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On the rim-types of hereditarily locally connected continua *

by

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Abstract. To every rational curve it is possible to assign a countable ordinal number called the rim-type of the curve. In this paper it is shown that for each ordinal number α that is at most countable there exists a hereditarily locally connected continuum of rim-type α.

It is known (see Kuratowski [1], p. 290) that the rim-type of a rational curve is an ordinal number that is strictly smaller than the first uncountable ordinal Ω. A continuum is regular if and only if it is of rim-type 1. Hereditarily locally connected continua are rational (see [3], p. 94) and regular continua are hereditarily locally connected. A. Lelek asked in a letter if the rim-type of every hereditarily locally connected continuum is at most 2. It is our purpose to prove that for each ordinal α such that $1 \leqslant \alpha < \Omega$ there exists a planar hereditarily locally connected continuum of rim-type α.

Our notation follows Whyburn [3]. A *continuum* is a compact, connected metric space. A continuum is said to be *hereditarily locally connected* if each of its subcontinua is locally connected.

If A is a subset of a space X we let A' denote the derived set of A. We let $A^{(0)} = A$. If α is the successor of the ordinal n we let $A^{(\alpha)} = (A^{(n)})'$. If α is a limit ordinal we let $A^{(\alpha)} = \bigcap \{A^{(n)} |\ n < \alpha\}$.

Let X be a rational continuum. Let α be the smallest ordinal number such that for each $x \in X$ and for each neighbourhood U of x there is a neighbourhood V of x such that $V \subset U$ and $(\partial V)^{(\alpha)} = \emptyset$ where ∂V denotes the boundary of V. The *rim-type* of X is defined to be α.

We shall need the following lemma:

* This research was supported in part by a grant from the National Research Council.

LEMMA. *If $Y_1, Y_2, \ldots$ is a sequence of pairwise disjoint hereditarily locally connected continua in E^3 whose diameters converge to 0 and if A is an arc which meets each Y_i then $X = A \cup Y_1 \cup Y_2 \cup \ldots$ is a hereditarily locally connected continuum.*

Proof. X is connected since $A, Y_1, Y_2, \ldots$ are connected sets and each Y_i meets A. X is compact since $A, Y_1, Y_2, \ldots$ are compact sets and the sequence Y_i is eventually in every neighbourhood of A. To prove that X is hereditarily locally connected it suffices to prove (by [3], p. 89) that X contains no continuum of convergence.

Let $\varepsilon > 0$ be given. Suppose $B_1, B_2, \ldots$ is a sequence of pairwise disjoint continua of X each of which has diameter at least ε. We may suppose that the sequence $B_1, B_2, \ldots$ converges to a continuum K in the space of closed subsets of X. One of the following two situations occurs. Either for each $\delta > 0$ the sequence $B_1, B_2, \ldots$ is eventually in the δ-neighbourhood of A or there is a natural number n and $\varepsilon_1 > 0$ such that infinitely many of the sets B_i meet Y_n in a connected set of diameter at least ε_1. In the second case we may suppose without loss of generality that each B_i meets Y_n in a connected set of diameter at least ε_1. In this case it follows from the fact that Y_n is hereditarily locally connected that almost every B_i meets K. If the second case fails to hold then it is easy to see that K is a line segment in A and almost every B_i meets K. This completes the proof that X does not contain a continuum of convergence.

THEOREM. *If α is an ordinal number such that $1 \leqslant \alpha < \Omega$ then there exists a planar hereditarily locally connected continuum of rim-type α.*

Proof. The proof is by transfinite induction. Let $X_1 = [0,1]\times\{0\}$. Then X_1 is of rim-type 1. Let α be an ordinal number such that $1 < \alpha < \Omega$. Suppose that for each ordinal number n such that $1 \leqslant n < \alpha$ X_n is a planar hereditarily locally connected continuum of rim-type n. Suppose also that for each n such that $1 \leqslant n < \alpha$ and n is the successor of an ordinal m the following hold:

(i) $[0,1]\times\{0\} \subset X_n \subset [0,1]\times[-1,1]$,

(ii) $A_n = X_n \cap (\{0\}\times[-1,1])$ and $B_n = X_n \cap (\{1\}\times[-1,1])$ are countable sets,

(iii) for each $A \subset X_n$ such that $A^{(m)} = \emptyset$ there exists an arc in $X_n \setminus A$ with one endpoint in A_n and the other in B_n.

We consider three cases.

Case 1. α is a limit ordinal. Let $\alpha_1, \alpha_2, \ldots$ be a strictly increasing sequence of ordinals which converges to α. For each i let Z_i be a hereditarily locally connected plane continuum of rim-type α_i and diameter less than $1/i$ such that for each $i \neq j$ $Z_i \cap Z_j = \{(0,0)\}$. Let $Z = Z_1 \cup Z_2 \cup \ldots$ Then Z is easily seen to be hereditarily locally connected continuum of rim-type α.

