Changing cofinality of cardinals

by

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Abstract. A forcing notion is defined by which the cofinality of certain kinds of measurable cardinals can be changed to any given value without collapsing any cardinals.

1. Introduction. Since the forcing method was invented in 1963, there was much work done on the different phenomena that one can create in a given model of set theory, by taking a Cohen extension of it.

One of the possible natural questions in this domain is whether one can find a Cohen extension of a given model in which the cardinals are the same as in the ground model but the cofinality of some cardinals differ. The interest in this question stems from the fact that there are some forcing constructions which create a given situation at regular cardinal but the corresponding problem for singular cardinals is open. A general approach to this kind of problem may be: Perform the required construction for a cardinal $\varkappa$ which is still regular and then make it singular. That ideas like that can give information on the famous singular cardinals problem is shown in [9] and [3].

The problem of finding a forcing notion for changing cofinality of a given cardinal without collapsing any cardinals was successfully attacked by K. Prikry in [5], in which he showed:

> *Let $\varkappa$ be a measurable cardinal in some countable model of* ZFC, *then there is a Cohen extension of the model, in which the cofinality of $\varkappa$ is ω, every bounded subset of $\varkappa$ is in the ground model and no cardinals are collapsed.*

Naturally there arises the problem of generalizing the Prikry construction, so as to change the cofinality of $\varkappa$ to cardinals different than ω and this is what is done in this work.

The author was informed, after this work was done, by an indirect private communication that L. Bukovsky has a different generalization of the Prikry construction, using different assumptions on $\varkappa$.

The construction we describe in this paper is weaker than Prikry's in the sense that in the extension we do have new bounded subsets of $\varkappa$. We give an informal argument why one can not expect to get Prikry's full result for every cardinal $\varkappa$ which has a certain property and $\alpha<\varkappa$. Silver's Singular Cardinals Theorem is:

Singular strong limit cardinal having cofinality $>w$ which violates G.C.H.; *is not the first singular cardinal which violates* G.C.H.

We start from a model in which there is $\varkappa$ having the required property $\varkappa>\alpha$ for some measurable cardinal α. Assume that in the model $2^{\beta}=\beta^{++}$ for every regular β and $2^{\beta}=\beta^{+}$ for singular β. This behavior of the function $\beta\to 2^{\beta}$ is consistent with most existing large cardinals as shown by J. Silver [9]. See also T. K. Menas [4]. If we could change the cofinality of $\varkappa$ to α without collapsing cardinals or adjoining bounded subsets of $\varkappa$, we are in contradiction to the fore-mentioned theorem.

The exact assumption that we make on $\varkappa$ for changing its cofinality to α is described in Section 2. Let us just note that Lemma 2.5 shows that $\varkappa$ which is $2^{\varkappa}$ super compact is more than enough for every $\alpha<\varkappa$.

Are these large cardinals really necessary? This is completely open (¹).

2. Preliminaries and notations. We shall try to make our notations as standart as possible so whenever some notation was not defined it is assumed to be standart (unless we are just careless).

Lower case Greek letters are reserved exclusively to denote ordinals. $\varkappa$ is a cardinal, whereas upper case Greek letters denote formulas in some concurrent language. $\mathrm{Dom}(f)$ is the domain of the function f. $f\restriction A$ is the restriction of the function f to A even when $A\nsubseteq\mathrm{Dom}(f)$, in that case $\mathrm{Dom}(f\restriction A)=\mathrm{Dom}(f)\cap A$. $\mathrm{cf}(\alpha)$ is the cofinality of α, and $\bar{A}$ the cardinality of A.

We assume that the reader is familiar with forcing techniques; $\Vdash$ denote weak forcing. We do not make any commitment whether we really start from a countable universe M and really construct a Cohen extension, $M[\mathscr{G}]$, or take a Boolean valued extension of the whole universe. Note that the order on the forcing conditions is assumed to be such that the larger the condition, the more information it gives about the corresponding generic extension. We may be confusing an element in the generic extension with its name or term which it realizes, which lies in the ground model (see [8]).

V is the class of all sets.

If U is a normal ultrafilter on $\varkappa$, it is called *normal* if

(a) U is $\varkappa$ complete,

(b) U is non-principle,

(c) U is closed under diagonal intersection, i.e., if $A_{\beta}\in U$ for every $\beta\in\varkappa$ then $\{\gamma|\ \gamma\in\bigcap_{\beta<\gamma}A_{\beta}\}\in U$.

(¹) Added in proof. The question is not open any more. Results of Jensen and of Dodd-Jensen show that at least the consistency of a measurable cardinal has to be assumed.

It is known that if $\varkappa$ is measurable it carries a normal measure. If U is a normal ultrafilter on $\varkappa$, $V^{\varkappa}/U$ is the ultrapower of the whole universe reduced by U. We denote by $V^{\varkappa}/U$ both the ultrapower and its transitive isomorph (see D. Scott [7]).

If $f\in V^{\varkappa}$ then $[f]_U$ is the element in $V^{\varkappa}/U$ represented by f (or the equivalence class of f modulo U).

LEMMA 2.1 (Łoś Theorem, [1]). *Let Φ be a formula of set theory; $f_1,\ldots,f_n\in V^{\varkappa}$; then*

$$V^{\varkappa}/U\models\Phi([f_1]_U,\ldots,[f_n]_U)\quad \textit{iff}\quad \{\alpha|\ \Phi(f_1(\alpha),\ldots,f_n(\alpha))\}\in U .$$

The following lemma is also well known.

