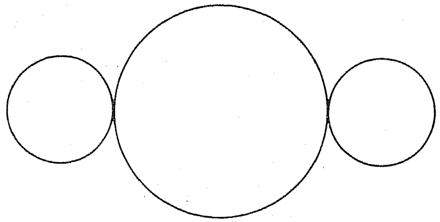

Give X the arc-length metric. The map is constructed roughly as follows: stretch each of the small circles onto the big circle; stretch each of the upper and lower semi-circles of the big circle first around a smaller circle, then across the other semi-circle, and finally around the other smaller circle.

For those who prefer a formula, we let $f\colon X\to X$ be defined by:

$$f(z)=\begin{cases} 2(z-\tfrac{3}{2}) & \text{if} \quad \operatorname{Re}(z)\geqslant 1\,,\\ 2(z+\tfrac{3}{2}) & \text{if} \quad \operatorname{Re}(z)\leqslant -1\,,\\ \tfrac{1}{2}z^{-6}-\tfrac{3}{2} & \text{if} \quad \tfrac{1}{2}\leqslant\operatorname{Re}(z)\leqslant 1\,,\\ z^3 & \text{if} \quad -\tfrac{1}{2}\leqslant\operatorname{Re}(z)\leqslant\tfrac{1}{2}\,,\\ -\tfrac{1}{2}z^{-6}+\tfrac{3}{2} & \text{if} \quad -1\leqslant\operatorname{Re}(z)\leqslant-\tfrac{1}{2}\,.\end{cases}$$

The reader can check that this is indeed a local expansion with no fixed points.

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On the *topology and its application

by

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Abstract. The purpose of the present paper is to study the relation between the set of the first category and the null set by introducing the *topology to T_1-space. As a result of this application, we made clearer the similarity and difference of the set having the Baire property and the measurable set in the sense of Lebesgue.

§ 1. Introduction. Let I be a T_1 space defined by the closure operation $X\to\overline{X}$. We denote by P some property about the subsets of I, and by $\boldsymbol{P}$ the family of all subsets of I which satisfy P. We say that a subset X has the *property* P at a point $p\in I$, if there exists a neighbourhood $V(p)$ of p such that $V(p)X\in\boldsymbol{P}$. We denote by X^* the set of points at which X does not have the property P, namely $X^* = \{p/\forall V(p),\ V(p)X\notin\boldsymbol{P}\}$. Assume that the family $\boldsymbol{P}$ is an ideal, i.e.,

$$\text{(1)}\quad \begin{array}{l}\text{(i) the conditions } X\in\boldsymbol{P} \text{ and } Y\subset X \text{ imply } Y\in\boldsymbol{P},\\ \text{(ii) the conditions } X\in\boldsymbol{P} \text{ and } Y\in\boldsymbol{P} \text{ imply } X+Y\in\boldsymbol{P},\end{array}$$

then the operation $X\to X^*$ has the following properties

$$\text{(2)}\quad \begin{array}{ll}\text{(a) } X^* \text{ is closed,} & \text{(b) if } X\subset Y, \text{ then } X^*\subset Y^*,\\ \text{(c) } X^{**}\subset X^*\subset\overline{X}\,, & \text{(d) if } G \text{ is open, then } GX^* = G(GX)^*,\\ \text{(e) } (XY)^*\subset X^*\cdot Y^*, & \text{(f) } X^*-Y^*\subset(X-Y)^*.\end{array}$$

Assume further that the property P satisfies the relation

$$\text{(3)}\qquad \{X\in\boldsymbol{P}\}\equiv\{XX^*=0\}\equiv\{X^*=0\}\,,$$

then

$$\text{(4)}\quad \begin{array}{l}\text{(a) } (X-X^*)^*=0, \text{ namely } X-X^*\in\boldsymbol{P},\\ \text{(b) } X^{**}=X^*,\\ \text{(c) if } Y\in\boldsymbol{P}, \text{ then } (X\pm Y)^*=X^*\end{array}$$

(see [2], [3]).

