Non-symmetric generalizations of theorems of Dyson and Livesay

by

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Abstract. The theorems in the title deal with a real-valued map f defined on the 2-sphere S^2. Dyson's theorem asserts the existence of two mutually orthogonal diameters all four of whose end points are mapped onto the same point by f. Livesay's theorem is more general in that it asserts the existence of two such diameters which are inclined to each other at a specified angle α, $0 < \alpha < \pi$. These theorems are generalized to higher dimensions, the sphere being replaced by any compact subset X of the Euclidean space R^{n+1} for which the origin lies in a bounded component of $R^{n+1} - X$. The results as well as the methods are similar to those in the author's earlier work in which a similar generalization of the Borsuk–Ulam theorem on antipodes was proved.

1. Introduction. This paper may be regarded as a continuation of [3] to which the reader will be referred frequently. In [3] the following theorem conjectured by Borsuk was proved.

THEOREM A. *Let X be a compact subset of the Euclidean space R^{n+1} which disconnects it in such a way that the origin lies in a bounded component of $R^{n+1} - X$. Then given any map $f\colon X \to R^n$ there exist two points x and y in X lying on opposite rays from the origin* (*that is, $y = -\lambda x$ for some $\lambda > 0$*) *such that $f(x) = f(y)$.*

The well-known Borsuk–Ulam theorem follows from this theorem by taking X to be the n-sphere S^n. The separation property of the set X in Theorem A may also be expressed by saying that "X separates 0 from ∞". Henceforth compact sets with this property will be called *Borsuk sets*. The theorems to be proved in this paper are generalizations of the theorems of Dyson [1] and Livesay [4] much the same way as Theorem A is a generalization of the Borsuk–Ulam theorem. Specifically they are,

THEOREM B. *Let X be a Borsuk set in R^{n+1} and f be a real-valued map on X. Then there exist n points $x_1, x_2, \ldots, x_n$ of X and n positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that,*

(i) $-\lambda_i x_i \in X$ *for each* $i = 1, 2, \ldots, n$,

(ii) $f(x_1) = f(x_2) = \ldots = f(x_n) = f(-\lambda_1 x_1) = \ldots = f(-\lambda_n x_n)$ *and*

(iii) x_i *and* x_j *are mutually orthogonal for all* $i \neq j$, $1 \leq i \leq n$, $1 \leq j \leq n$.

THEOREM C. *Let a Borsuk set X in R^{2k+1}, a map $f\colon X\to R^k$ and a number a between 0 and π be given. Then there exists at least one subset of the form $\{x_1, x_2, -\lambda_1 x_1, -\lambda_2 x_2\}\subset X$ such that $\lambda_1>0$, $\lambda_2>0$, $f(x_1)=f(x_2)=f(-\lambda_1 x_1)=f(-\lambda_2 x_2)$ and the angle between x_1 and x_2 is a.*

The theorem of Dyson follows from Theorem B by taking $n=2$ and $X=S^2$ while that of Livesay follows from Theorem C by taking $k=1$ and $X=S^2$. Yang ([6], [7]) has proved generalizations of theorems of Dyson and Livesay and we shall use these in our proofs. In fact, as in [3] our technique will be to construct a certain T-space $(A(X);T)$ from the Borsuk set X, to compute its Smith index and then to apply Yang's results.

In Section 2 we review and improve some of the results from [3]. Section 3 is devoted to a brief discussion of Euclidean T-spaces and Yang's results which will be needed later. In Section 4 we prove the desired generalizations of the theorems of Dyson and Livesay.

The author is indebted to R. Mark Goresky for an important suggestion leading to the proof of Theorem (2.4) which gives an affirmative answer to a question that was left open in [3], p. 19.