LEMMA 2.2. *Let U be a normal ultrafilter on $\varkappa$.*

(a) *If $A\subseteq\varkappa$ and f is defined as $f(\alpha)=A\cap\alpha$, then $[f]_U=A$.*

(b) *If $f(\alpha)\subseteq\alpha$ for every $\alpha\in\varkappa$, then there exists $A\subseteq\varkappa$ such that*

$$\{\alpha|\ A\cap\alpha=f(\alpha)\}\in U .$$

DEFINITION. Let U, Z be normal ultrafilters on $\varkappa$; then $U<Z$ if $U\in V^{\varkappa}/Z$. It can be shown that $<$ is a partial well order of the set of normal ultrafilters on $\varkappa$. We just use the fact that it is a partial order.

LEMMA 2.3. (Absoluteness Lemma). *Let $U<Z$, $U=[f]_Z$ then*

(a) *$A\in U$ iff $A\subseteq\varkappa$ and $\{\alpha|\ A\cap\alpha\in f(\alpha)\}\in Z$.*

(b) *If $S\in V^{\varkappa}/Z$, then $V^{\varkappa}/Z\models S$ is a normal ultrafilter on $\varkappa$ iff $V\models S$ is a normal ultrafilter on $\varkappa$.*

(c) *$\{\alpha|\ f(\alpha)$ is a normal ultrafilter on $\alpha\}\in Z$.*

(d) *If $T<U$ $T=[g]_U$, then $T<Z$ and $V^{\varkappa}/Z\models T<U$ and $T=[g]_U$.*

Proof. (a), (b) and (c) follow easily from Lemmas 2.1 and 2.2 and from the fact that every function from $\varkappa$ into $V^{\varkappa}/U$ is in $V^{\varkappa}/U$ (see [7]).

(d) follows from the fact that g can be assumed to be in $V^{\varkappa}/U$ (by changing g on a set which is not in U) and from (a).

As described in [7] there is a natural elementary embedding of V into $V^{\varkappa}/U$ denoted by $*$ $(a\to a^*)$. In particular, $\varkappa<\varkappa^*$ and $\alpha\leqslant\alpha^*\leqslant(2^{\alpha})^{+}$ for every ordinal α.

The condition on $\varkappa$ we are going to use to change the cofinality of $\varkappa$ to α is: There exists a sequence $U_0<U_1<\ldots<U_{\gamma}<\ldots(\gamma<\alpha)$ of normal ultrafilters on $\varkappa$. We consider the following lemma as an argument for the consistency of this condition.

LEMMA 2.4. *If $\varkappa$ is $2^{\varkappa}$ supercompact, then there exists a sequence of normal ultrafilters on $\varkappa$*

$$U_0<U_1<\ldots U_{\gamma}<\ldots\quad(\gamma<\varkappa) .$$

Note. $\varkappa$ is 2 *supercompact* if there exist a transitive class M and a mapping j such that $j(\alpha)=\alpha$ for $\alpha<\varkappa$, $j(\varkappa)>\varkappa$, $M^{2^{\varkappa}}\subseteq M$ and j is an elementary embedding of V into M ([6]).

Proof. In [6] the following statement was promised to be published (due to Solovay).

Let $\varkappa$ be $2^{\varkappa}$ supercompact and let A be a set of normal ultrafilters on $\varkappa$, $\bar{\bar{A}}\leqslant\varkappa$; then there is a normal ultrafilter on $\varkappa$, Z such that for all $U\in A$ $U<Z$.

The lemma follows immediately from the statement by building the sequence $U_0<U_1$, by induction, applying the statement at each stage.

3. The forcing conditions. Let $\varkappa$ be a measurable cardinal for which there exists a $<$ increasing sequence of length α $(\alpha<\varkappa)$. Let $U_0<U_1<\ldots<U_\gamma<\ldots\gamma<\alpha$ be a sequence of normal ultrafilters on $\varkappa$ which will be fixed for the rest of this paper. In this section we present the forcing notion that will change the cofinality of $\varkappa$ to $\mathrm{cf}(\alpha)$ without collapsing any cardinals.

Before we do it we have to fix some notations. Since $U_\beta\in V^{\varkappa}/U_\gamma$ for $\beta<\gamma<\alpha$, we can find a function $f_\beta^\gamma\in V^{\varkappa}$ which represents U_β in the ultrapower $V^{\varkappa}/U_\gamma$ (i.e., $U_\beta=[f_\beta^\gamma]_{U_\gamma}$). Clearly, by the Łoś Theorem (Lemma 2.1) and the fact that each U_γ is a $\varkappa$ complete ultrafilter we get:

$$A_\gamma=\{\delta|\ \forall\beta<\gamma\,\forall\eta<\beta\ f_\beta^\gamma(\delta)\text{ is a normal ultrafilter on }\delta\ \text{and } f_\eta^\gamma(\delta)<f_\beta^\gamma(\delta)\}\in U_\gamma.$$

We use the Absoluteness Lemma for $<$ (Lemma 2.3). Define

$$B_\gamma=\{\delta|\ \delta\in A_\gamma,\ \forall\beta<\gamma\forall\eta<\beta[f_\eta^\beta\restriction\delta]_{f_\beta^\gamma(\delta)}=f_\eta^\gamma(\delta)\}\quad\text{for}\quad\gamma>0,$$

$$B_0=\{\delta|\ \delta<\varkappa\text{ and }\delta\text{ is inaccessible}\}.$$

Note that if $\delta\in A_\gamma$ $f_\eta^\gamma(\delta)<f_\beta^\gamma(\delta)$. Hence there exist a function in V^δ which represents $f_\eta^\gamma(\delta)$ in $V^\delta/f_\beta^\gamma(\delta)$. $\delta\in B_\gamma$ means that this function can be picked to be the restriction of f_η^β to δ.

