Multicoherent spaces

by

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Abstract. A. H. Stone has offered a conjecture concerning a characterization of multicoherent spaces. In this paper we offer another conjecture characterizing these spaces and establish our conjecture whenever our space is weakly-finitely multicoherent or compact. We use this result to show that if our space is weakly-finitely multicoherent, then Stone's conjecture obtains. Finally we show that if every completion of a locally connected metric space X is multicoherent, so also is X.

Let X denote a locally connected, connected normal space. By a *region* we mean a connected open subset of X and by a *continuum* we mean closed and connected subset of X. For $A \subset X$, $b_0(A)$ denotes the number of components of X less one (or ∞ if this number is infinite). The *degree of multicoherence*, $r(X)$, of X is defined by

$$r(X) = \sup\{b_0(H \cap K):\ X = H \cup K \text{ and } H \text{ and } K \text{ are subcontinua of } X\}.$$

If $r(X) = 0$, X is said to be *unicoherent* and we say that X is multicoherent otherwise. If $0 < r(X) < \infty$, we say that X is *finitely multicoherent* and if $0 < r(X) \leqslant \infty$ but $b_0(H \cap K) < \infty$ for any representation $X = H \cup K$, where H and K are continua, we say that X is *weakly-finitely multicoherent*. Thus every finitely multicoherent space is weakly-finitely multicoherent. A. H. Stone has shown that there exists a large class of spaces which are weakly-finitely multicoherent, but not finitely multicoherent [7].

Let $n > 2$ be an integer and let $S(n)$ denote the following statement: $S(n)$: X is multicoherent iff there exists non-empty continua $A_1, \ldots, A_n$ such that

(i) $X = \bigcup_{i=1}^{n} A_i$,

(ii) no three of the A_i's have a point in common, and

(iii) $A_i \cap A_j \neq \emptyset$ iff $|i(\mathrm{mod}\, n) - j(\mathrm{mod}\, n)| \leqslant 1$.

In a private communication A. H. Stone conjectured that $S(n)$ is true for all $n > 2$ and he stated that he had established $S(n)$ for all $n > 2$ whenever X was finitely multicoherent or compact (1). A. D. Wallace established $S(3)$ when X is a compact,

(1) The compact case remains unsolved.

locally connected, connected metric space in [8] and A. H. Stone announced $S(3)$ in [4] for any locally connected, connected normal space.

In [3] it was shown that $S(4)$ is equivalent to $(*)$: X is unicoherent iff every pair of non-empty disjoint continua can be separated by a continuum.

We say that a set $A\subset X$ is *C-separated* (in X) provided that there exist disjoint subcontinua L and M of X such that $A\subset L\cup M$ and $A\cap L\neq\Phi\neq A\cap M$. We say that a space X has *Property* C if every separated closed set is C-separated. In [1] it was shown that if X has Property C, then $(*)$ holds, and hence $S(4)$ holds in such spaces. In [3] it was shown that every locally connected, locally compact, connected paracompact Hausdorff space has Property C and in [2] it was shown that every connected metric space with a Property S metric has Property C.

In this paper we offer a sequence of conjectures, $T(n)$, $n\geqslant 2$, characterizing multicoherent spaces. We show that

(i) for $n>1$, $S(2n)$ implies $T(n)$,

(ii) $S(4)$ is equivalent to $T(2)$,

(iii) $S(6)$ is equivalent to $T(3)$,

(iv) if X is weakly-finitely multicoherent or compact, then $T(n)$ holds for all $n\geqslant 2$, and

(v) if X is weakly-finitely multicoherent, then $S(n)$ holds for all $n>2$.

A disadvantage of Stone's conjecture is the inability to use induction to establish it. Of course in spaces where Stone's conjecture holds, a beautiful representation of the space is obtained. We use finite-induction to establish our conjecture in Lemma 3 and then show that Stone's conjecture follows from ours whenever the space is weakly-finitely multicoherent. Finally we show that any unicoherent, connected, locally connected (respectively, Property S) metric space has a unicoherent completion (respectively, compactification).

Let $n\geqslant 2$ be an integer and let $T(n)$ denote the following statement:

$T(n)$: X is multicoherent if and only if there exists pairwise disjoint non-empty continua $B_1,\dots,B_n$ such that

(i) no B_i separates X,

(ii) no component of the complement of the union of the B_i's has limit points in three of the B_i's and

(iii) for each i, $1\leqslant i\leqslant n$, $B_{i(\mathrm{mod}\,n)}\cup B_{(i+1)\mathrm{mod}\,n}$ separates X.