In the following, we shall now assume that P satisfies the conditions (1) and (3) and that every single element-subset of I belongs to $\boldsymbol{P}$. Two important examples of the family $\boldsymbol{P}$ of this kind are the family of the sets of the first category and the family of the sets of measure zero (in the sense of Lebesgue).

In this paper, introducing the notion of *topology, we shall investigate some properties of the set of the first category and that of Lebesgue measure zero collectively and make clear a relation between the set having the Baire property and the measurable set.

§ 2. **The *topology.** For any subset X in our T_1 space 1, we define the *closure $\tilde{X}$ of X by $\tilde{X} = X+X^*$. Defining this way, we see easily by the properties of X^* that $\tilde{X}\subset\bar{X}$ and that $\tilde{X}$ satisfies the axioms of T_1 space, and defines a new topology on the set 1, which we call *topology. Our set 1 has now two topologies on it; hereafter the notion on *topology will be specified by *.

Definition 1. A subset $X\subset 1$ is **closed*, **boundary* and **nowhere dense* if and only if $\tilde{X} = X$, $\widetilde{1-X} = 1$ and $\widetilde{1-\tilde{X}} = 1$ respectively.

To be more precise, we should say that, for example, X is *closed with respect to the property P. However, we will omit "with respect to P" in the following, when no confusion is expected. [1]

Theorem 1. *A subset $X\subset 1$ is *closed if and only if X is the union of a closed set and a set belonging to $\boldsymbol{P}$.*

Proof. Let X be a *closed set. We have $X = \tilde{X}\supset X^*$. Therefore, X is the union of a closed set X^* and $X-X^*$ which belongs to $\boldsymbol{P}$ by (4). Conversely, let X be the union of a closed set F and a set A in $\boldsymbol{P}$. It follows from (2) and (4) that $X^* = (F+A)^* = F^*\subset F\subset X$. Hence X is *closed.

Corollary. *A subset $X\subset 1$ is *open if and only if X is the difference of an open set and a set belonging to $\boldsymbol{P}$.*

Lemma 1. *For every open set G and every $X\subset 1$, we have $(GX)^* = (GX^*)^* = \overline{GX^*}$*

Proof. Let G be an open set. First we have $GX^*\subset(GX)^*$ by (2) (d), which implies $(GX^*)^*\subset(GX)^{**} = (GX)^*$ by (2) and (4). $(G(X-X^*))^* = 0$ by (4), then $(GX)^*\subset(GX^*)^*$ holds by (2)(f) and we have $(GX^*)^* = (GX)^*$. Secondly, $GX^*\subset(GX)^*$ implies $\overline{GX^*}\subset(GX)^*$ by (2)(a). On the other hand, (2)(c) implies $\overline{GX^*}\supset(GX^*)^* = (GX)^*$, from which follows $\overline{GX^*} = (GX)^*$ immediately.

Corollary. *An open set G belonging to $\boldsymbol{P}$ is contained in $1-1^*$.*

Proof. Putting $X = 1$ in Lemma 1, it follows that $G^* = \overline{G1^*}$ for any open set G. Hence if $G\in\boldsymbol{P}$ is open, then $G^* = 0$ by (3). Therefore we have $G\subset 1-1^*$.

Lemma 2. *If a subset $*G$ is *open, then we have $\widetilde{*GX^*} = \overline{*GX^*}$ for any subset X.*

Proof. Let $*G$ be a *open set. Then $*G$ is of the form $*G = G-A$ where G is open and $A\in\boldsymbol{P}$ by the above Corollary. Hence it follows from Lemma 1 that $\widetilde{*GX^*} = \widetilde{(G-A)X^*} = \widetilde{GX^*-A} = GX^*-A+(GX^*)^* = \overline{GX^*}$. Therefore we have $\overline{*GX^*} = \overline{(G-A)X^*} = \overline{GX^*-A}\subset\overline{GX^*} = \widetilde{*GX^*}$. Clearly $\widetilde{*GX^*}\subset\overline{*GX^*}$ holds, and thus $\widetilde{*GX^*} = \overline{*GX^*}$ follows.