LEMMA 3.1. $B_\gamma\in U_\gamma$.

Proof. If $B_\gamma\notin U_\gamma$ then, since $A_\gamma\in U_\gamma$ and $\gamma<\varkappa$, there exists $\beta<\gamma$ and $\eta<\beta$ such that

$$C=\{\delta|\ \delta\in A_\gamma,\ [f_\eta^\beta\restriction\delta]_{f_\beta^\gamma(\delta)}\neq f_\eta^\gamma(\delta)\}\in U_\gamma;$$

$U_\beta<U_\gamma$, $A_\beta\in U_\beta$, hence by Lemma 2.3 (U_β is represented in $V^{\varkappa}/U_\gamma$ by f_β^γ),

$$D=\{\delta|\ A_\beta\cap\delta\in f_\beta^\gamma(\delta),\ \delta\in A_\gamma\}\in U_\gamma.$$

Therefore, $D\cap C\in U_\gamma$. Fix $\delta\in D\cap C$. Since $A_\beta\cap\delta\in f_\beta(\delta)$ and, for $\varrho\in A_\beta\cap\delta$, $f_\eta^\beta(\varrho)$ is a normal ultrafilter on ϱ, by the Łoś Theorem $[f_\eta^\beta\restriction\delta]_{f_\beta^\gamma(\delta)}$ is a normal ultrafilter on δ. $\delta\in C$, hence this ultrafilter is not $f_\eta^\gamma(\delta)$, which is also a normal ultrafilter on δ.

Therefore there exists $E_\delta\subseteq\gamma$, $E_\delta\in f_\eta^\gamma(\delta)$ but $E_\delta\notin[f_\eta^\beta\restriction\delta]_{f_\beta^\gamma(\delta)}$, which implies, by Lemma 2.3,

$$F_\delta=\{\varrho|\ \varrho<\delta,\ E_\delta\cap\varrho\notin f_\eta^\beta(\varrho)\}\in f_\beta^\gamma(\delta).$$

By Lemma 2.2, there are $E\subseteq\varkappa$, $F\subseteq\varkappa$ such that

$$K=\{\delta|\ \delta\cap E=E_\delta\ \delta\cap F=F_\delta\}\in U_\gamma.$$

Using Lemma 2.3 again,

$$E\in U_\eta,\quad F\in U_\beta.$$

(Since, for $\delta\in D\cap C\cap K$, $E\cap\delta\in f_\eta^\gamma(\delta)$, $F\cap\delta\in f_\beta^\gamma(\delta)$ and $[f_\eta^\gamma]_{U_\gamma}=U_\eta$, $[f_\beta^\gamma]_{U_\gamma}=U_\beta$.) For $\varrho\in F$, $E\cap\varrho\notin f_\eta^\beta(\delta)$. But $[f_\eta^\beta]_{U_\beta}=U_\eta$ which implies $E\notin U_\eta$, hence a contradiction. ■

We are now ready to describe the set of the forcing condition $\mathscr{P}$.

$\mathscr{P}$ is the set of all pairs of the form (g,G) which satisfies conditions (I)-(IV).

I. *g is an increasing function from a finite subset of α into $\varkappa$ such that $g(\beta)\in B_\beta$, $\alpha<g(\beta)$. (In particular, $g(\beta)$ is always inaccessible.) G is a function from $\alpha-\mathrm{Dom}(g)$ into $P(\varkappa)$.*

The intended interpretation is that g supplies partial information about a function which enumerates a closed unbounded subset of $\varkappa$ of order type α. Let us denote the intended function by $\mathscr{G}$. $G(\gamma)$ is the set of possible candidates for extending g to γ, hence defining $g(\gamma)$.

II. *If $\gamma\in\alpha$ and for every $\beta\in\mathrm{Dom}(g)$, $\beta<\gamma$ then $G(\gamma)\in U_\gamma$.*

The set of possible candidates for being $g(\gamma)$ is large in the sense of U_γ.

III. *If $\gamma\in\alpha-\mathrm{Dom}(g)$ and $\mathrm{Dom}(g)-\gamma\neq 0$, then $G(\gamma)\in f_\gamma^\beta(g(\beta))$ for $\beta=\min(\mathrm{Dom}(g)-\gamma)$.*

Once we decided what is $g(\beta)$ the set of possible candidates for defining $g(\gamma)$ is drastically reduced. (Since $g(\gamma)$ should be less than $g(\beta)$ to a bounded subset of $\varkappa$, but it should be still large in the sense of $g(\beta)$.) It is of measure 1 with respect to some ultrafilter on $g(\beta)$ which reflect some of the properties of U_γ, namely $f_\gamma^\beta(g(\beta))$.

IV. *For $\gamma\in\alpha-\mathrm{Dom}(g)$, $G(\gamma)\subseteq B_\gamma$, if $\varrho<\gamma$ then $G(\gamma)\cap g(\varrho)=\emptyset$.*

This restriction is mainly due to technical reasons. The intuitive remarks above motivate the following partial ordering of $\mathscr{P}$: (Note that we adapt the convention that *larger condition* means more information or smaller element of the corresponding Boolean algebra.)