2. The antipodal space $A(X)$. As in [3] by a space we shall mean a compact metric space and by an involution a fixed-point-free homeomorphism of a space onto itself having period two. Also all homology groups will be understood to have Z_2 coefficients. An involution on a space X will usually be denoted by T and the pair $(X;T)$ will be called a *T-space*. A map between two T-spaces which preserves the action of their involutions will be called a *T-map*. For the definition and properties of the invariant (also called *equivariant* or *symmetric*) homology theory on the category of T-spaces and of the Smith index of a T-space we refer the reader to Smith [5] or Yang [6]. We shall also need the properties of regular polyhedra defined and studied in [3].

Given any subset S of R^{n+1} we define its antipodal space $A(S)$ which consists of all ordered pairs of points in S which lie on opposite rays from the origin; that is to say, $A(S)=\{(x,y)\in S\times S\,|\,y=-\lambda x \text{ for some } \lambda>0\}$. If S is compact so is $A(S)$. There is a natural homeomorphism T of $A(S)$ onto itself defined by $T(x,y)=(y,x)$. This homeomorphism is an involution provided the origin is not in S. We are interested in the Smith index of the T-space $(A(X);T)$ in the case where X is a Borsuk set in R^{n+1}. In [3] it was proved that $\operatorname{Ind}(A(X);T)\leqslant n$ and that equality holds if X is a regular polyhedron. The question whether equality holds for an arbitrary Borsuk set X was left open. We shall settle it here by expressing the T-space $(A(X);T)$ as the inverse limit of a sequence of T-spaces each having index n. To do this first we must establish a continuity property for the Smith index.

We note that if M is a countable directed set and $\{X_a, T_a\}_{a\in M}$ is an inverse system of T-spaces in which all maps of the form $X_a\to X_\beta$ for $\beta<a$ in M are T-maps, then the inverse limit X of the system $\{X_a\}_{a\in M}$ is a compact metric space with a natural involution T. Indeed if we consider X as a subspace of the product $\prod_{a\in M} X_a$, then T acts on X by $T(\{x_a\})=\{T_a x_a\}$. With this action of T all the projections $X\to X_a$ are obviously T-maps and the T-space $(X;T)$ is the inverse limit of the system $\{X_a; T_a\}_{a\in M}$ in the category of T-spaces.

(2.1) THEOREM. *The equivariant homology groups as well as the index homomorphisms are compatible with inverse limits. In other words, if $\{X_a; T_a\}_{a\in M}$ is an inverse system of T-spaces with limit $(X;T)$, then for each integer p, $H_p(X;T)$ is isomorphic to the inverse limit of the inverse system of groups $\{H_p(X_a;T_a)\}_{a\in M}$ and, moreover, the index homomorphism $\vartheta_p\colon H_p(X;T)\to Z_2$ coincides with the homomorphism induced by the family of the index homomorphisms $\{\vartheta_p\colon H_p(X_a;T_a)\to Z_2\}_{a\in M}$.*

Proof. The proof of the continuity of the equivariant homology groups proceeds along a standard, although lengthy argument analogous to the one used in proving the continuity property of the ordinary Čech homology theory, see for example ([2], Chapter X). We omit the details but note that the only essential difference between the equivariant homology theory and the ordinary Čech homology theory is that in the definition finite open T-coverings (i.e. coverings whose members are permuted among themselves under the action of T) are used instead of ordinary coverings; see Smith [5]. A similar modification in the proof yields the desired result.

To prove the assertion about the index homomorphism we note that by the properties of inverse limits there is a unique homomorphism $g\colon h_p(X;T)\to Z_2$ which makes the following diagram commute for each $a\in M$.

$$\begin{array}{ccc} H_p(X;T) & \longrightarrow & H_p(X_a;T_a) \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle \vartheta_p} \\ & & Z_2 \end{array}$$

Here the horizontal arrow indicates the homomorphism induced by the projection map $X\to X_a$. On the other hand by naturality of the index homomorphism ([3], p. 17) the diagram above commutes if we take g to be the index homomorphism ϑ_p. Therefore the index homomorphism must coincide with the homomorphism induced by the family of index homomorphisms. This completes the proof.

