An addition to "Substructures of reduced powers"

by

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Abstract. In the paper there is given an answer to the problem "Is for every relational structure $\mathfrak{A}$ and any two filters $\mathcal{F}$ and $\mathcal{G}$ the limit power $\mathfrak{A}|(\mathcal{F}\otimes\mathcal{G})$ an elementary extension of $\mathfrak{A}|(\mathcal{F}\otimes^s\mathcal{G})$?

In this note we give an answer to the following question: is for every relational structure $\mathfrak{A}$ and any two filters $\mathcal{F}$ and $\mathcal{G}$ the limit power $\mathfrak{A}|(\mathcal{F}\otimes\mathcal{G})$ an elementary extension of $\mathfrak{A}|(\mathcal{F}\otimes_s\mathcal{G})$? The question was motivated by the fact that $\mathfrak{A}|(\mathcal{F}\otimes\mathcal{G})\cap\cap\mathfrak{A}|(\mathcal{F}\otimes_T\mathcal{G}) = \mathfrak{A}|(\mathcal{F}\otimes_s\mathcal{G})$ and the structures $\mathfrak{A}|(\mathcal{F}\otimes\mathcal{G})$, $\mathfrak{A}|(\mathcal{F}\otimes_T\mathcal{G})$ and $\mathfrak{A}|(\mathcal{F}\otimes_s\mathcal{G})$ contain the same structure as an elementary substructure (see [1]).

We assume the notation and terminology of [1] but for simplicity we write $\mathfrak{A}|\mathcal{F}$ instead of $\mathfrak{A}^I|\mathcal{F}$. We say that a partition $\mathcal{P}$ of I is given by an equivalence relation $F\subseteq I^2$ if $\mathcal{P}$ is the set of equivalence classes of F. $\mathcal{P}$ is a $\mathcal{F}$-partition if $\mathcal{P}$ is given by F for some $F\in\mathcal{F}$. A pair (x, y) of elements of a Boolean algebra $\mathcal{B}$ is a decomposition of an element z of $\mathfrak{B}$ if $x\cap z$ is atomless $y\cap z$ is atomic and $z\subseteq x\cup y$. We say that (x, y) is a common decomposition of a family $\{z_i: i\in I\}$ of elements of a Boolean algebra if for every $i\in I$ (x, y) is a decomposition of z_i. An element z is decomposable in a Boolean algebra $\mathfrak{B}$ if there is a decomposition (x, y) of z such that $x, y\in\mathfrak{B}$. Let us notice that the fact that z is decomposable can be expressed by an elementary formula $(\mathrm{E}x)(\mathrm{E}y)\big(x\cup y = z\wedge\varphi_0(x)\wedge\varphi_1(y)\big)$ where $\varphi_0(x)$ says "x is atomless" and $\varphi_1(x)$ says "x is atomic". If $\mathcal{F}$ is a filter on I^2 then we identify the elements of $2|\mathcal{F}$ with the corresponding subsets of I. (i, j) is an ordered pair, if $p = (i, j)$, then $(p)_0 = i$ and $(p)_1 = j$. C is the Cantor set.

Definition. Let $\mathcal{F}$ be a filter on I^2. We say that $\mathcal{F}$ is an *Olympia filter* if there is a $\mathcal{F}$-partition $\mathcal{P}$ of I such that every element of $\mathcal{P}$ is decomposable in $2|\mathcal{F}$ and there is not a common decomposition of $\mathcal{P}$ in $2|\mathcal{F}$.

Lemma. *Let $\mathcal{F}$ be an Olympia filter on I^2. Then there is a set J such that if $\mathcal{G}$ is a filter of all subsets of J^2, then $2|(\mathcal{F}\otimes\mathcal{G})$ is not an elementary extension of $2|(\mathcal{F}\otimes_s\mathcal{G})$.*