$(h,H)\leqslant(g,G)$ ((g,G) extends (h,H)) if

I. $h\subseteq g$,

II. $\forall\beta\in\mathrm{Dom}(g)-\mathrm{Dom}(h)\ \ g(\beta)\in H(\beta)$,

III. $\forall\beta\in\alpha-\mathrm{Dom}(g)\ \ G(\beta)\subseteq H(\beta)$.

In similar situations the following lemma is trivial. However, here we need a little argument.

LEMMA 3.2. *For every condition (h,H) and for $\gamma\in\alpha-\mathrm{Dom}(h)$, there exists a set A such that $A\in U_\gamma$ if $\mathrm{Dom}(h)\subseteq\gamma$, $A\in f_\gamma^\beta(h(\beta))$ if $\mathrm{Dom}(h)-\gamma\neq\emptyset$ and*

$$\beta=\min(\mathrm{Dom}(h)-\gamma)$$

such that: for every $\delta\in A$ there is an extension of (h,H), (g,G) with $g = h\cup\{\langle\gamma,\delta\rangle\}$ and $G\restriction\beta'+1 = H\restriction\beta'+1$ for $\beta'\in\gamma\cap\mathrm{Dom}(h)$.

Proof. Assume $\gamma\notin\mathrm{Dom}(h)$. (Otherwise we are done.) We give the argument for the case: $\exists\beta\in\mathrm{Dom}(h)\ \gamma<\beta$. The argument for the other case is completely analogous and even simpler. Let β be the minimal element in $\mathrm{Dom}(h)-\gamma$. Fix $\eta<\gamma$; such that there is no $\delta\in\mathrm{Dom}(h)$, $\eta\leqslant\delta<\gamma$.

$H(\eta)\in f_\eta^\beta(h(\beta))$ and both $f_\eta^\beta(h(\beta))$ and $f_\gamma^\beta(h(\beta))$ are normal ultrafilters on $h(\beta)$ such that

$$f_\eta^\beta(h(\beta))<f_\gamma^\beta(h(\beta)).$$

Hence, by Lemma 2.3,

$$D_\eta = \{\delta|\ \delta<\beta,\ H(\eta)\cap\delta\in f_\eta^\gamma(\delta)\}\in f_\gamma^\beta(h(\beta)).$$

We used the fact that $h(\beta)\in B_\beta$ which implies

$$[f_\eta^\gamma\restriction h(\beta)]_{f_\gamma^\beta(h(\beta))} = f_\eta^\beta(h(\beta)).$$

Let D_η be $h(\beta)$ in the case where there exists $\varrho\in\mathrm{Dom}(h)$, $\eta\leqslant\varrho<\gamma$, $H(\gamma)\in f_\gamma^\beta(h(\beta))$ and $D_\eta\in f_\gamma^\beta(h(\beta))$ for every $\eta<\gamma$; therefore

$$\bigcap_{\eta<\gamma} D_\eta\cap H(\gamma) = A\in f_\gamma^\beta(h(\beta)),$$

A is the required A. Let $\delta\in A$. Define:

$$g = h\cup\{\langle\gamma,\delta\rangle\},$$

and for $\varrho\neq\gamma$,

$G(\varrho) = H(\varrho)\cap\delta$ if $\varrho<\gamma$ and there is no $\eta\in\mathrm{Dom}(h)$, $\varrho<\eta<\gamma$.

$G(\varrho) = H(\varrho)$ otherwise.

(g,G) is trivially an extension of (h,H) which has γ in its domain. (g,G) can be verified to be a condition the main point being:

$G(\varrho)\in f_\varrho^\gamma(\delta)$ for $\varrho<\gamma$ for which there is no $\varrho<\eta<\gamma$, $\eta\in\mathrm{Dom}(h)$. (This follows from $\delta\in D_\varrho$.) ■

The following lemma easily follows from the fact that α is a limit ordinal.

LEMMA 3.3. *For every condition (g,G) and an ordinal $\delta<\varkappa$ there exists an extension (h,H) of (g,G) such that $\delta<\bigcup\mathrm{Rang}(h)$, and*

$$(h\restriction\beta+1, H\restriction\beta+1) = (g\restriction\beta+1, G\restriction\beta+1)\quad \textit{where}\quad \beta = \bigcup\mathrm{Dom}(g).$$

Proof. Since α is a limit ordinal and $\mathrm{Dom}(g)$ is finite, there exists $\gamma\in\alpha$ $\bigcup\mathrm{Dom}(g)<\gamma$. Let G' be like G except that $G'(\gamma) = G(\gamma)-(\delta+1)$. (g,G') is a condition which extends (g,G). ($G'(\gamma)$ is expected to be in U_γ and indeed it is because $\delta<\varkappa$ and $G(\gamma)\in U_\gamma$.) Use Lemma 3.2 to extend (g,G') to a condition (h,H) such that $\gamma\in\mathrm{Dom}(h)$, but, since $h(\gamma)\in G'(\gamma)$, $\delta<h(\gamma)\Rightarrow\delta<\bigcup\mathrm{Dom}(h)$. ■

Lemmas 3.2 and 3.3 combined imply that a generic filter in $\mathscr{P}$ naturally generates a function from α into $\varkappa$ whose domain is cofinal in $\varkappa$. We shall not distinguish between the function and the generic filter. The main point in the construction is that the cardinals of M are the same as the cardinals of $M[\mathscr{G}]$ and no new subsets of α are introduced in $M[\mathscr{G}]$. Hence if α was regular in M, it is still regular in $M[\mathscr{G}]$ and $\mathrm{cf}(\varkappa) = \alpha$, but no cardinal was collapsed.