LEMMA 1. *If $X=A_1\cup A_2\cup A_3$ where each A_i is a continuum and $A_i\cap A_j\neq\emptyset$ for $1\leqslant i\leqslant j\leqslant 3$ and $A_1\cap A_2\cap A_3=\emptyset$, then X is multicoherent.*

Proof. Obvious.

LEMMA 2. *If $n>2$ is an integer, then $S(n+1)$ implies $S(n)$ and $T(n)$ implies $T(n-1)$.*

Proof. If $A_1,\dots,A_n,A_{n+1}$ satisfy the conditions (i)-(iii) of $S(n+1)$, then $A_1,A_2,\dots,A_n\cup A_{n+1}$ satisfies the conditions (i)-(iii) of $S(n)$.

If $B_1,B_2,\dots,B_n$ satisfy conditions, (i)-(iii) of $T(n)$, then $B_1,B_2,\dots,B_{n-2},B_{n-1}^1$ satisfy the conditions of $T(n-1)$ where B_{n-1}^1 is the union of B_{n-1} and B_n together with all of the components of $X/(\bigcup_{i=1}^{n}B_i)$ that have limit points in both B_n and B_{n-1}.

THEOREM 1. *If $n>1$ is an integer, then $S(2n)$ implies $T(n)$.*

Proof. In light of Lemmas 1 and 2 we need only show that when X is multicoherent, then there exists a collection of pairwise disjoint non-empty continua satisfying (i)-(iii) of $B(n)$. Suppose X is multicoherent. Then by $S(2n)$ there exist continua $A_1,\dots,A_{2n}$ satisfying (i)-(iii) of $S(2_n)$. Note that $\bigcup_{i=3}^{2n-1}A_i$ and A_1 are disjoint continua and so by the proof of Lemma 8 of [1] there exists continua B_1 and C_1 such that $X=B_1\cup C_1$, $\mathrm{Fr}\,B_1=\mathrm{Fr}\,C_1$ misses $A_1\cup(\bigcup_{i=3}^{2n-1}A_i)$, neither B_1 or C_1 separates X and $A_1\subset B_1$, $\bigcup_{i=3}^{2n-1}\subset C_1$. If $n=2$, let B_2 be A_3 together with all of the components of $X\setminus A_3$ that miss B_1.

Otherwise suppose $B_1,\dots,B_k$, $k<n-1$, have been defined so that they satisfy the conditions of $T(k)$ and $A_{2i-1}\subset B_i$ for $1\leqslant i\leqslant k$ and $(\bigcup_{i=2k+1}^{n-1}A_i)\cap(\bigcup_{i=1}^{k}B_i)=\emptyset$. Then there exist continua B_{k+1} and C_{k+1} such that

$$A_{2k+1}\subset B_{k+1},\quad [(\bigcup_{i=1}^{2k-1}A_i)\cup(\bigcup_{i=2k+3}^{2n}A_i)]\subset C_{k+1},$$

$$\mathrm{Fr}\,B_{k+1}=\mathrm{Fr}\,C_{k+1}\quad\text{is a subset of } A_{2k}\cup A_{2k+2}$$

and neither B_{k+1} or C_{k+1} separates X. Then $\{B_1,\dots,B_{k+1}\}$ satisfies the conditions (i)-(iii) of $T(k+1)$. The result now follows by induction.

For completeness we prove the following:

Let X be any connected, locally connected normal space. Then $S(3)$ holds.

Proof. By Lemma 1, we need only prove the necessity part of $S(3)$. To this end suppose that $X=H\cup K$ where H and K are continua and $H\cap K=A\cup B$ is a separation. By the proof of Lemma 3 of [2] there exists regions R_1 and R_2 of K such that $A\cap R_1\neq\emptyset\neq B\cap R_2$ and $\mathrm{Fr}\,R_1=\mathrm{Fr}\,R_2$ misses $A\cup B$, and K/R_2 is connected. Then H, $\overline{R}_2$, K/R_2 satisfies the conditions (i)-(iii) of $S(3)$.

THEOREM 2. *$S(4)$ is equivalent to $T(2)$.*

Proof. We need only show that if there exists non-empty disjoint continua B_1 and B_2 satisfying (i)-(iii) of $T(2)$, then there exists continua $A_1,\dots,A_4$ satisfying the conditions (i)-(iii) of $S(4)$. To this end suppose B_1 and B_2 satisfy conditions (i)-(iii) of $T(2)$. Then by the second part of the proof of Theorem 1 of [3], we obtain the desired continua $A_1,\dots,A_4$.

