Extension of ZF-models to models with the scheme of choice

by

Andrzej Zarach (Wrocław)

Abstract. In this paper we show that if M is a countable, standard model of ZF, then there exists a countable, standard model N of $\mathrm{ZFC}^-+\mathrm{V}=\mathrm{HC}$ (ZF theory without the power set axiom + the choice scheme + "every set is countable") such that $M\subseteq N$ and M, N are models of the same height.

§ 0. Introduction. The following theorem was announced by D. B. Morris [1]. Let M be a c.s.m. for ZFC. There exists a c.s.m. N for ZF such that $M\subseteq N$, N has the same cardinals as M, and the following statement is true in N: For each α there exists a set X such that X is a countable union of countable sets and the power set of X can be partitioned into $\aleph_\alpha$ nonempty sets.

Notice that there is no transitive model M_1 of ZFC for which N would be a submodel with the same ordinals; M_1 would ensure the partitioning of $2^{\aleph_0}$ into any number of nonempty sets.

Let T be a theory obtained by removing the power set axiom of ZF and adding the sentence "every set is countable". We already know that not every c.s.m. M for ZF can be extended to a c.s.m. N for ZF + the choice scheme such that $On^M = On^N$, but every c.s.m M for ZF can be extended to a c.s.m N for T + the choice scheme such that $On^M = On^N$.

The following part of the paper is devoted to the proof of the above-mentioned fact. We assume that the reader is familiar with the method of unramified forcing due to Shoenfield [2] and also that he know the paper [3].

§ 1. Collapsing onto ω.

Definition 1. Let Q be a set. Then

$$K_Q(a) = \{K_Q(b)\colon\ (\exists p)_Q(\langle b, p\rangle \in a)\}\,.$$

The definition is inductive with respect to the rank of the set.

Definition 2. Let M be a transitive set. Then

$$M[Q] = \{K_Q(a)\colon\ a\in M\}\,.$$

Definition 3. M is a $\mathcal{3}$-model if M is c.s.m. for

$$\mathrm{ZF}^- + \text{the collection scheme.}$$

Let M be a c.s.m. for ZF and let R_α^M be the αth set in the von Neumann hierarchy (in M).

In order to obtain a model for ZF^- of the same height as M we shall add to the model M a class of generic functions f_α such that $f_\alpha\colon \omega \xrightarrow[\text{onto}]{} R_\alpha^M$ for $\alpha \in On^M$.

DEFINITION 4. Let C be the class of all finite functions f such that $\mathrm{dom}(f) \subseteq On^M \times \omega$ and if $\langle\langle\alpha, n\rangle, x\rangle \in f$ then $x \in R_\alpha^M$; $p \leqslant q \overset{\text{dfn}}{\equiv} q \subseteq p$ for $p, q \in C$. $P_{(\alpha)} = p \cap (\alpha \times \omega \times R_\alpha^M)$ and $p^{(\alpha)} = p - p_{(\alpha)}$ for $p \in C$, $\alpha \in On^M$. $p \wedge q = p \cup q$ for compatible conditions p, q.

LEMMA 1. *$\langle C, \leqslant\rangle$ is a coherent continuous notion of forcing. $C_\alpha = \{p_{(\alpha)}\colon p \in C\}$ and $C^\alpha = \{p^{(\alpha)}\colon p \in C\}$* (see [3]).

Proof. If $\lambda > 0$ is a limit ordinal, then $C_\lambda = \bigcup_{\alpha<\lambda} C_\alpha$ and $C = \bigcup_{\alpha \in On^M} C_\alpha$.

DEFINITION 5. G is *C-3-generic* over M if G is C-generic over M and if for every set of dense sections (see [3]) $\{D_a\}_{a\in b}$ there exists a function $W \in M$ such that $\mathrm{dom}(W) = b$, $W(a) \subseteq D_a$ for $a \in b$ and $(a)_b(W(a) \cap G \neq \emptyset)$.

MAIN LEMMA. *If $\langle C, \leqslant\rangle$ is a coherent continuous notion of forcing, then every C-generic is a 3-generic.*

Proof. By $\Delta'(p)$ we shall denote the least ordinal α such that $p \in C_\alpha$. Let $\{D_a\}_{a\in b}$ be a set of dense section.

By transfinite recursion we can uniformly define sequences of F_α^a for $a \in b$, $\alpha \in On$. Let $F_0^a = \{p \in D_a\colon \Delta'(p) \text{ — minimal}\}$.

$F_\beta^a = \{p \in D_a\colon p$ incompatible with any $q \in \bigcup_{\alpha<\beta} F_\alpha^a$ and $\Delta'(p)$-minimal$\}$. Then $p \in F_\beta^a \Rightarrow \Delta'(p) \geqslant \beta$ and $p_1 \in F_{\alpha_1}^a \,\&\, p_2 \in F_{\alpha_2}^a \,\&\, \alpha_1 < \alpha_2 \Rightarrow \Delta'(p_1) < \Delta'(p_2) \,\&\, p_1, p_2$ are incompatible. Hence we infer that the class $F^a = \bigcup_{\alpha \in On} F_\alpha^a$ is maximal, i.e., if $p \in D_a$ then there exists a $q \in F^a$ such that p, q are compatible. Obviously $F^a \subseteq D_a$. Now let $h_a(p) = \min_\beta (\exists q)_{F_\beta^a}(p, q \text{ compatible})$. The mapping h_a is defined for any condition p. Indeed, if p is a condition, then from the density of D_a it follows that there exists a $p' \in D_a$ such that $p' \leqslant p$. The construction of F^a implies that there exists a $q \in F^a$ such that p' and q are compatible; hence q and p are compatible. Let $g_a(\alpha) = \min_\beta (h_a(C_\alpha) \subseteq \beta)$. By the continuity of the notion of forcing we infer that for every α there exists a $\sigma \geqslant \alpha$ such that $C_\sigma = \bigcup_{\beta<\sigma} C_\beta$ and $g_a(\sigma) \leqslant \sigma$. Subsequently $\bigcup_{\beta \in On} F_\beta^a \subseteq C_\sigma$ for σ greater than the level starting from which C is continuous. Hence in view of the comprehension scheme, every F^a is a set and $W(a) = F^a$, $a \in b$, is a function whose existence we require in Definition 5.