4. The hard core. In this section we introduce the technical machinery needed for the proof of the main (and the only) result.

DEFINITION. Let (g,G) be a condition $\beta<\alpha$; then

$$(g,G)_\beta\quad\text{is}\quad(g\restriction\beta+1, G\restriction\beta+1)$$

and

$$(g,G)^\beta\quad\text{is}\quad(g-g\restriction\beta+1, G-G\restriction\beta+1).$$

If (g,G) and (h,H) are conditions, then $(g,G)_\beta{}^\frown(h,H)^\beta$ is the unique condition (i,I) (if it exists) which satisfies $(i,I)_\beta = (g,G)_\beta$, $(i,I)^\beta = (h,H)^\beta$. If $\mathscr{G}\subseteq\mathscr{P}$ is a generic filter, then

$$\mathscr{G}_\beta = \{(g,G)_\beta|\ (g,G)\in\mathscr{G}\},$$
$$\mathscr{G}_\beta = \{(g,G)^\beta|\ (g,G)\in\mathscr{G}\}.$$

Note that if $g(\beta)$ is defined then $\overline{\overline{\{(h,H)_\beta|\ (g,G)\leqslant(h,H)\}}}\leqslant 2^{g(\beta)}$.

LEMMA 4.1. *Let $\beta<\alpha$ and let $(g,G_\gamma)_{\gamma<\delta}$ be a sequence of conditions ($\delta<\varkappa$); such that*

$$\gamma\neq\gamma'\Rightarrow(g,G_\gamma) = (g,G_{\gamma'})_\beta$$

and for every ϱ, $\beta<\varrho$, $\varrho\in\mathrm{Dom}(g)\Rightarrow\delta<g(\varrho)$. Then there exists a condition (g,H) which extends each (g,G_γ) and $(g,H)_\beta = (g,G_\gamma)_\beta$ for $\gamma<\delta$.

Proof. Define $H(\varrho) = \bigcap_{\gamma<\delta} G_\gamma(\varrho)$, $\varrho\in\alpha-\mathrm{Dom}(g)$; (g,H) is a condition since for $\varrho\leqslant\beta$ $G_\gamma(\varrho)$ is a constant and for $\beta<\varrho$ $H(\varrho)$ is supposed to be a member of some normal ultrafilter on $\varkappa$ or on some $g(\eta)$ for $\beta<\eta$. In any case this ultrafilter is δ^+ complete, so $H(\varrho)$ is of the right form.

For technical simplicity we shall assume that f_γ^β are extended to be defined also for $\varkappa$ by the natural definition:

$$f_\gamma^\beta(\varkappa) = U_\gamma.$$

LEMMA 4.2 (Diagnalization Lemma). *Let $(g,G)\in P$. Let $\gamma<\alpha$, $\gamma\notin\mathrm{Dom}(g)$. Let ϱ = the minimal $\varrho\in\mathrm{Dom}(g)$, $\gamma<\varrho$ if there exists such an ordinal and α if not. Let $\eta = g(\varrho)$ if $\varrho<\alpha$ and $\varkappa$ otherwise. $Z = f_\gamma^\varrho(\eta)$ if $\varrho<\alpha$, $Z = U_\gamma$ otherwise. Let $A\in Z$. For every $\xi\in A$, let (g^ξ, H^ξ) be an extension of (g,G) where $g^\xi = g\cup\{\langle\gamma,\xi\rangle\}$. Let $\beta\in\gamma\cap\mathrm{Dom}(g)$.*

Then there exists a condition (g,H), $(g,G)\leqslant(g,H)$, $(g,G)_\beta = (g,H)_\beta$, such that for every (j,J), $(g,H)\leqslant(j,J)$, $\gamma\in\mathrm{Dom}(j)$, there exists $\xi\in A$ such that $(g^\xi,H^\xi)\leqslant(j,J)$.

Proof. Let δ be the maximal member of $\mathrm{Dom}(g)$ which is $<\gamma$. (If no such δ exists, the following proof applies with trivial modifications.) Define for $\varrho \in A$: $\mathfrak{J}(\xi) = (g^{\xi}, H^{\xi})_{\delta}$. $\mathfrak{J}$ induces a partition of A into at most $2^{g(\delta)}$ classes. Since $2^{g(\delta)} < \eta$ and Z is a normal ultrafilter on η, $A \in Z$, there is $B \subseteq A$, $B \in Z$, such that $\mathfrak{J}$ is constant for $\xi \in B$. Denote this constant by $(i, I)_{\delta}$. Let $\delta < \mu < \gamma$; by definition of $\mathscr{P}$, $H^{\xi}(\mu) \in f^{\gamma}_{\mu}(g^{\xi}(\gamma)) = f^{\gamma}_{\mu}(\xi)$. Note that $[f^{\gamma}_{\mu}]_Z = f^{\varrho}_{\gamma}(\eta)$. By Lemmas 2.2 and 2.3 there exist $A_{\mu} \in Z$, $D_{\mu} \in f^{\varrho}_{\gamma}(\eta)$, such that for $\xi \in A_{\mu}$ $D_{\mu} \cap \xi = H^{\xi}(\mu)$. We are ready to define H:

$$H(\mu) = \begin{cases} I(\mu) & \text{for } \mu \leqslant \delta, \\ D_{\mu} & \text{for } \delta < \mu < \gamma, \\ \{\nu \mid \nu \in \bigcap_{\xi < \nu, \xi \in A} H^{\xi}(\mu)\} & \text{for } \delta < \mu, \end{cases}$$

$$H(\gamma) = G(\gamma) \cap B \bigcap \bigcap_{\delta < \mu < \gamma} A_{\mu};$$