Corollary. *If the space 1 satisfies $1^* = 1$, then $\widetilde{*G} = \overline{*G}$ for any *open set $*G$.*

Note that $p\in\widetilde{X-p}$ is equivalent to $p\in X^*$. Therefore let p be a *accumulation point of a subset X, namely $p\in\widetilde{X-p}$, then X^* is nothing but the *derived set of X, i.e., the set of all *accumulation points of X. X is *dense in itself if $X\subset X^*$. A *scattered set is a set not containing *dense in itself non-empty subset.

Let X be a subset of 1. $X = X-X^*+XX^*$ implies $XX^*\subset X^* = (XX^*)^*$ by (4), then XX^* is a *dense in itself set contained in X. Hence, if a subset X is a *scattered set, then $XX^* = 0$, and $X\in\boldsymbol{P}$ is necessary by (3). Conversely, if $X\in\boldsymbol{P}$, any subset Y of X also belongs to $\boldsymbol{P}$, and $Y^* = 0$. Namely X can not contain any non-empty subset Y with $Y\subset Y^*$. Thus any $X\in\boldsymbol{P}$ is a *scattered set. We have proved the following:

Theorem 2. *A subset X is a *scattered set if and only if X belongs to $\boldsymbol{P}$.*

It is well known that, in any topology, a scattered set is the union of an open set and a nowhere dense set. Therefore the following is an immediate corollary of Theorem 2.

Proposition. *X belonging to $\boldsymbol{P}$ is represented in the form $X = *G+*N$, where $*G$ and $*N$ are *open and *nowhere dense, respectively.*

We shall study the property of $*G$ and $*N$ in the above proposition. $*G$ is of the form $*G = G-A$ by Corollary of Theorem 1, and it belongs to $\boldsymbol{P}$ as a subset of X. Then $G\in\boldsymbol{P}$, so that $G\subset 1-1^*$ by Corollary of Lemma 1, and $*G\subset 1-1^*$ follows. Next, $*N_1 = *N(1-1^*)$ is a set belonging to $\boldsymbol{P}$ and contained in $1-1^*$. Then $(1-*N_1)^*\cdot *N_1 = 0$, and $*N_1$ is not *nowhere dense. $*N_1$ is a subset of $*N$, then $*N_1 = 0$ and $*N\subset 1^*$ follows. Thus we have

Theorem 3. *X belonging to $\boldsymbol{P}$ is decomposed into the union $X = *G+*N$, where $*G$ is *open contained in $1-1^*$, and $*N$ is *nowhere dense contained in 1^*,*

Corollary 1. *Every set belonging to $\boldsymbol{P}$ is *nowhere dense if and only if $1^* = 1$ holds true.*

Corollary 2. *If the space 1 satisfies $1^* = 1$, then the complement of every set belonging to $\boldsymbol{P}$ is dense.*

Proof. By Theorem 3, $X\in\boldsymbol{P}$ can be represented in the form $*G+*N$, then $1-X = (1-*N)-*G$. Here $D = 1-*N$ is dense, since $1 = \widetilde{1-*\tilde{N}}\subset\overline{1-*N}$, then $1-X = D-A$, where $A = *G\in\boldsymbol{P}$ as a subset of $1-1^*$. In our space $A = \emptyset$, then $1-X$ is dense.

The above Corollary 2 generalizes the statement that, in a Baire space, the complement of any set of the first category is dense.

§ 3. **The *nowhere dense set.** Throughout this section, our space 1 satisfies the condition $1^* = 1$.

Lemma 3. *A nowhere dense set is a *nowhere dense set.*

Proof. Let X be a nowhere dense set. We see that $1 = \overline{1-\overline{X}} = \widetilde{1-\overline{X}} \subset \widetilde{1-\tilde{X}}$ by Corollary of Lemma 2, and that X is *nowhere dense.