THEOREM 3. *$S(6)$ is equivalent to $T(3)$.*

Proof. We need only show that if B_1, B_2, B_3 satisfy the conditions of $T(3)$, then there exist continua $A_1, \ldots, A_6$ satisfying the conditions of $S(6)$. Let $B = B_1 \cup B_2 \cup B_3$ and for $(i,j) \in \{(1,2),(2,3),(3,1)\}$ let $H(i,j)$ be the union of all of the components of X/B that have limit points in both B_i and B_j. Since no B_i separates X, $X/B = H(1,2) \cup H(2,3) \cup H(3,1)$ and since no component of X/B has limit points in three of the B_i's, the sets $H(1,2)$, $H(2,3)$, $H(3,1)$ are pairwise disjoint. Let $Y = B_1 \cup H(1,2) \cup B_2$, $Z = B_2 \cup H(2,3) \cup B_3$, and $W = B_3 \cup H(3,1) \cup B_1$. Then Y, Z, W are continua and $X = Y \cup Z \cup W$. We can represent Y as the union of two continua A_1 and A_2 such that $B_1 \subset A_1$, $B_2 \subset A_2$ and $A_1 \cap A_2 \subset H(1,2)$. (In order to see this, let U be any open subset of Y containing B_1 whose closure misses B_2. Let Q be the component of U that contains B_1, P the component of Y/Q that contains B_2 and let R be the component of $Y/\overline{P}$ that contains B_1. Set $A_1 = \overline{R}$ and $A_2 = Y/R$. Then since Y is locally connected at every point of $Y/(B_1 \cup B_2)$, $\mathrm{Fr} A_1 = \mathrm{Fr} A_2 \subset Y/(B_1 \cup B_2)$). Likewise let $Z = A_3 \cup A_4$ where $B_2 \subset A_3$, $B_3 \subset A_4$ and $A_3 \cap A_4 \subset H(2,3)$ and $W = A_5 \cup A_6$, where $B_3 \subset A_5$, $B_1 \subset A_6$ and $A_1 \cap A_6 \subset H(1,3)$. Then $\{A_1, \ldots, A_6\}$ satisfies the conditions (i)-(iii) of $S(6)$.

LEMMA 3. *Suppose $n \geqslant 2$ is an integer and X is weakly-finitely multicoherent or compact and $B_1, \ldots, B_n$ are non-empty, pairwise disjoint continua that satisfy the conditions* (i)-(iii) *of $T(n)$. Then there exists non-empty pairwise disjoint continua $B'_1, \ldots, B'_{n+1}$ satisfying the conditions* (i)-(iii) *of $T(n+1)$. Furthermore the B'_i's can be chosen so that $B_1 \subset B'_1$, $B'_i = B_i$ for $2 \leqslant i < n$, $B_n \subset B'_n$ and B'_{n+1} is a subset of the union of the components of $X/(\bigcup_{i=1}^{n} B_i)$ that have limit points in both B_1 and B_n.*

Proof. Let H be the union of all the components of $X/(B_1 \cup B_n)$ that have limit points in both B_1 and B_n and let $Y = B_1 \cup H \cup B_n$. Then by conditions (i)-(iii) of $T(n)$, Y is a continuum and $Y \cap B_i = \emptyset$ for $i \neq 1, n$. We consider two cases:

Case I. There exists a continuum T lying entirely in H that separates B_1 and B_n in Y.

In this case let B'_{n+1} be the union of T together with all of the components of H/T that fail to have limit points in either B_1 or B_2. Then $B_1, B_2, \ldots, B_n, B'_{n+1}$ satisfies the conclusion of our Lemma.

Case II. There does not exist a continuum lying in H that separates B_1 and B_n in Y.

However in this case, since X is either weakly-finitely multicoherent or compact (and locally connected) there does exist a finite collection, say $C_1, \ldots, C_k$ of continua each lying in H such that $C = \bigcup_{i=1}^{k} C_i$ separates B_1 and B_n in Y and for any i, $1 \leqslant i \leqslant k$. C/C_i fails to separate B_1 and B_n in Y. For each i, let D_i be the union of C_i together with all of the components of X/C_i with limit points only in C_i. Then $D_1, \ldots, D_k$ is a collection of pairwise disjoint continua in Y such that no D_i separates Y and the union of any proper subcollection of $\{D_i\}_{i=1}^{k}$ fails to separate B_1 and B_n in Y. Let $D = \bigcup_{i=1}^{k} D_i$. Note that every component of $Y/(D \cup B_1 \cup B_n)$ has either limit points in both D and B_1 or limit points in both D and B_n. Let Q be the union of B_n together with all the components of Y/D with limit points in B_n and let P be B_1 together with all the components of Y/D that have limit points in B_1. Then P and Q are regions in Y and $Y = P \cup D \cup Q$ where P, Q, D are pairwise disjoint and furthermore for any i, $1 \leqslant i \leqslant k$, $D_i \cap \overline{P} \neq \emptyset \neq D_i \cap \overline{Q}$. One can thus find a continuum L in X such that L contains $(\bigcup_{i=2}^{k} D_i) \cup B_n$, $L \cap D_1 = \emptyset$, $L \cap B_i = \emptyset$ for $i \neq n$, and L fails to separate D_1 and B_1 in Y. (In order to do this, first choose a continuum M in X that contains $D_1 \cup B_1$ and such that M misses $Q \cup (\bigcup_{i=2}^{k} D_i)$. Then choose L in the complement of M.)