(g, H) is the required condition. It is an extension of (g, G) because each (g^{ξ}, H^{ξ}) was. $H(\mu)$ is always in the right ultrafilter as can easily be verified. (We used the fact that all these ultrafilters are normal.) If $(g, H) \leqslant (i, I)$ and $\gamma \in \mathrm{Dom}(i)$, then $i(\gamma) = \xi$, $\xi \in B \subseteq A$, and then (i, I) is an extension of (g^{ξ}, H^{ξ}) as can be again verified. ■

LEMMA 4.3. *Let $\beta < \alpha$, Φ a statement in the forcing language, $(g, G) \in P$ and $\beta < \alpha_1 < \ldots < \alpha_k < \alpha$, $\alpha_i \notin \mathrm{Dom}(g)$; then there exists H such that*:

(a) $(g, G) \leqslant (g, H)$,

(b) $(g, G)_{\beta} = (g, H)_{\beta}$,

(c) *if* $(g, H) \leqslant (i, I)$, $(i, I)_{\beta} = (g, H)_{\beta}$, $(i, I) \Vdash \Phi$, $\mathrm{Dom}(i) - \mathrm{Dom}(g) = \{\alpha_1, \ldots, \alpha_k\}$ *for some* (i, I), *then* $(g, H) \Vdash \Phi$.

Proof (by induction on k). Let $k = 0$. If there exists H such that $(g, G) \leqslant (g, H)$, $(g, H)_{\beta} = (g, G)_{\beta}$ and $(g, H) \Vdash \Phi$, then we already have our (g, H). If there is no such H define $H = G$.

Assume the lemma for $k-1$ and prove it for k. Let ϱ be the minimal in $\mathrm{Dom}(g)$ $\alpha < \varrho$ and $\eta = g(\varrho)$ if there is such an ordinal. Otherwise, $\eta = \varkappa$, $Z = f^{\varrho}_{\alpha_1}(\varrho)$ if ϱ exists, $Z = U_{\alpha_1}$ otherwise. Let A be a set which satisfies the assumptions of Lemma 3.2; for (g, G) and the ordinal α_1. For $\xi \in A$ define (g^{ξ}, G^{ξ}) as follows:

$$g^{\xi} = g \cup \{\langle \alpha_1, \xi \rangle\},$$

$$G^{\xi}(\mu) = \begin{cases} G(\mu) \cap \xi & \text{for } \mu < \alpha_1, \\ G(\mu) - \xi & \text{for } \alpha_1 < \mu. \end{cases}$$

Apply the present lemma to (g^{ξ}, G^{ξ}) and $\alpha_2, \ldots, \alpha_k$ which we can do by the induction assumption. We get (g^{ξ}, H^{ξ}) such that:

(a) $(g^{\xi}, H^{\xi})_{\beta} = (g^{\xi}, G^{\xi})_{\beta} = (g, G)_{\beta}$ (note that $\beta < \alpha_1$),

(b) $(g^{\xi}, G^{\xi}) \leqslant (g^{\xi}, H^{\xi})$,

(c) if $(g^{\xi}, H^{\xi}) \leqslant (i, I)$, $(i, I)_{\beta} = (g, G)_{\beta}$, $(i, I) \Vdash \Phi$, $\mathrm{Dom}(i) - \mathrm{Dom}(g) = \{\alpha_2, \ldots, \alpha_k\}$, then $(g^{\xi}, H^{\xi}) \Vdash \Phi$.

$H(\alpha_1) \in Z$, hence since Z is an ultrafilter on η, either $\{\xi \mid (g^{\xi}, H^{\xi}) \Vdash \Phi\} \in Z$ or $\{\xi \mid (g^{\xi}, H^{\xi}) \text{ not} \Vdash \Phi\} \in Z$. In the first case, let $A' = \{\xi \in A \mid (g^{\xi}, h^{\xi}) \Vdash \Phi\}$, in the second $A' = \{\xi \in A \mid (g^{\xi}, H^{\xi}) \text{ not} \Vdash \Phi\}$. Apply the Diagonalization Lemma (Lemma 4.2) and get a condition (g, H), $(g, G) \leqslant (g, H)$. Note that the proof of Lemma 4.2 yields that $(g, H)_{\beta} = (g, G)_{\beta}$. (Since for every $\xi \in A$ $(g^{\xi}, H^{\xi})_{\beta} = (g, G)_{\beta}$.) Assume $(g, H) \leqslant (i, I)$, $(i, I)_{\beta} = (g, G)_{\beta}$, $(i, I) \Vdash \Phi$ and $\mathrm{Dom}(i) - \mathrm{Dom}(g) = \{\alpha_1, \ldots, \alpha_k\}$. By definition of (g, H) there exists $\xi \in A'$ such that $(g^{\xi}, H^{\xi}) \leqslant (i, I)$. Since $(i, I)_{\beta} = (g^{\xi}, H^{\xi})_{\beta}$ and $\mathrm{Dom}(i) - \mathrm{Dom}(g^{\xi}) = \{\alpha_2, \ldots, \alpha_k\}$, by definition of (g^{ξ}, H^{ξ}), we have $(g^{\xi}, H^{\xi}) \Vdash \Phi$. Hence by definition of A', for every $\xi \in A'$ $(g^{\xi}, H^{\xi}) \Vdash \Phi$.

