Compacta with the shape of finite complexes

by

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Abstract. In Theorems 1 and 2 we give necessary and sufficient conditions for a compactum to have the shape of a finite complex.

The theory of shape, as introduced by Borsuk, is the Čech homotopy theory of compact metric spaces (compacta). It is natural to ask: which compacta have the shape of compact polyhedra? In this note we give some answers.

THEOREM 1. *Let X be a finite-dimensional compactum which is $1-UV$. Then X has the shape of a compact polyhedron if and only if its Čech cohomology with integer coefficients is finitely generated.*

THEOREM 2. *Let X be a finite-dimensional compactum. Then X has the shape of a compact polyhedron if and only if X can be embedded in some sphere S^n in such a way that $S^n \setminus X$ is homeomorphic to the interior of a compact topological manifold.*

The proofs are not difficult once the relevant literature is known. But this literature involves a substantial amount of mathematics, and we feel Theorems 1 and 2 throw some light on the geometrical meaning of shape.

Now some definitions. A compactum X is $1-UV$ if for some (and hence every) embedding of X in an ANR, Z, each neighborhood U of X in Z contains a neighborhood V such that every map of S^1 into V becomes homotopically trivial in U. (Note that if X is movable and the first, or fundamental, shape group of X is trivial, then X is $1-UV$). A compact subset X of S^n satisfies the *cellularity criterion* (CC) [7] if each neighborhood U of X contains a neighborhood V such that every map of S^1 into $V \setminus X$ becomes homotopically trivial in $U \setminus X$.

Theorem 1 will follow easily from the

LEMMA. *Let X be a compact, connected subset of S^n ($n \geqslant 6$) which satisfies CC and does not separate any of its neighborhoods. Then X has the shape of a compact polyhedron if and only if its Čech cohomology with integer coefficients is finitely generated.*

Proof. "Only if" is obvious: we prove "if". First we need a basic system of simply connected neighborhoods of X. Let W be any piecewise linear (PL) manifold

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neighborhood of X. By surgery (see [8], p. 16) there is a smaller compact PL manifold neighborhood U of X whose boundary ∂U is simply connected. By Van Kampen's Theorem, U is simply connected.

Next we prove that $U\setminus X$ is simply connected. Let B^2 be the 2-ball with boundary ∂B^2, and let $\varphi\colon (B^2, \partial B^2)\to(U\ U\setminus X)$ be a map. We must show that there is a map $\varphi'\colon B^2\to U\setminus X$ which agrees with φ on ∂B^2. Let $Y=\varphi^{-1}(X)$. Y is a compact subset of $B^2\setminus\partial B^2$. Since X satisfies CC there is a neighborhood V of X in U such that any loop in $V\setminus X$ contracts in $U\setminus X$. Choose a compact PL submanifold N of $B^2\setminus\partial B^2$ which contains Y in its interior and which lies in $\varphi^{-1}(V)$. ∂N is a finite union of simple closed curves and $\varphi(\partial N)\subset V\setminus X$. Let P be the closure of the component of $B^2\setminus N$ which contains ∂B^2. P is a PL submanifold of B^2, $P\cap Y=\emptyset$ and $\varphi(\partial P\setminus\partial B^2)\subset V\setminus X$. Since each component of the closure of $B^2\setminus P$ is a disk whose boundary lies in ∂P it is possible to extend $\varphi|P$ to a map $\varphi'\colon B^2\to U\setminus X$ as required.

Next we apply the main theorem of [2] (or the more general versions in [8] or [9]). Because there are basic neighborhoods U as above, $S^n\setminus X$ is "simply connected at infinity" in the sense of [2]. Since the Čech cohomology of X is finitely generated, Alexander Duality implies that the singular homology of $S^n\setminus X$ is finitely generated. Since $n\geqslant 6$, it follows that $S^n\setminus X$ is PL homeomorphic to the interior of a compact PL manifold M. Since ∂M is PL collared, there is a compact PL manifold neighborhood V of X in S^n, and a homeomorphism $h\colon V\setminus X\to\partial M\times(0,1]$ which maps ∂V onto $\partial M\times\{1\}$.

We claim that X and V have the same shape. In order to use the terminology of Borsuk [1] we remove a point from $S^n\setminus V$ and work in euclidean n-space E^n instead of S^n. We must define fundamental sequences in E^n between X and V which are fundamentally homotopically inverse to one another. The identity map 1_{E^n} of E^n defines a fundamental "inclusion" sequence $\underline{i}\colon X\to V$. We define a fundamental "retraction" sequence $\underline{r}\colon V\to X$ as follows. Let $\varphi_m\colon I\to I$ be the PL map defined by $\varphi_m(0)=0$, $\varphi_m(1/2m)=1/2m$, $\varphi_m(1/2)=1/m$ and $\varphi_m(1)=1$. Define $r_m\colon E^n\to E^n$ to agree with the identity on $(E^n\setminus V)\cup X$ and to agree with $h^{-1}\circ(1_{\partial M}\times\varphi_m)\circ h$ on V. Let $\underline{r}=\{r_m\}$. It is easy to see that the compositions $\underline{r}\circ\underline{i}$ and $\underline{i}\circ\underline{r}$ are fundamentally homotopic to the appropriate identity sequences.

Proof of Theorem 1. Embed X in S^n, $n\geqslant 6$ being large enough, so that $S^n\setminus X$ is uniformly locally 1-connected: see for example Proposition 1.3 of [5]. Then X clearly satisfies the hypotheses of the Lemma and the result follows.

Proof of Theorem 2. We first note that for $n\geqslant 6$, $S^n\setminus X$ is homeomorphic to the interior of a compact (topological) manifold if and only if it is PL homeomorphic to the interior of a compact PL manifold. The non-obvious "only if" part of this statement follows from the Product Structure Theorem of [6]. Thus, in proving the "if" part of Theorem 2 we may assume that $S^n\setminus X$ is PL homeomorphic to the interior of a compact PL manifold. Applying the argument in the second half of the proof of the Lemma to (each component of) X, we obtain a compact PL manifold neighborhood V of X which is shape equivalent to X.

To prove the "only if" part of Theorem 2, we suppose X has the shape of a compact polyhedron P. Following [3] or [5] there is an integer n such that if X is suitably embedded in S^n and if P is PL embedded in S^n, the complements $S^n\setminus X$ and $S^n\setminus P$ are homeomorphic. Hence $S^n\setminus X$ is homeomorphic to the interior of a compact manifold.

Concluding remarks and questions:

1. What are the analogous theorems for infinite-dimensional compacta?

2. We have addressed the question: When does a finite-dimensional compactum have the shape of a compact polyhedron? For $1-UV$ compacta, we have answered by giving intrinsic conditions on the compactum (Theorem 1). But without Property $1-UV$ we can only give embedding conditions (Theorem 2). Are there reasonable intrinsic conditions? (Note: the problem of verifying our embedding conditions is the subject of [8] and [9]).

3. A related question is: When does a compactum have the (Fox) shape of a (not necessarily compact) ANR? For movable compacta a slightly weaker question is answered in Theorem 2 of [4].

4. Theorem 2 should be compared with Theorem 2.5 of [10].

Added in proof (June 1976). In the more than two years since this paper was written additional work has been done. See note added in proof to [4].

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