Then if B'_n is the union of L together with all of the components of X/L that have limit points only in L and if we set $B'_{n+1} = D_1$ for $2 \leqslant i < n$, $B'_n = B_n$, we have that (a) for $1 < i \leqslant n+1$, no B_i separates X and (b) each of the sets $B'_2 \cup B'_3$, $B'_3 \cup B'_4, \ldots, B'_n \cup B'_{n+1}$ separates X. In order to complete the proof we need to enlarge B_1 to a continuum B'_1 so that $B'_{n+1} \cup B'_1$ and $B'_1 \cup B'_2$ separates X, no component of $X/(\bigcup_{i=1}^{n+1} B'_i)$ has limit points in three of the B_i's, and B'_i fails to separate X. To this end let R_n and R_{n+1} be regions in X that contain B'_n and B'_{n+1} respectively and such that $\overline{R}_n \cap \overline{R}_{n+1} = \emptyset$ and $(\overline{R}_n \cup \overline{R}_{n+1}) \cap (B_1 \cup B'_2 \cup B'_3 \cup \ldots \cup B'_{n-1}) = \emptyset$. Since X is either weakly-finitely multicoherent or compact, there exist finitely many continua $F_1, \ldots, F_s$ such that $\mathrm{Fr}(R_1 \cup R_2) \cap P$ is a subset of $F_1 \cup \ldots \cup F_s$ and each of the sets F_i, $1 \leqslant i \leqslant s$, lies entirely in P. Now let F be any continuum containing $B_1 \cup F_1 \cup \ldots \cup F_s$ such that $F \cap B'_i = \emptyset$ for $1 < i \leqslant n+1$ and set B'_1 to be the union of F together with all of the components of X/F with limit points only in F. Then $B'_1, \ldots, B'_n$, B'_{n+1} satisfy the conclusion of our Lemma.

LEMMA 4. *If X is weakly-finitely multicoherent or compact, then $T(2)$ holds in X.*

Proof. Suppose X is weakly-finitely multicoherent and $X = H \cup K$ where $H \cap K$ is separated. Then since $b_0(H \cap K) < \infty$, there exists disjoint continua L and M such that $H \cap K \subset L \cup M$, $H \cap K \cap L \neq \emptyset \neq H \cap K \cap M$. Let B_1 be the union of L together with all of the components of X/L that have limit points only in L and let B_2 be the union of M together with all of the components of X/M that only have limit points in M. Then $\{B_1, B_2\}$ satisfy conditions (i)-(iii) of $T(2)$. The compact case is similarly established.

THEOREM 4. *If X is weakly-finitely multicoherent or compact, then $T(n)$ holds in X for all $n \geqslant 2$.*

The proof follows immediately from Lemmas 3 and 4.

LEMMA 5. *Let Z be a connected normal space, let A and B be non-empty disjoint subcontinua of Z such that Z is locally connected at every point of $Z/(A \cup B)$. Suppose* *that $T_1, \ldots, T_k$ is a collection of continua such that for any i, $1 \leqslant i \leqslant k$, T_i separates $U_i = A \cup (\bigcup_{j=1}^{i-1} T_j)$ and $V_i = B \cup (\bigcup_{j=i+1}^{k} T_j)$ in Z but fails to separate either U_i or V_i. Then $Z = \bigcup_{i=1}^{k+1} A_i$ where $A \subset A_1$, $B \subset A_{k+1}$, no three of the A_i's have a point in common, $A_j \cap (A \cup B) = \emptyset$ if $j \neq 1$, $k+1$ and $A_i \cap A_j \neq \emptyset$ if and only if $|i-j| \leqslant 1$.*

Proof. Suppose $k = 1$. Let P be the component of X/T_1 that contains A and let $A_2 = X/P$ and $A_1 = P \cup T_1$. Then $\{A_1, A_2\}$ satisfies the conclusion of our lemma. Now suppose that $\{T_1, \ldots, T_k\}$ satisfies the hypotheses of our lemma. Choose A_1 and A_2 as in the case for $k = 1$. Then $\{T_2, \ldots, T_k\}$ satisfies the hypothesis of our lemma relative to the disjoint continua A_1 and B. The result now follows by induction.