We claim $(g, H) \Vdash \Phi$. Otherwise, there is $(g, H) \leqslant (i, I)$ and $(i, I) \Vdash \Phi$. Without loss of generality we can assume $\alpha_1 \in \mathrm{Dom}(i)$; hence $(g^{\xi}, H^{\xi}) \leqslant (i, I)$ for some $\xi \in A'$, but $(g^{\xi}, H^{\xi}) \Vdash \Phi$, a contradiction. So (g, H) is the required condition. ■

LEMMA 4.4. *Let β, Φ, and (g, G) be like in Lemma* 4.3. *Then there exists H such that*:

(a) $(g, G) \leqslant (g, H)$,

(b) $(g, G)_{\beta} = (g, H)_{\beta}$,

(c) *if* $(g, H) \leqslant (i, I)$, $(i, I)_{\beta} = (g, H)_{\beta}$ *and* $(i, I) \Vdash \Phi$ *for some* (i, I), *then* $(g, H) \Vdash \Phi$.

Proof. For every $\alpha_1, \ldots, \alpha_k < \alpha$, $\alpha_i \notin \mathrm{Dom}(g)$, we can get a condition $(g, H^{\alpha_1 \ldots \alpha_k})$ which satisfies the conclusion of Lemma 4.3. Since the cardinality of possible sequences $\alpha_1 < \ldots < \alpha_k < \alpha$ is the cardinality of α, we can apply Lemma 4.1 and get (g, H) which is a common extension of $(g, H^{\alpha_1 \ldots \alpha_k})$, $(g, H)_{\beta} = (g, H)_{\beta}$. Let $(g, H) \leqslant (i, I)$; if $(i, I)_{\beta} = (g, H)_{\beta}$, $(i, I) \Vdash \Phi$, then let $\alpha_1, \ldots, \alpha_k$ be an increasing enumeration of $\mathrm{Dom}(i) - \mathrm{Dom}(g)$. (i, I) is an extension of $(g, H^{\alpha_1 \ldots \alpha_k})$, hence by the definition of $(g, H^{\alpha_1 \ldots \alpha_k})$: $(g, H^{\alpha_1 \ldots \alpha_k}) \Vdash \Phi$. Therefore, $(g, H) \Vdash \Phi$. ■

LEMMA 4.5. *Let β, (g, G) and Φ be like in Lemma* 4.3, $\beta \in \mathrm{Dom}(g)$. *Then there exists (g, H) such that*:

(a) $(g, G) \leqslant (g, H)$,

(b) $(g, G)_{\beta} = (g, H)_{\beta}$,

(c) *if* $(g, G) \leqslant (i, I)$ *and* (i, I) *decides* Φ, *then* $(i, I)_{\beta}{}^{\frown}(g, G)^{\beta}$ *decides* Φ (*the same way, of course*).

This lemma means that the truth of Φ depends just on $\mathscr{G}_{\beta}$, after we decided that (g, H) is true.

(i, I) decides Φ if either $(i, I) \Vdash \Phi$ or $(i, I) \Vdash \neg\Phi$.

Proof. The cardinality of possible $(i, I)_{\beta}$ for $(g, G) \leqslant (i, I)$ is at most $2^{g(\beta)}$. Let (t_{γ}, T_{γ}), $\gamma < 2^{g(\beta)}$, be an enumeration of $\{(i, I)_{\beta} \mid (g, G) \leqslant (i, I)\}$. Define a sequence (g, H_{γ}), $\gamma < 2^{g(\beta)}$, where $(g, H_{\gamma})_{\beta} = (g, G)_{\beta}$. Apply Lemma 4.4 to $(t_{\gamma}, T_{\gamma})^{\frown}(g, G)^{\beta}$ and Φ to get $(t_{\gamma} \cup g, H'_{\gamma})$ such that for every (i, I), if $(t_{\gamma} \cup g, H'_{\gamma}) \leqslant (i, I)$, $(i, I) \Vdash \Phi$, $(i, I)_{\beta} = (t_{\gamma}, T_{\gamma})$, then $(t_{\gamma} \cup g, H_{\gamma}) \Vdash \Phi$.

Apply Lemma 4.4 again to $(t_{\gamma} \cup g, H'_{\gamma})$ and Φ to get $(t_{\gamma} \cup g, H''_{\gamma})$ such that for every (i, I) if $(t_{\gamma} \cup g, H''_{\gamma}) \leqslant (i, I)$, $(i, I) \Vdash \Phi$ and $(i, I)_{\beta} = (t_{\gamma}, T_{\gamma})$, then

$(t_\gamma \cup g, H''_\gamma) \Vdash \Phi$. (g, H_γ) is $(g, G)_\beta\widehat{\ }(t_\gamma g, H''_\gamma)^\beta$, (g, H) is a common extension of (g, H_γ) for $\gamma < 2^{g(\beta)}$, which exists by Lemma 4.1. (If $\beta < \varrho < \alpha$ then $g(\varrho)$ is inaccessible, $g(\varrho) > g(\beta)$, and hence $2^{g(\beta)} < g(\varrho)$.)

