *N is a retract of M*.

Example. Let A be an annulus and let p be a boundary point of A. By [16], $A\times E$ is an E-manifold. Then $(M, N) = (A\times E, \{p\}\times E)$ is an E-manifold pair with N a Z-set in M which cannot be embedded in E such that N is the topological boundary of M in E and bicollared in E. In fact, assume that M can be embedded such a way. Let $k\colon N\times(-2, 2)\to E$ be an open embedding such that $k(x, 0) = x$ for each $x\in N$. Since $N\cong E$ is contractible, there exists a deformation $d\colon N\times I\to N$ such that $d_0 = \mathrm{id}$ and $d_1(N)$ is one point. We define a deformation $d^*\colon E\times I\to E$ by

$$d^*(y, t) = \begin{cases} y & \text{for}\quad y\notin k(N\times[-1, 1]),\\ k\big(d(x, (1-|s|)t), s\big) & \text{for}\quad y = k(x, s)\in k(N\times[-1, 1]).\end{cases}$$

Then E is homotopic to the space $d^*(E)$ which is the one point union of $d^*(M)$ and another space and hence $\pi_*(M)\cong\pi_*(d^*(M)) = 0$ because $\pi_*(d^*(E))\cong\pi_*(E) = 0$. This contradict to $\pi_1(M)\neq 0$.

Addendum. The method in this paper can apply to all E-manifold pairs where E is a locally convex linear metric space such that $E\cong E^\omega$ or $\cong E^\omega_f$, using the result of Toruńczyk in [17] and the relative Stability Theorem (cf. Lemma 2) by the generalized Anderson–McCharen Homeomorphism Extension Theorem. Details are in [14] where we shall find a little more mild sufficient condition. And R. D. Edwards recently announced that compact AR's are Hilbert cube factors. Using this result, the method in this paper can also apply to compact Q-manifold pairs.

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