(2.2) COROLLARY. *Suppose each T-space $(X_a; T_a)$ has index n and that the equivariant homology groups $H_p(X_a; T_a)$ are finite for all p and a. Then the inverse limit $(X; T)$ has Smith index n.*

Proof. By definition, the Smith index of a T-space $(X; T)$ is the largest integer p for which the index homomorphism $\vartheta_p : H_p(X; T)\to Z_2$ is an epimorphism. In the present case by (2.1) the index homomorphism $\vartheta_n : H_n(X; T)\to Z_2$ is induced by the family of index homomorphisms $\{\vartheta_n : H_n(X_a; T_a)\to Z_2\}_{a\in M}$ each of which is an epimorphism. The assumption about finiteness of $H_n(X_a; T_a)$ implies (see [2], p. 226) that the index homomorphism $\vartheta_n : H_n(X; T)\to Z_2$ is an epimorphism and hence that $\mathrm{Ind}(X; T)\geqslant n$. On the other hand because we have T-maps $X\to X_a$ it follows from [3], Theorem 2.3, that $\mathrm{Ind}(X; T)\leqslant \mathrm{ind}(X_a; T_a)$. Thus $\mathrm{Ind}(X; T) = n$.

An important case where this corollary is applicable is when each X is a polyhedron. Note that no assumption is made regarding finiteness of the equivariant homology groups of the limit $(X; T)$.

Given a Borsuk set X in R^{n+1} we wish to apply the above corollary to compute $\mathrm{Ind}(A(X); T)$. For this we shall construct a decreasing sequence of polyhedra, $Q_1\supset Q_2\supset Q_3\supset \ldots$ such that $\bigcap_{i=1}^{\infty} Q_i = A(X)$ and each Q_i is a T-space of index n, the involution on Q_i being induced by that on Q_1. First, given a Borsuk set X we fix some closed annulus $N = A(r, R) = \{x\in R^{n+1} | r\leqslant ||x||\leqslant R\}$, where the real numbers r and R are so chosen that $0<r<R$ and the interior of N contains X. The antipodal space $A(N)$ is homeomorphic to $S^n\times J\times J$, where J is the closed interval $[r, R]$. Indeed the map φ: $A(N)\to S^n\times J\times J$ defined by $\varphi(x, y) = (x/||x||, ||x||, ||y||)$ is a homeomorphism. There is an involution T on $S^n\times J\times J$ defined by $T(x, s, t) = (-x, t, s)$. The homeomorphism φ is clearly a T-map because if $(x, y)\in A(N)$, then $x/||x|| = -y/||y||$. Since the n-sphere S^n with its antipodal involution can be equivariantly embedded in $S^n\times J\times J$ we conclude that $(A(N); T)$ has Smith index n.

In the construction of the sequence of polyhedra $\{Q_i\}$ the first polyhedron Q_1 will be $A(N)$. To construct the succesive ones we need the following Lemma.

(2.3) LEMMA. *If B is an invariant closed subset of $A(N)$ and V is an equivariant neighborhood of B in $A(N)$, then there exists a compact polyhedron Q such that $Q\subset V$ and Q is itself an equivariant neighborhood of B in $A(N)$. ("Equivariant" and "invariant" mean the same.)*

Proof. Clearly we may suppose that V is open and that $V\subsetneq A(N)$. We let $\varepsilon = d(B, A(N)-V)$, where d is any metric on $A(N)$ which is compatible with the action of T on $A(N)$. Next we construct an equivariant triangulation of $A(N)$ whose mesh is less than $\varepsilon/2$ and let Q be the union of all closed simplices which meet B. Clearly Q has the desired properties and the lemma is proved.