THEOREM 5. *If X is weakly-finitely multicoherent, then $S(n)$ holds for all $n > 2$.*

Proof. Let $N > 2$ be an integer. By Theorem 4, exist continua B_1^2 and B_2^2 satisfying (i)-(iii) of $T(2)$. Let $X \setminus (B_1^2 \cup B_2^2) = H^2 \cup K^2$ and $Y^2 = B_1^2 \cup H^2 \cup B_2^2$. Suppose that no subcontinuum of Y^2 separates B_1^2 and B_2^2 in Y^2. Then by the proof of Lemma 3, there exists non-empty continua B_1^3, B_2^3, B_3^3 such that $\{B_i^3\}_{i=1}^3$ satisfy the conditions (i)-(iii) of $T(3)$ and $B_1^2 \subset B_1^3$, $B_2^2 \subset B_2^3$ and $B_3^3 \subset H^2$, and in particular $H^3(1, 2)$, the union of the components of $X \setminus (\bigcup_{i=1}^{3} B_i^3)$ that have limit points in both B_1^3 and B_3^3, is non-empty.

Suppose again that no subcontinuum of $Y^3 = B_1^3 \cup H^3(1, 3) \cup B_3^3$ separates B_1^3 and B_3^3 in Y^3 where $H^3(1, 3)$ is the union of the components of $X \setminus (\bigcup_{i=1}^{3} B_i^3)$ that have limit points in both B_1^3 and B_3^3. Then as above there exist continua B_1^4, B_2^4, B_3^4, B_4^4 satisfying the conditions (i)-(iii) of $T(4)$, $B_1^3 \subset B_1^4$, $B_2^3 = B_2^4$, $B_3^3 \subset B_3^4$, $B_4^4 \subset H^3(1, 3)$ and

$$H^4(1, 3) = \text{union of all components of } X \setminus (\bigcup_{i=1}^{4} B_i^4)$$

that have limit points in both B_1^4 and B_4^4 is non-empty.

Continue in this fashion, using the proof of Lemma 3, to construct a sequence of collections $\{B_i^n\colon 1 \leqslant i \leqslant n\}_{n=2}^{\infty}$ of non-empty, pairwise disjoint continua such that each of the collections $\{B_i^n\}_{i=1}^n$ satisfies the conditions (i)-(iii) of $T(n)$ for each $n \geqslant 2$, $B_1^n \subset B_1^{n+1}$, $B_n^n \subset B_n^{n+1}$ and for each i, $1 < i < n$, $B_i^{n+1} = B_i^n$. For each $n > 1$, let $B^n = \bigcup_{i=1}^{n} B_i^n$ and for each triple (n, i, j) where $1 \leqslant i < j \leqslant n$, let $H^n(i, j)$ be the union of all components of $X \setminus B^n$ that have limit points in both B_i^n and B_j^n. We note that from our construction we can choose the collections so that $H^{n+1}(1, i) \subset H^n(1, i)$ for $1 \leqslant i \leqslant n$ and $H^n(i, j) = \emptyset$ if $i+1 < j < n$. Then since $H^{n+1}(1, n) \cap H^{n+1}(1, n+1) = \emptyset$, the collection $\{H^{n+1}(1, n)\}$ is a collection of pairwise disjoint sets such that

for any non-empty member, say $H^{m+1}(1, m)$, $H^{m+1}(1, m)$ has limit points in both B_1^{m+1} and B_m^{m+1}.

We next observe that if for some $k \geqslant 2$, there does not exist a subcontinuum of $Y^k = B_k^k \cup H^k(1, k) \cup B_1^k$ that separates B_k^k and B_1^k in Y^k, then $H^{k+1}(1, k)$ is non-empty.

Suppose that this is the case for infinitely many $k \geqslant 2$. Then for each k, where $H^{k+1}(1, k) \neq \emptyset$, $Z^k = B_k^{k+1} \cup H^{k+1}(1, k) \cup B_1^{k+1}$ can be represented as the union of two continua W_1^k and W_k^k where $\operatorname{Fr} W_1^k = \operatorname{Fr} W_k^k$ is a non-empty subset of $H^{k+1}(1, k)$ that misses $B_k^{k+1} \cup B_1^{k+1}$ and $W_1^k \supset B_1^{k+1}$ and $W_k^k \supset B_k^k$. Then

$$W_1 = \bigcup \{W_1^k\colon k \geqslant 2 \text{ and } H^{k+1}(1, k) \neq \emptyset\}$$

and

$$\begin{aligned} W_\infty &= (\bigcup \{W_k^k\colon k \geqslant 2 \text{ and } H^{k+1}(1, k) \neq \emptyset\}) \cup (\bigcup_{k=2}^{\infty} H^{k+1}(k, k+1)) \\ &= W_2^2 \cup H^3(2, 3) \cup W_3^3 \cup H^4(3, 4) \cup \ldots \\ &\quad \ldots \cup H^{k-1}(k-1, k) \cup W_k^k \cup H^k(k, k+1) \cup \ldots \end{aligned}$$

are connected sets such that $X = \overline{W}_1 \cup \overline{W}_\infty$. Furthermore for each $k \geqslant 2$, $\operatorname{Fr} W_1^k \subset \overline{W}_1 \cap \overline{W}$ and so $b_0(\overline{W}_1 \cap \overline{W}_\infty) = \infty$.