Proof. Let $\mathscr{P} = \{P_j: j \in J\}$ be an $\mathscr{F}$-partition of I with no common decomposition in $2|\mathscr{F}$ such that P_j is decomposable in $2|\mathscr{F}$ for $j \in J$. We put

$$Z = \bigcup \{P_j \times \{j\}: j \in J\}.$$

Of course $Z \subseteq I \times J$ and $Z \in 2|(\mathscr{F} \otimes_s \mathscr{G})$. We claim that Z is decomposable in $2|(\mathscr{F} \otimes \mathscr{G})$ but is not decomposable in $2|(\mathscr{F} \otimes_s \mathscr{G})$. In fact let $(P_{0,j}, P_{1,j})$ be a decomposition of P_j. It is obvious that if $X_0 = \bigcup \{(P_{0,j} \cap P_j) \times \{j\}: j \in J\}$ and $X_1 = \bigcup \{P_{1,j} \cap P_j) \times \{j\}: j \in J\}$, then (X_0, X_1) is a decomposition of Z in $2|(\mathscr{F} \otimes \mathscr{G})$. On the other hand assume that (Z_0, Z_1) is a decomposition of Z in $2|(\mathscr{F} \otimes_s \mathscr{G})$. Then $(\{i: (i,j) \in Z \cap Z_0\}, \{i: (i,j) \in Z \cap Z_1\})$ is a common decomposition of $\mathscr{P}$ contrary to the assumption.

Now we shall prove that there is an Olympia filter.

Example. Let $T = \{(i,j): i \leqslant j, j < \omega\}$ and let $I = \mathbf{C} \times T \cup T$. Now we shall define a filter $\mathscr{F}$ on I^2. Let for $i < \omega$

$$P_i = \bigcup \{(C \times \{t\}) \cup \{t\}: t \in T \quad \text{and} \quad (t)_1 = i\}, \quad \mathscr{P} = \{P_i: i < \omega\}$$

and

$$B_i = \bigcup \{(C \times \{t\}) \cup \{t\}: t \in T, (t)_0 < i\}.$$

Also for any $t \in T$ and for any closed and open subset U of $\mathbf{C}$ let $B_{t,U} = U \times \{t\}$. We define $\mathscr{F}$ as the filter generated by the set

$$\mathscr{F}_0 = \{F_P\} \cup \{F_i: i < \omega\} \cup \{F_{t,U}: t \in T, U \supseteq C \;\; U \text{ closed and open}\}$$

where $F_P = \bigcup \{P_i^2: i < \omega\}$, $F_i = B_i^2 \cup (I - B_i)^2$ and $F_{t,U} = B_{t,U}^2 \cup (I - B_{t,U})^2$.

Of course for $i < \omega$ P_i is decomposable in $2|\mathscr{F}$ by $(p_i \cap B_i, P_i - B_i)$. Also there is no common decomposition of $\mathscr{P}$. In fact let (X, Y) be a pair of elements of $2|\mathscr{F}$. Since $\mathscr{F}$ is generated by $\mathscr{F}_0$ there is a finite set $s_1 \subset \omega$ and a finite set s_2 of pairs (t, U) such that $t \in T$, U is a closed and open subset of C and X and Y are unions of elements of the partition given by

$$F = \bigcap \{F_i: i \in s_1\} \cap \bigcap \{F_{t,U}: (t, U) \in s_2\} \cap F_P.$$

Equivalently

$$X = \bigcup \{R_k: k \in K_X\} \quad \text{and} \quad Y = \bigcup \{R_k: k \in K_Y\} \tag{1}$$

where for $k \in K_X \cup K_Y$ R_k is of the form

$$P_j \cap \bigcap \{B_i^{q(i)}: i \in s_1\} \cap \bigcap \{B_{t,U}^{h(t,U)}: (t, U) \in s_2\} \tag{2}$$

for some $q \in {}^2 s_1$ and $h \in {}^2 s_2$ $(V^1 = V, V^0 = -V)$. Let

$$i_0 = \max\{i: i \in s_1 \text{ or } ((j, i) \;\; U) \in s_2 \text{ for some } j \leqslant i \text{ and } U \text{ a closed and open subset of } C\}.$$

Assume that $P_{i_0+1} \cap Y$ contains all atoms P_{i_0+1}. Then $(i_0, i_0+1) \in Y$ and by (1) and (2) $Y \cap P_{i_0+1} \subseteq C \times (i_0, i_0+1)$. But the last element is atomless hence (X, Y) is not a decomposition of P_{i_0+1}.

Reference

[1] B. Węglorz, *On iterated limit powers*, Fund. Math., in press.

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