Returning to the construction of the polyhedra $\{Q_i\}$ we first fix a descending sequence $U_1\supset U_2\supset U_3\supset \ldots$ of closed neighborhoods of X in R^{n+1} such that $U_1 = N$, each U_i is a neighborhood of U_{i+1} for $i\geqslant 1$ and $\bigcap_{i=1}^{\infty} U_i = X$. Then clearly $A(U_i)$ is a neighborhood of $A(U_{i+1})$ in $A(N)$ and $\bigcap_{i=1}^{\infty} A(U_i) = A(X)$. We have already set $Q_1 = A(N)$. By notation we also set $U_0 = N$. To construct Q_i for $i\geqslant 2$ we proceed inductively. Suppose $Q_1\supset Q_2\supset \ldots \supset Q_{i-1}$ have been constructed so that Q_{i-1} is an equivariant neighborhood of $A(U_i)$ in $A(N)$ and $Q_{i-1}\subset A(U_{i-2})$. Then $Q_{i-1}\cap A(U_{i-1})$ is an equivariant neighborhood of $A(U_i)$ in $A(N)$, and so by (2.3) there is a polyhedron Q_i such that Q_i is an invariant neighborhood of $A(U_i)$ and *a fortiori* of $A(U_{i+1})$ and also $Q_i\subset Q_{i-1}\cap A(U_{i-1})$. This completes the inductive step. The descending sequence of polyhedra $\{Q_i\}$ thus obtained clearly has the property that $\bigcap_{i=1}^{\infty} Q_i = A(X)$ and so $A(X)$ is the inverse limit of this sequence. Also each Q_i being a compact polyhedron, all groups $H_p(Q_i; T)$ are finite. In order to apply (2.2) we now only need to show that the Smith index of each $(Q_i; T)$ is n. For this, we know from [3], p. 31, that each U_i contains a regular polyhedron P_i. Hence we have inclusions $A(P_i)\subset A(U_i)\subset Q_i\subset A(N)$ for each i; and therefore the Smith indices of these T-spaces are in the same order ([3], p. 18). But it is known ([3], p. 31) that $\mathrm{Ind}(A(P_i); T) = n$. Since we also know that $\mathrm{Ind}(A(N); T) = n$ it follows that $\mathrm{Ind}(Q_i; T) = n$ for all i. All the hypotheses of Corollary (2.2) are now satisfied and we have the following theorem which is the goal of this section.

(2.4) THEOREM. *If X is a Borsuk set in R^{n+1}, then* $\mathrm{Ind}(A(X); T) = n$.

Remark 1. In the proof just given it is also true that $\mathrm{Ind}(A(U_i); T) = n$ for all i. But we do not know if the groups $H_p(A(U_i); T)$ are finite or not. On the other hand although each $A(P_i)$ is a polyhedron and therefore the groups $H_p(A(P_i); T)$ are known to be finite, the construction of these polyhedra $\{P_i\}$ does not guarantee that they form a descending sequence. It is therefore necessary to construct the sequence $\{Q_i\}$ which satisfies all three hypotheses of (2.2) simultaneously.

Remark 2. It would be interesting to have a short proof of Theorem (2.4) which is based directly on the fact that X separates 0 from ∞ and which does not go through the process of approximating the set X.

3. Euclidean T-spaces and Yang's results. The antipodal space $A(X)$ and its involution T discussed in the last section can be looked at slightly differently. Obviously $A(X)$ is a subset of the Euclidean space

$R^{2n+2} = R^{n+1} \times R^{n+1}$. Let Π be the diagonal $(n+1)$-plane in $R^{n+1} \times R^{n+1}$, i.e. $\Pi = \{(x, x) | \; x \in R^{n+1}\}$. The involution T on $A(X)$ maps (x, y) onto (y, x) and therefore coincides with the reflection into Π. The T-space $(A(X); T)$ is therefore a Euclidean T-space in the sense of the following definition.

(3.1) DEFINITION. A T-space $(X; T)$ is called a *Euclidean T-space* if $X \subset R^m$ for some m and T coincides with the reflection into some t-plane Π in R^m, $0 \leqslant t \leqslant m$.