Indeed, if $p \in F^a - C_\sigma$ then there exist a $p_{(\sigma)} \in C_\sigma$ and a $p^{(\sigma)} \in C^\sigma$ such that $p = p_{(\sigma)} \wedge p^{(\sigma)}$. Since $C_\sigma = \bigcup_{\beta<\sigma} C_\beta$, then there exists a $\mu < \sigma$ such that $p_{(\sigma)} \in C_\mu$. Hence there exists a $q \in C_{g_a(\mu)} \subseteq C_\sigma$ such that $q \in F^a$ and q is compatible with p. Subsequently q is compatible with p, which contradicts

$$\Delta'(q) < \Delta'(p) \quad \text{and} \quad p, q \in \bigcup_{\beta \in On} F_\beta^a. \quad \text{Q.E.D.}$$

Note. To prove this fact in [3] we needed the assumption that in C there exists a definable well-ordering. In the sequel $\langle C, \leqslant\rangle$ will denote the notion of forcing given in Definition 4.

LEMMA 2. *Let G be C-generic over M. Then $M[G_{\alpha+1}] \models$ "R_α^M is countable".*

Proof. Let $G_\alpha = G \cap C_\alpha$. Then G_α is C_α-generic over M and $G_\alpha = \{p_{(\alpha)}\colon p \in G\}$. Obviously $G_\alpha \in M[G_\alpha]$ and $M[G] = \bigcup_{\alpha \in On^M} M[G_\alpha]$. Let

$$G'_\alpha = \{p \in G_{\alpha+1}\colon (\beta, n, x)(\langle\langle\beta, n\rangle, x\rangle \in p \Rightarrow \beta = \alpha)\}$$

and $G_\alpha^* = \{\langle n, x\rangle\colon (\exists p)_{G'_\alpha}(\langle\langle\beta, n\rangle, x\rangle \in p)\}$. Then $G'_\alpha \in M[G_{\alpha+1}]$, $G_\alpha^* \in M[G_{\alpha+1}]$ and $G_\alpha^*\colon \omega \xrightarrow[\text{onto}]{} R_\alpha^M$.

LEMMA 3. *Let G be C-generic over M. If $a \in M$ and $\Delta(a) \leqslant \alpha$, $\mathrm{rank}(a) < \alpha$, then $x = K_G(a) = K_{G_\alpha}(a) \in M[G_\alpha]$ and $M[G_\alpha] \models$ "x is countable".*

Remark. $\Delta(a) = \bigcup \{\max(\Delta(b), \Delta'(p))\colon \langle b, p\rangle \in a\}$.

Proof. If $\Delta(a) \leqslant \alpha$ and $a \in M$, then is evident that $K_G(a) = K_{G_\alpha}(a) \in M[G_\alpha]$. Let $\beta = \mathrm{rank}(a)$ and $\mathrm{Rg}(a) = \{b\colon (\exists p)_C(\langle b, p\rangle \in a)\}$. Then $\mathrm{Rg}(a) \subseteq R_\beta^M$ and $G_\beta^* \in M[G_{\beta+1}] \subseteq M[G_\alpha]$. Hence there exists a function $f \in M[G_\alpha]$ such that $f\colon \mathrm{Rg}(a) \xrightarrow{1-1} \omega$. Since $G_\alpha \in M[G_\alpha]$, $M \subseteq M[G_\alpha]$, $M[G_\alpha] \models$ ZF and "the functional K_{G_α} is definable in $M[G_\alpha]$", we infer that $w = \{\langle K_{G_\alpha}(b), f(b)\rangle\colon \langle b, p\rangle \in a \,\&\, p \in G_\alpha\}$ is an element of $M[G_\alpha]$ and $w \subseteq x \times \omega$. Moreover, for every $u \in x$ there exists at least one n such that $\langle u, n\rangle \in w$ and $\langle z_1, n_1\rangle \in w \,\&\, \langle z_2, n_2\rangle \in w \,\&\, z_1 \neq z_2 \Rightarrow n_1 \neq n_2$. Let $l(z) = \min_n(\langle z, n\rangle \in w)$ for $z \in x$. It is a 1-1 function belonging to $M[G_\alpha]$.

LEMMA 4. *$ZF^- +$ the collection scheme$+ V = HC \vdash ZFC^- + V = HC$.*

Now we have

THEOREM. *Let M be a c.s.m. for ZF. Then there exists a c.s.m. N such that $M \subseteq N$, $On \cap M = On \cap N$ and $N \models ZFC^- + V = HC$.*

Proof. It suffices to put $N = M[G]$. Indeed, G is C-3-generic over M; hence (see [3]) $M[G]$ is a 3-model. $M[G] = \bigcup_{\alpha \in On^M} M[G_\alpha]$ and by Lemma 3 $M[G] \models V = HC$.

References

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INSTITUTE OF MATHEMATICS, TECHNICAL UNIVERSITY, Wrocław

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