If $(g, H) \leqslant (i, I)$ and (i, I) decides Φ, then there exists γ such that $(t_\gamma, T_\gamma) = (i, I)_\beta$; but this means that $(i, I)_\beta\widehat{\ }(g, H)^\beta \geqslant (t_\gamma \cup g, H''_\gamma)$ which by definition of H''_γ implies: since $(i, I)_\beta = (t_\gamma, T_\gamma)$, $(t_\gamma \cup g, H''_\gamma)$ decides Φ. Hence $(i, I)_\beta\widehat{\ }(g, H)^\beta$ decides Φ. ■

LEMMA 4.6. *Let $(g, G) \in \mathscr{P}$, let $\Phi(x)$ be a formula in the forcing language with the only free variable x, $\beta < \alpha$ and δ an ordinal. $\beta \in \mathrm{Dom}(g)$. If $\beta < \varrho$, $\varrho \in \mathrm{Dom}(g)$, then $\delta < g(\varrho)$. Then there exists (g, H), $(g, G) \leqslant (g, H)$, such that the conclusion of Lemma 4.5 holds for (g, H) and $\Phi(\varrho)$ for every $\varrho < \delta$ simultaneously.*

Proof. For every $\varrho < \delta$ we get (g, H^ϱ) which satisfies the conclusion of Lemma 4.5 with respect to $\Phi(\varrho)$. By Lemma 4.1 we get (g, H) which is a common extension of all (g, H^ϱ) $\varrho < \delta$. ■

5. Cardinals are preserved. In this section we conclude the proof by showing that a cardinal in M is still a cardinal in $M[\mathscr{G}]$. Let

$$\mathscr{P}^\beta_\eta = \{(g, G)_\beta \mid (g, G) \in \mathscr{P}, g(\beta) = \eta\},$$

$\beta < \alpha$, $\eta < \varkappa$. (Thus $\mathscr{P}^0_\eta$ can be identified with $\{\eta\}$.) $\mathscr{P}^\beta_\eta$ can be partially ordered by: $(g, G)_\beta \leqslant (h, H)_\beta$ if

(a) $g \restriction \beta+1 \subseteq h \restriction \beta+1$,

(b) for $\gamma \in \mathrm{Dom}(h \restriction \beta+1) - \mathrm{Dom}(g \restriction \beta+1)$ $h(\gamma) \in G(\gamma)$,

(c) for $\gamma \in \beta+1$, $\gamma \notin \mathrm{Dom}(h)$ $H(\gamma) \subseteq G(\gamma)$.

LEMMA 5.1. *$\langle \mathscr{P}, \leqslant \rangle$ satisfies the $\varkappa^+$ chain condition. $\langle \mathscr{P}^\beta, \leqslant \rangle$ satisfies the η^+ chain condition.*

Proof. It follows from Lemma 4.1 that (g, H) and (g, G) are always compatible. Thus the cardinality of a set of mutually incompatible conditions is at most the cardinality of possible g's, which is $\varkappa$. The argument for $\mathscr{P}^\beta_\eta$ is the same with the obvious analogue of Lemma 4.1.

LEMMA 5.2. *Let $\mathscr{G}$ be generic filter in $\mathscr{P}$ and $\beta < \alpha$, $(g, G) \in \mathscr{G}$, $g(\beta) = \eta$; then $\mathscr{G}_\beta$ is generic in $\mathscr{P}^\beta_\eta$.*

This is the famous product lemma. (See, for example, Shoenfield [8].)

LEMMA 5.3. *Let $\mathscr{G}$ be generic filter in $\mathscr{P}$ $\beta < \alpha$, $(g, G) \in \mathscr{G}$, $g(\beta) = \eta$, $\delta \leqslant \eta$; then if $a \subseteq \delta$, $a \in M[\mathscr{G}]$, then $a \in M[\mathscr{G}_\beta]$.*

Proof. This actually is a restatement of Lemma 4.6. ■

It follows from Lemma 5.3 that if $a \in M[\mathscr{G}]$, $a \subseteq \delta$ for some $\delta < g(0)$, then $a \in M$ because $\mathscr{G}_0$ is always in M. It follows that $P^M(\alpha) = P^{M[\mathscr{G}]}(\alpha)$. Hence if α is regular in M it is still regular in $M[\mathscr{G}]$. In this case $\mathrm{cf}^{M[\mathscr{G}]}(\varkappa) = \alpha$. ■

THEOREM 5.4. *M and $M[\mathscr{G}]$ have the same cardinals.*

Proof. Let γ be the first cardinal in M which is not a cardinal in $M[\mathscr{G}]$. γ is a successor cardinal in M. (Otherwise by definition of γ it is a limit of cardinals in $M[\mathscr{G}]$, hence a cardinal in $M[\mathscr{G}]$). $\gamma < \varkappa$ by Lemma 5.1, $\gamma \notin \mathrm{Rang}(g)$ if $(g, H) \in \mathscr{G}$ because γ is not inaccessible; hence since $\mathscr{G}$ defines a normal function from α into $\varkappa$, there exists $\varrho < \alpha$ such that for some $(g, H) \in \mathscr{G}$ $g(\varrho) < \gamma < g(\varrho+1)$. By Lemma 5.3 every subset of γ is a member of $M[\mathscr{G}_{\varrho+1}]$. $M[\mathscr{G}_{\varrho+1}]$ is the same as: $M[\mathscr{G}_\varrho]$ as can easily be verified. $\mathscr{G}_\varrho$ is generic in $\mathscr{P}^\varrho_{g(\varrho)}$ which satisfies $g(\varrho)^+$ chain condition. Hence γ is still a cardinal in $M[\mathscr{G}_\varrho]$; this contradicts the fact that some subset of γ codes a mapping of γ onto a smaller ordinal. ■

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