We have already given an example of a Euclidean T-space. The n-sphere S^n with its antipodal involution is another example. Here the plane Π is zero dimensional. Actually every T-space $(X; T)$ in which X is finite dimensional can be realized as a Euclidean T-space. Indeed, let $f\colon X \to R^k$ be an embedding for some k. Let $m = 2k$ and Π be the diagonal k-plane in $R^k \times R^k$. Define $g\colon X \to R^m$ by $g(x) = (f(x), f(Tx))$. Clearly g is an embedding. The set $g(X)$ has an involution T on it defined by $T(x, y) = (y, x)$ and g is a T-map. Moreover, T coincides with the reflection into Π; and thus $(g(X); T)$ is a Euclidean T-space. The map $h\colon X \to g(X)$ defined by g is an equivariant homeomorphism.

We consider a Euclidean space R^m and a fixed t-plane Π through the origin. Terms such as "perpendicular", "angle between two vectors" will be understood to be with respect to the usual inner product on R^m. If x is a point of R^m not in Π, then the $(t+1)$-plane spanned by Π and x will be denoted by Πx. The unit vector in the direction of the perpendicular line to Π through x will be denoted by $u(x)$. If x and y are two points of R^m not in Π, then the angle between Πx and Πy is defined to be the angle between the unit vectors $u(x)$ and $u(y)$. If this angle is a, then we say that Πx and Πy intersect at an angle a. If $a = \pi/2$, then Πx and Πy are said to be *mutually orthogonal* at Π.

We remark that if $R^m \subset R^n$, then the inner product on R^m is the restriction of the inner product on R^n. Consequently the angle between Πx and Πy is the same whether they are considered as $(t+1)$-planes in R^m or in R^n. Thus the integer m plays no role as long as R^m contains Π, x and y.

Our primary interest is of course in the case where $m = 2n+2$ and where Π is the diagonal $(n+1)$-plane. Note that the orthogonal complement of Π in R^{2n+2} is the $(n+1)$-plane Φ defined by $\Phi = \{(x, -x) \in R^{n+1} \times R^{n+1} | \; x \in R^{n+1}\}$. An easy calculation shows that if (x, y) is in $R^{n+1} \times R^{n+1}$, then the perpendiculars from (x, y) to Π and Φ meet them respectively at points $((x+y)/2, (x+y)/2)$ and $((x-y)/2, (y-x)/2)$. If $(x, y) \notin \Pi$, then the unit vector $u(x, y)$ along the direction of the perpendicular to Π through (x, y) is therefore, $((x-y)/\sqrt{2}\,||x-y||, (y-x/\sqrt{2}\,||y-x||))$.

Suppose now X is a Borsuk set in R^{n+1}. An element of the antipodal space $A(X)$ is of the form $(x, -\lambda x)$ for some $x \in X$ and for some $\lambda > 0$.

Certainly $(x, -\lambda x) \notin \Pi$ and so Π and $(x, -\lambda x)$ together span an $(n+2)$-plane in $R^{n+1} \times R^{n+1}$ denoted by $\Pi(x, -\lambda x)$. Given two such points $(x_1, -\lambda_1 x_1)$ and $(x_2, -\lambda_2 x_2)$ we want to relate the angle between the $(n+2)$-planes $\Pi(x_1, -\lambda_1 x_1)$ and $\Pi(x_2, -\lambda_2 x_2)$ to the angle between the vectors x_1 and x_2. The relationship turns out to be surprizingly simple as proved in the following theorem.