$*\mathrm{Int}\, X$ denotes the *interior of X: $*\mathrm{Int}\, X = 1-\widetilde{1-X}$.

Lemma 4. *$\overline{*\mathrm{Int}\, \tilde{X}}$ is the set of all points at which X is not locally *nowhere dense.*

Proof. Let p be a point of $\overline{*\mathrm{Int}\, \tilde{X}}$ and G an open neighbourhood of p, then $0 \neq G\cdot\overline{*\mathrm{Int}\, \tilde{X}}$ implies $0 \neq G\cdot *\mathrm{Int}\, \tilde{X} \subset G\tilde{X} \subset \widetilde{GX}$. The set $G\cdot *\mathrm{Int}\, \tilde{X}$ is a *open set as a product of two *open sets, then $1-\widetilde{\widetilde{GX}} \subset \widetilde{1-G\cdot *\mathrm{Int}\, \tilde{X}} = 1-G\cdot *\mathrm{Int}\, \tilde{X} \neq 1$. Hence X is not *nowhere dense at p. Conversely, if $p \in 1-\overline{*\mathrm{Int}\, \tilde{X}}$, then the set $G = 1-\overline{*\mathrm{Int}\, \tilde{X}}$ is an open neighbourhood of p. Now we have $\widetilde{GX} \subset \tilde{G}\tilde{X} \subset (\tilde{G}-G)+ +G\cdot\tilde{X}$, and $\tilde{G}-G$ is *nowhere dense. On the other hand $*\mathrm{Int}(G\tilde{X}) = *\mathrm{Int}\, G\cdot *\mathrm{Int}\, \tilde{X} = G(1-(1-G)^*)\cdot *\mathrm{Int}\, \tilde{X} = 0$, hence $G\tilde{X}$ is *boundary. These being so, $\widetilde{GX}$ is *boundary as a subset of the union of a *nowhere dense set $\tilde{G}-G$ and a *boundary set $G\tilde{X}$. Therefore GX is *nowhere dense. Lemma 4 is thereby proved.

From Corollary 1 of Theorem 3 and Lemma 4, it follows that

$$\overline{*\mathrm{Int}\, \tilde{X}} \subset X^*, \tag{5}$$

for any subset $X \subset 1$. The set $1-\tilde{X}$ is *open, hence $*\mathrm{Int}\, \tilde{X} = 1-\widetilde{1-\tilde{X}} = 1-\overline{1-\tilde{X}} = \mathrm{Int}\, \tilde{X}$ by Corollary of Lemma 2. Hence it follows from (5) that $\mathrm{Int}\, \tilde{X} = \mathrm{Int}(\mathrm{Int}\, \tilde{X}) \subset \mathrm{Int}(\overline{*\mathrm{Int}\, \tilde{X}}) \subset \mathrm{Int}\, X^*$, and that

$$\overline{*\mathrm{Int}\, \tilde{X}} = \overline{\mathrm{Int}\, X^*}, \tag{6}$$

where X is any subset of 1. From these facts, we can see that, for any subset $X \subset 1$, the set of points at which X is not locally *nowhere dense is a closed domain.

Lemma 5. *Any *nowhere dense set X is the union of a nowhere dense set and a set belonging to $\boldsymbol{P}$.*

Proof. $\tilde{X}$ is *closed, then $\tilde{X} = (1-G)+A$ for some open set G and $A \in \boldsymbol{P}$, by Theorem 1. If X is *nowhere dense, then $1 = \widetilde{1-\tilde{X}} \subset \overline{G-A} \subset \overline{G}$. Hence $1-G$ is nowhere dense and $X = (1-G)X+AX$, where $(1-G)X$ is nowhere dense and $AX \in \boldsymbol{P}$.

Theorem 4. *If $1^* = 1$ and every nowhere dense set belongs to $\boldsymbol{P}$, then $X \in \boldsymbol{P}$ if and only if X is *nowhere dense.*

This follows from Lemma 5 and Corollary 1 of Theorem 3.