This is a contradiction, hence for some $k_0 \geqslant 2$, every pair of the sets B_j^j and B_1^j can be separated by a continuum in $H^j(1, j)$ whenever $j \geqslant k_0$. It then follows that $Z = B_{k_0}^{k_0} \cup H^{k_0}(1, k_0) \cup B_1^{k_0}$ satisfies the conditions of Lemma 5, and so $Z = \bigcup_{i=1}^{N} A_i$ where each A_i is non-empty, $B_{k_0}^{k_0} \subset A_N$, $B_1^{k_0} \subset A_1$ and $A_j \cap (B_1^{k_0} \cup B_{k_0}^{k_0}) = \emptyset$ if $1 \neq j$ and $N \neq j$, no three of the A_i's have a point in common and $A_i \cap A_j \neq \emptyset$ iff $|i-j| \leqslant 1$. Then if

$$\begin{aligned} A_N &= (\bigcup_{i=1}^{k_0} B^{k_0}) \cup (\bigcup_{i=1}^{k_0-1} H^{k_0}(i, i+1)) \\ &= B_1^{k_0} \cup H^{k_0}(1, 2) \cup B_2^{k_0}(1, 2) \cup B_2^{k_0} \cup H^{k_0}(2, 3) \cup B_3^{k_0} \cup \ldots \\ &\quad \ldots \cup H^{k_0}(k_0-1, k_0) \cup B_{k_0}^{k_0}, \end{aligned}$$

$\{A_1, \ldots, A_N\}$ satisfies the condition of $S(N)$.

COROLLARY (5.1). *If X is a locally connected, connected normal space and $r(X) = \infty$, then either $r(X)$ is attained or $S(n)$ holds for all $n > 2$.*

Remark (5.2). Let X be weakly-finitely multicoherent and let k_0 be chosen as in the proof of Theorem 5. Then as we noted in the proof of Theorem 5, X is very well behaved between $B_{k_0}^{k_0}$ and $B_1^{k_0}$, i.e. on the continuum Z. In fact if X is separable one can show, using the techniques of [1], that there exists a continuous function $f\colon Z \to [0, 1]$ such that $f(B_{k_0}^{k_0}) = 1$, $f(B_1^{k_0}) = 0$, and for some dense subset D

of $(0, 1)$, $f^{-1}(d)$ is connected for every $d \in D$. If X is also compact, f can be shown to be a monotone map.

Proof. By Remark 5 of [1], there exists a non-alternating mapping $f\colon Z \to [0, 1]$ such that $f(B_{k_0}^{k_0}) = 1$, $f(B_1^{k_0}) = 0$ and by the main result of [10]; there exists a dense subset D of $(0, 1)$ such that if $d \in D$ and U is an open subset of Z which meets $f^{-1}(d)$, then d is interior to $f(U)$. Then as noted in Theorem 1 of [1] for any $d \in D$, $f^{-1}(d)$ is an irreducible separating set in Z, i.e. if A is a proper subset of $f^{-1}(d)$, $Z \setminus A$ is connected. Now if for some $d \in D$, $f^{-1}(d)$ fails to be connected, we may, as in the proof of Theorem 5, construct $B_1^{k_0+1}, \ldots, B_{k_0+1}^{k_0+1}$s o that $H^{k_0+1}(1, k_0)$ is non-empty. This contradicts our selection of k_0 and thus $f^{-1}(d)$ must be connected for each $d \in D$. In case X is also compact, it follows from Lemma 9 of [1] that f is monotone.

Definition. A metric d for a space X is said to have *Property* S (alternately (X, d) has Property S), if for any $\varepsilon > 0$, there exists finitely many connected sets of d-diameter less than ε whose union is X. In [9] Property S is shown to imply local connectedness. In [2] it was shown that every connected metric space with Property S has Property C and it was shown that every rim-compact, locally connected, separable connected metric space has a compatible Property S metric and thus $T(2)$ and $S(4)$ hold such spaces. An example was given in [2] showing that not every unicoherent locally connected, connected separable metric space has Property C, and hence not every such space has a compatible Property S metric.