(3.2) THEOREM. *Let x_1, x_2 be non-zero vectors in R^{n+1} and let λ_1, λ_2 be positive real numbers. Then the angle between the $(n+2)$-planes $\Pi(x_1, -\lambda_1 x_1)$ and $\Pi(x_2, -\lambda_2 x_2)$ in $R^{n+1} \times R^{n+1}$ is the same as the angle between the vectors x_1 and x_2 in R^{n+1}.*

Proof. First we observe the relation between the usual inner product on R^{n+1} and that on $R^{2n+2} = R^{n+1} \times R^{n+1}$. Using the same notation $\langle \circ, \circ \rangle$ for both of these we have $\langle (y_1, z_1), (y_2, z_2) \rangle = \langle y_1, y_2 \rangle + \langle z_1, z_2 \rangle$ for all y_1, y_2, z_1, z_2 in R^{n+1}. Let a be the angle between the $(n+2)$-planes $\Pi(x_1, -\lambda_1 x_1)$ and $\Pi(x_2, -\lambda_2 x_2)$ and let β be the angle between the vectors x_1 and x_2. Then by definition, $\cos a = \langle u(x_1, -\lambda_1 x_1), u(x_2, -\lambda_2 x_2) \rangle$, where $u(x_i, -\lambda_i x_i)$, $i = 1, 2$, are unit vectors defined above. Using the formulas for these unit vectors as mentioned above as well as the relation for the inner product just observed, a straightforward computation yields that $\cos a = \langle x_1/||x_1||, x_2/||x_2|| \rangle$. But the right-hand side of this equation equals $\cos\beta$ by definition of β. Hence $\cos a = \cos\beta$. The convention about angles requires that $0 \leqslant a \leqslant \pi$ and $0 \leqslant \beta \leqslant \pi$. Therefore $a = \beta$ and the theorem is proved.

We conclude this section by stating those theorems of Yang which will be needed in the next section. To be fair we remark that the results obtained by Yang are more general. However, to state them in their full generality would necessitate the introduction of still more definitions and notations. We avoid this because the versions given below, while not most general, are sufficient for our purpose.

(3.3) THEOREM. *Let $(X; T)$ be a Euclidean T-space in R^m of index s and with T induced by the reflection into some t-plane Π, $0 \leqslant t \leqslant m$. Let f be a real-valued map on X. Then there are $s+1$ points $x_1, x_2, \ldots, x_{s+1}$ of X such that $f(x_1) = f(x_2) = \ldots = f(x_s) = f(x_{s+1})$ and, moreover, the $(t+1)$-planes $\Pi x_1, \Pi x_2, \ldots, \Pi x_{s+1}$ are mutually orthogonal at Π.*

(See [6], p. 278.)

(3.4) THEOREM. *Let $(X; T)$ be a Euclidean T-space in R^m of index $\geqslant 2k$ (for some integer k) and with T induced by the reflection into some t-plane Π. Let f be a map of X into R^k. Then there exists a pair of points x_1 and x_2 in X such that $f(x_1) = f(x_2) = f(Tx_1) = f(Tx_2)$ and Πx_1 and Πx_2 intersect at a preassigned angle a, $0 < a < \pi$. If, moreover, $a \neq \pi/2$, then there exist at least two such pairs.*

(See [7], p. 281.)

(3.5) THEOREM. *Let $(X; T)$ be a T-space of index n and let f be a map of X into the Euclidean k-space R^k, where $0 \leqslant k \leqslant n$. Let $X_k = \{x \in X | f(x) = f(Tx)\}$. Then X_k is T-invariant, compact and $(X_k; T)$ has index $\geqslant n-k$.*

(See [6], p. 270.)

4. The main results. In this section we prove the desired generalizations of the theorems of Dyson and Livesay. We already have all the machinery needed and it is only a matter of putting it together.

(4.1) THEOREM. *Let X be a Borsuk set in R^{n+1} and let f be a real-valued map on X. Then there exist n points $x_1, x_2, \ldots, x_n$ of X and n positive real numbers $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that,*

(i) $-\lambda_i x_i \in X$ *for each* $i = 1, 2, \ldots, n$,

(ii) $f(x_1) = f(x_2) = \ldots = f(x_n) = f(-\lambda_1 x_1) = f(-\lambda_2 x_2) = \ldots = f(-\lambda_n x_n)$,

(iii) x_i *and* x_j *are mutually orthogonal for all* $i \neq j$, $1 \leqslant i \leqslant n$; $1 \leqslant j \leqslant n$.