Corollary. *The equation $X^* = \overline{\mathrm{Int}\, X^*}$ holds for all $X \subset 1$ if and only if every nowhere dense set belongs to $\boldsymbol{P}$.*

Proof. In the space satisfying $1^* = 1$, our theorem follows from Theorem 4 and (6). Let X be a subset in a general space, then $X = X_1+X_2$, where $X_1 \subset 1-1^*$ and $X_2 \subset 1^*\cdot 1^{**} = 1^*$ by (4)(b), therefore, by the above fact, $X_2^* = \overline{\mathrm{Int}\, X_2^*}$ holds, and we have $X^* = \overline{\mathrm{Int}\, X^*}$, since $X^* = X_2^*$.

§ 4. Characterization of *topology.

Lemma 6. *Let $\boldsymbol{P}$ be the family of scattered sets, then $X^* = \overline{\ker X}$, where $\ker X$ is the largest dense in itself set contained in X, and $\boldsymbol{P}$ satisfies the relation* (3).

Proof. Let p be a point of $\overline{\ker X}$ and $V(p)$ an open neighbourhood of p, then $V(p)X \supset V(p)\ker X$ and $V(p)\ker X$ is not scattered by the definition of $\ker X$. Conversely, if $p \notin \overline{\ker X}$, for $V(p)$ such that $V(p)\overline{\ker X} = \emptyset$, $V(p)X$ is scattered by the definition of $\ker X$, and we have $X^* = \overline{\ker X}$. In our case, evidently, $X \in \boldsymbol{P} \to X^* = 0 \to XX^* = 0$ holds, and $XX^* = 0$ implies $\ker X = 0$, since $0 = XX^* = X\overline{\ker X} \supset \ker X$.

It seems of interest to determine just what properties characterize *topology. If $\mathscr{I}_0 \subset \mathscr{I}$ are T_1-topologies in 1, and $\boldsymbol{P}$ is the family of $\mathscr{I}$-scattered sets, then $\boldsymbol{P}$ satisfies our conditions assumed in § 1 relative to $\mathscr{I}_0$. Then it follows from Theorems 1 and 2.

Theorem 5. *$\mathscr{I}$ is the *topology corresponding to $\mathscr{I}_0$ and some ideal $\boldsymbol{P}_0$ satisfying conditions assumed in § 1 relative to $\mathscr{I}_0$ if and only if $\mathscr{I}_0 \subset \mathscr{I}$, $\boldsymbol{P}_0 = \boldsymbol{P}$, each member of $\boldsymbol{P}$ is $\mathscr{I}$-closed, and each member of $\mathscr{I}$ is of the form $G-A$, $G \in \mathscr{I}_0$, $A \in \boldsymbol{P}$.*

§ 5. The set having the Baire property. We shall denote by P the property to be of the first category. In ordinary topology, a subset X is the difference of two closed sets if and only if $\overline{X}-X$ is closed. Therefore, in the *topology with respect to the first category, it follows:

Lemma 7. *A subset X is of the form $X = {}^*F_1 - {}^*F_2$, where each ${}^*F_i$ is *closed, if and only if $\tilde{X}-X$, i.e. X^*-X, is *closed.*

Clearly we can assume without loss of generality that ${}^*F_1 \supset {}^*F_2$ in Lemma 7.

Lemma 8. *A *boundary set which is not of the first category does not have the Baire property.*

Proof. Let X be a *boundary set. Then we have $\overline{X} \subset (1-X)^*$ by (2)(a), and $X^*\cdot(1-X)^* = X^*$ follows. In our case, $X^* \neq 0$, and then X^* is not nowhere dense, since X^* is a closed domain by Corollary of Theorem 4. Thus $X^*\cdot(1-X)^*$ is not nowhere dense, and X does not have the Baire property. Lemma 8 is proved.