Definition. We say that a metric d for a space X is a *connected metric* (or simply a *c-metric*) if for any $\varepsilon > 0$ and $x \in X$, $S_d(x, \varepsilon) = \{y \mid d(x, y) < \varepsilon\}$ is a connected set.

Lemma 6. *If (Y, ϱ) is a metric space and X is a dense subset of Y such that $\varrho | X = d$ is a connected metric, then Y is ϱ is a c-metric and hence Y is locally connected and furthermore if U is any connected open subset of Y, $U \cap X$ is connected in X.*

Proof. Let $y \in Y$ and $\varepsilon > 0$. Let $\{x_i\}_{i=1}^{\infty}$ be a sequence of points in X such that $\varrho(y, x_i) < \varepsilon/i$. It then follows that for any $i \geqslant 1$, the closure of any sphere $S_\varrho(x_i, \varepsilon - \varepsilon/i)$ is a subset of $S_\varrho(y, \varepsilon)$. We next assert that $S_\varrho(y, \varepsilon)$ is a subset of the union of the closures of the collection $\{S_\varrho(x_i, \varepsilon - \varepsilon/i)\}_{i=1}^{\infty}$. To see this let $x \in S_\varrho(y, \varepsilon)$, and let $\alpha = \varrho(x, y)$. Choose i so that $\varepsilon/i < \varepsilon - \alpha$. Then

$$\varrho(x, x_i) \leqslant \varrho(x, y) + \varrho(y, x_i) < \alpha + \varepsilon/i < \alpha + (\varepsilon - \alpha) = \varepsilon .$$

Thus

$$S_\varrho(y, \varepsilon) = \bigcup_{i=1}^{\infty} \overline{S_\varrho(x_i, \varepsilon/i)} = \bigcup_{i=1}^{\infty} \overline{S_\varrho(x_i, \varepsilon/i) \cap X} = \bigcup_{i=1}^{\infty} \overline{S_d(x_i, \varepsilon/i)}$$

(the closures all being taken in Y) and since each of the sets $\overline{S_d(x_i, \varepsilon/i)}$ is connected and contains y, $S_\varrho(y, \varepsilon)$ is connected.

We now argue that for any $y \in Y$ and $\varepsilon > 0$, $S_\varrho(y, \varepsilon) \cap X$ is connected. In order to see this suppose that $S_\varrho(y, \varepsilon) \cap X = H \cup K$ is a separation in X. Then there exists a point $z \in Y \setminus X$ such that $z \in S_\varrho(y, \varepsilon) \cap \overline{H} \cap \overline{K}$ (the closures taken in Y). There exists $\delta > 0$ such that any ϱ-sphere of radius δ that contains z lies entirely in $S_\varrho(y, \varepsilon)$. Let $x_1 \in S_\varrho(z, \frac{1}{4}\delta) \cap H$ and $x_2 \in S_\varrho(z, \frac{1}{4}\delta) \cap K$. Then $S_d(x_1, \frac{1}{2}\delta) = S_\varrho(x_1, \frac{1}{2}\delta) \cap X$ is a connected set containing z that lies entirely in $S_\varrho(y, \varepsilon) \cap X$. Likewise $S_d(x_2, \frac{1}{2}\delta)$ is a connected subset of $S_\varrho(y, \varepsilon) \cap X$ that contains z, so that $C = S_d(x_1, \frac{1}{2}\delta) \cup S_d(x_2, \frac{1}{2}\delta)$ is a connected subset of $X \cap S_\varrho(y, \varepsilon)$ that meets H and K. This is a contradiction, hence $S_\varrho(y, \varepsilon) \cap X$ is connected.

It now follows very easily that for any connected open subset U of Y, $U \cap X$ is connected. For let $a, b \in U \cap X$. For each $y \in Y$, let $S_\varrho(y, \varepsilon_y)$ be a d-sphere that lies entirely in Y. Then by the Simple Chain Theorem of [11] there exists a finite collection $y_1, \ldots, y_n$ of elements of Y such that

$$a \in S_\varrho(y_1, \varepsilon_{y_1}),\ S_\varrho(y_i, y_{\varepsilon_i}) \cap S_\varrho(y_{i+1}, y_{\varepsilon_{i+1}}) \neq \emptyset$$

for each i, $1 \leqslant i < n$ and $b \in S_\varrho(y_n, y_{\varepsilon_n})$. Then $\bigcup_{i=1}^{n} [S_\varrho(y_i, y_{\varepsilon_i} \cap X]$ is a connected set containing a and b that lies entirely in $U \cap X$. Hence $U \cap X$ is connected.

Notation. For any metric d on a space X let $(\tilde{X}, \tilde{d})$ be the completion of (X, d).