Proof. We form the antipodal space $A(X)$ and define a real-valued map $g\colon A(X) \to R$ by $g(x, y) = f(x)$. Let $B(X) = \{(x, y) \in A(X) | g(x, y) = g(y, x)\}$. Note that a point $(x, -\lambda x)$ in $A(X)$ is in $B(X)$ if and only if $f(x) = f(-\lambda x)$. We apply (2.4) and (3.5) to the T-space $(A(X); T)$ and to the map g on it and get that the Smith index of $(B(X); T)$ is at least $n-1$. Also the involution T on $B(X)$ coincides with the reflection into the diagonal $(n+1)$-plane Π in $R^{n+1} \times R^{n+1}$. Hence by applying (3.3) to $(B(X); T)$ and the map g we get the existence of n points (x_i, y_i), $i = 1, 2, \ldots, n$, in $B(X)$ such that, $g(x_1, y_1) = \ldots = g(x_n, y_n)$ and the $(n+2)$-planes $\Pi(x_1, y_1), \ldots, \Pi(x_n, y_n)$ are mutually orthogonal at Π. Now $y_i = -\lambda_i x_i$ for some $\lambda_i > 0$ for each $i = 1, 2, \ldots, n$ and as noted above $g(x_i, y_i) \in B(X)$ implies that $f(x_i) = f(y_i)$. Thus the x_i's and λ_i's satisfy conditions (i) and (ii) of the conclusion of the theorem. Also, by Theorem (3.2), $\Pi(x_i, -\lambda_i x_i)$ and $\Pi(x_j, -\lambda_j x_j)$ are mutually orthogonal at Π if and only if the vectors x_i and x_j are perpendicular to each other. This shows that condition (iii) is also satisfied and the proof is complete.

(4.2) THEOREM. *Let a Borsuk set X in R^{2k+1}, a map $f\colon X \to R^k$ and a number a between 0 and π be given. Then there exists at least one subset of the form $\{x_1, x_2, -\lambda_1 x_1, -\lambda_2 x_2\} \subset X$ such that $\lambda_1 > 0$, $\lambda_2 > 0$, $f(x_1) = f(x_2) = f(-\lambda_1 x_1) = f(-\lambda_2 x_2)$ and the angle between x_1 and x_2 is a. If, moreover, $a \neq \pi/2$, then there exist at least two such subsets.*

Proof. Once again we form the space $A(X)$ and define $g\colon A(X) \to R^k$ by $g(x, y) = f(x)$. By (2.4) and (3.4) we get two points $(x_1, -\lambda_1 x_1)$ and $(x_2, -\lambda_2 x_2)$ in $A(X)$ such that $g(x_1, -\lambda_1 x_1) = g(x_2, -\lambda_2 x_2) = g(-\lambda_1 x_1, x_1) = g(-\lambda_2 x_2, x_2)$ and the angle between the $(2k+2)$-planes $\Pi(x_1, -\lambda_1 x_1)$ and $\Pi(x_2, -\lambda_2 x_2)$ is a. This of course means that $f(x_1) = f(x_2) = f(-\lambda_1 x_1) = f(-\lambda_2 x_2)$ and, in view of (3.2), that the angle between x_1 and x_2 is a. This gives a desired subset of X. If $a \neq \pi/2$, then by (3.4) there are at least two distinct pairs of points in $A(X)$ satisfying the conclusion of (3.4) and these give at least two distinct subsets of X having the desired properties. The theorem is now completely proved.

As observed earlier (4.1) reduces to Dyson's theorem [1] if we take $n = 2$ and $X = S^2$ while Livesay's result [4] follows from (4.2) by taking $k = 1$ and $X = S^2$.

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