Note that, for any subset X, X^*-X is *boundary, since $1 = \tilde{X}+1-\tilde{X} \subset \tilde{X}+\widetilde{1-\tilde{X}} = \widetilde{X+1-\tilde{X}} = \widetilde{1-(\tilde{X}-X)}$. Hence, by Lemma 8, if X^*-X has the Baire property, this is nothing but of the first category. A *closed set has the Baire property by Theorem 1, so that, in Lemma 7, X^*-X is *closed if and only if $(X^*-X)^* = 0$, namely X has the Baire property, and we have the following:

Theorem 6. *A subset X has the Baire property if and only if it is of the form $X = {}^*F_1 - {}^*F_2$, where each ${}^*F_i$ $(i = 1, 2)$ is *closed, and ${}^*F_1 \supset {}^*F_2$.*

Definition 2. A set E of the form

$$E = {}^*F_1 - {}^*F_2 + {}^*F_3 - \ldots + {}^*F_{2n-1} - {}^*F_{2n} + \ldots, \tag{7}$$

where $\{{}^*F_i\}$ forms a decreasing sequence of *closed sets, is said to be **resolvable with respect to P*.

${}^*F_{2n-1} - {}^*F_{2n}$ $(n = 1, 2, \ldots)$ have the Baire property by Theorem 6, then a *resolvable set E with respect to the first category defined by (7) also have the same property, since the class of sets having the Baire property is a σ-algebra. Hence we have

THEOREM 7. *X has the Baire property if and only if X is *resolvable with respect to the first category.*

§ 6. The measurable set. We shall take the property to be of Lebesgue measure zero as P. We denote by an ${}^*F_\sigma$ set the union of a countable family of *closed sets, and by a ${}^*G_\delta$ set the intersection of a countable family of *open sets. Then, by Theorem 1, an ${}^*F_\sigma$ set is of the form F_σ set plus a nullset, and a ${}^*G_\delta$ set is of the form G_δ set minus a nullset. Evidently the inverse of each of these holds true. Hence we obtain the following:

THEOREM 8. *X is measurable if and only if X is the set both ${}^*F_\sigma$ and ${}^*G_\delta$ with respect to Lebesgue measure zero.*

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Topological completeness of first countable Hausdorff spaces II *

by

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Abstract. This article continues the study of basic completeness, a concept introduced in part I. It analyzes basic completeness into more primitive components: pararegularity (a generalization of regularity), monotonic completeness (a natural form of topological completeness), and base closurewise of countable order (a form of uniform first countability). The analysis finds expression in these theorems: 1. A space is basically complete if and only if it is T_1, locally monotonically complete, and has a base closurewise of countable order. 2. A space is basically complete if and only if it is a pararegular monotonically complete T_0-space having a base of countable order.

In addition there are results concerning pararegularity and monotonic completeness. It is shown that a pararegular space which is pseudo-m-complete (a modification of Oxtoby's pseudo-completeness) satisfies the Baire category theorem. The technique of primitive sequences exposited in I is further elaborated and applications are made. A number of examples are given.

This paper analyzes the concept of basic completeness, introduced in [22], into more primitive components: pararegularity, monotonic completeness, and base closurewise of countable order. These isolate, respectively, features of regularity, of completeness, and of uniform first countability. Each of them is discussed in a separate section where examples are given and relations to other concepts are established. A Baire category theorem is proved for pararegular spaces satisfying a weak completeness condition. The final section presents some characterizations of basic completeness in terms of these components.

The first section continues the development of the technique of primitive sequences initiated in I. Here some results are established in general form which are used in the subsequent proofs and which are useful in other contexts as well. This section may be regarded as a complement to Section 2 of I. These two sections begin a systematic presentation of a powerful technique for dealing with monotonically contracting sequences. In particular, they have application to spaces and concepts whose definitions involve monotonically contracting sequences of open coverings such as the spaces which are the subject of this investigation.

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* This paper is a continuation of [22] which will be referred to herein as I. We use the notation, definitions, and results of I throughout: references such as Lemma I.2.1 are to Lemma 2.1 of I.

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