In [9], G. T. Whyburn showed that for any connected, locally connected metric space (X, d) there was a c-metric d_c for X such that the identity map $h\colon (X, d) \to (X, d_c)$ is a homeomorphism and h^{-1} is uniformly continuous. Furthermore if (X, d) has Property S, so has (X, d_c) and h is also uniformly continuous.

Theorem 6. *Let (X, d) be a connected, locally connected metric space. Then if any of the spaces (X, d), (X, d_c) or $(\tilde{X}, \tilde{d}_c)$ is weakly-finitely multicoherent, then for all $n > 2$, $S(n)$ holds in (X, d).*

Proof. Let $N > 2$ be an integer. Since (X, d) and (X, d_c) are homeomorphic, we need only suppose that $(\tilde{X}, \tilde{d}_c)$ is weakly-finitely multicoherent and show that $S(N)$ holds in (X, d_c).

By our supposition and Theorem 5, $S(N)$ holds in $(\tilde{X}, \tilde{h}_c)$, that is there exist non-empty continua $A_1, A_2, \ldots, A_N$ satisfying conditions (i)-(iii) of $S(N)$. By the normality and local connectedness of $\tilde{X}$, there exists regions $R_1, R_2, \ldots, R_N$ such that for each i, $1 \leqslant i \leqslant N$, $A_i \subset R_i$ and $\{\overline{R}_i\}_{i=1}^{N}$ satisfies the conditions of $S(N)$. Then by Lemma 6, each of the sets $U_i = R_i \cap X$ is connected and by the denseness of (X, d_c) in $(\tilde{X}, \tilde{d}_c)$, the collection $\{\overline{R_i \cap X}\}_{i=1}^{N}$ (the closures being taken in (X, d_c)) satisfy the conditions of $S(N)$ in (X, d_c). This completes the proof.

Theorem 7. *Let (X, d) be a locally connected, connected metric space. Then if X is unicoherent, so also is $(\tilde{X}, \tilde{d}_c)$. Hence if (X, d) also has Property* S, *$(\tilde{X}, \tilde{d}_c)$ is a unicoherent, locally connected, compactification of X.*

Proof. Suppose that $\tilde{X} = H \cup K$ and $H \cap K$ is separated. Then if U and V are regions of $(\tilde{X}, \tilde{d}_c)$ containing H and K respectively such that $\overline{U} \cap \overline{V}$ is separated, $H_0 = \overline{U \cap X}$ and $K_0 = \overline{V \cap X}$ (the closures taken in (X, d_c) are contained in X with $H_0 \cap K_0$ a separated set. Since (X, d_c) and (X, d) are homeomorphic, this is a contradiction. Hence $(\tilde{X}, \tilde{d}_c)$ is unicoherent.

If (X, d) also has Property S, by (9.1.5) of [9], (X, d_c) has Property S and hence is totally bounded. Then $(\tilde{X}, \tilde{d}_c)$ is compact. This completes the proof.

Corollary (7.1). *If (X, d) is a connected, locally connected (Property S) metric space and every locally connected metric completion (compactification) of X is multicoherent, then X is multicoherent.*

Remark. If we can show that whenever (X, d) is a multicoherent locally connected, connected multicoherent Property S metric space, there exists a Property S metric δ such that $(\tilde{X}, \tilde{\delta}_c)$ is multicoherent, then we can show that for all $n>1$, $T(n)$ holds in all locally connected, Property S metric spaces. (This is because $(\tilde{X}, \tilde{\delta}_c)$ would by compact and by Theorem 4, $T(n)$ holds in compact spaces).

If we can establish that in every compact, locally connected, connected metric spaces $S(n)$ holds for all $n>2$, then under the circumstances above we could establish $S(n)$ for all locally connected, connected Property S metric spaces.

Example (1). Let C denote the complex numbers, $D_0 = \{z\colon 0<|z|\leqslant 1\}$ and $D_1 = \{z\colon \frac{1}{2}<|z|\leqslant 1\}$. Then D_0 is homeomorphic to D_1, $\overline{D}_0$ is unicoherent while $\overline{D}_1$ is not. Thus for this *nice* example we can find such a δ_c as in the remark.

Example (2). For each $i\geqslant 1$, let $L_i = \{(x, y)\colon\ x = 1/i$ and $0\leqslant y\leqslant 1$. Let T = line segment joining $(0, 1)$ to $(1, 1)$ and B = line segment joining $(0, 0)$ to $(1, 0)$. Then $X = B \cup T \cup (\bigcup_{i=1}^{\infty} L_i)$ is locally connected and fails to have a locally connected compactification (it is also not rim-compact). Hence X fails to have a Property S metric. Necessary and sufficient conditions for a locally connected, connected, metric space to have a locally connected metric compactification are not known.

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Accepté par la Rédaction le 26. 6. 1974