Case 2. α is the successor of the ordinal number n where n is not a limit ordinal. Let

$$E_+ = ([0,1]\times[0,1])\setminus\{(0,0),(1,0)\}$$

and

$$E_- = ([0,1]\times[-1,0])\setminus\{(0,0),(1,0)\}\,.$$

Let $Y_0 = [0,1]\times\{0\}$. Suppose that for each $j = 1, \ldots, k$ Y_j has been defined and

$$Y_j = f_{j1}(X_n) \cup \ldots \cup f_{jm_j}(X_n)$$

where each f_{ji} is a homeomorphism of X_n into one of E_+ and E_- such that for all j, j' and i, i' such that $1 \leqslant j,\ j' \leqslant k$, $1 \leqslant i \leqslant m_j$ and $1 \leqslant i' \leqslant m_{j'}$

a) $f_{ji}(X_n) \cap f_{j'i'}(X_n) = \emptyset$ if $i \neq i'$ or $j \neq j'$,

b) $Y_j \cap (\{0\}\times[-1,1]) = f_{j1}(A_n)$,

c) $Y_j \cap (\{1\}\times[-1,1]) = f_{jm_j}(B_n)$,

d) $Y_j \cap Y_0 = f_{j1}(B_n) \cup \ldots \cup f_{j,m_j-1}(B_n) \cup f_{j2}(A_n) \cup \ldots \cup f_{jm_j}(A_n)$,

e) diameter $f_{ji}(X_n) < 1/j$,

f) if $i \in \{2, \ldots, m_j\}$ and $(x,0) \in f_{ji}(A_n)$ and $(y,0) \in f_{j,i-1}(B_n)$ then $y < x$,

g) and if $i \in \{2, \ldots, m_j-1\}$ and $(x,0) \in f_{ji}(A_n)$, $(y,0) \in f_{ji}(B_n)$ then $x < y$.

Let $0 < b_0 < 1/(k+1) \leqslant 1-1/(k+1) < a_1 < 1$ such that

$$\big(([0,b_0] \cup [a_1,1])\times\{0\}\big) \cap \bigcup_{j=1}^{k} Y_j = \emptyset\,.$$

For $x \in [b_0, a_1]$ let $0 < a_x < x < b_x < 1$ such that

$$(a_x,0),(b_x,0) \notin \bigcup_{j=1}^{k} Y_j,$$

$[a_x, b_x]\times\{0\}$ meets at most one of the sets $f_{ji}(X_n)$ for $j \in \{1, \ldots, k\}$ and $i \in \{1, \ldots, m_j\}$ and the points $(a_x, 0)$ and $(b_x, 0)$ can be joined in $(E_- \cup E_+)\setminus \bigcup_{j=1}^{k} Y_j$ by an arc of diameter less than $1/(k+1)$. Since $[b_0, a_1]$ is compact there is a smallest finite set $\{x_1, \ldots, x_{m_{k+1}}\}$ such that $b_0 = x_1 < \ldots < x_{m_{k+1}} = a_1$ and $[b_0, a_1] \subset [a_{x_1}, b_{x_1}[\cup]a_{x_2}, b_{x_2}[\cup \ldots \cup]a_{x_{m_{k+1}}}, b_{x_{m_{k+1}}}]$.

We may suppose without loss of generality that if $([a_{x_p}, b_{x_p}]\times\{0\}) \cap f_{ji}(X_n) \neq \emptyset$ for some (j, i) then $f_{ji}(X_n) \subset E_-$ if p is odd and $f_{ji}(X_n) \subset E_+$ if p is even. It is now clearly possible to define homeomorphisms $f_{k+1,i}$ for each $i = 1, \ldots, m_{k+1}$ of X_n into E_+ if i is odd and into E_- if i is even such that the conditions a)-g) are all satisfied. Let $Y_{k+1} = \bigcup_{i=1}^{m_{k+1}} f_{k+1,i}(X_n)$.

Let $X_\alpha = Y_0 \cup Y_1 \cup \ldots$ By the lemma X_α is a hereditarily locally connected continuum.

We check that the rim-type of X_α is no greater than α. If $x \in X_\alpha \setminus Y_0$ then x has a set homeomorphic to X_n for a neighbourhood. Since the rim-type of X_n is n there exist arbitrarily small neighbourhoods V of x in X_α such that $(\partial V)^{(n)} = \emptyset$. If $x \in Y_0$ then it follows from the construction that there exist arbitrarily small neighbourhoods V of x such that $\partial V \cap Y_0$ contains at most two points and for each $j\,(\partial V \cap Y_j)^{(n)} = \emptyset$. Since $\partial V \cap Y_j$ is open in ∂V for each $j\,(\partial V)^{(n)} \subset \partial V \cap Y_0$. Since $\partial V \cap Y_0$ is finite $(\partial V)^{(\alpha)} \subset (\partial V \cap Y_0)' = \emptyset$.

Next we show that rim-type of X_α is at least α. Let A be a set in X_α which separates $(x, 0)$ and $(y, 0)$ in X_α where $x, y \in [0, 1]$. Since X_α is completely normal we may suppose without loss of generality that A is a closed set. Just suppose that $A^{(n)} = \emptyset$.

There exist at most finitely many $j \in \{1, 2, \ldots\}$ and $i \in \{1, \ldots, m_j\}$ such that $\big(f_{ji}(X_n) \cap A\big)^{(m)} \neq \emptyset$ for each ordinal $m < n$ since the sets $f_{ji}(X_n)$ are pairwise disjoint and their diameters converge to zero. We may suppose without loss of generality, therefore, that for each $j \in \{1, 2, \ldots\}$ and for each $i \in \{1, \ldots, m_j\}$ there is an ordinal $m < n$ such that $(f_{ji}(X_n) \cap A)^{(m)} = \emptyset$. Thus by condition (iii) on the continuum X_n for each j, i there exists an arc K_{ji} in $f_{ji}(X_n) \setminus A$ with one endpoint of K_{ji} in $f_{ji}(A_n)$ and the other in $f_{ji}(B_n)$.

Let $Y = Y_0 \cup \bigcup \{K_{ji} |\ j \in \{1, 2, \ldots\}$ and $i \in \{1, \ldots, m_j\}\}$. It is not difficult to see that Y is a continuum and that the set of local cutpoints of Y is contained in $\bigcup K_{ji}$.

Now, $Y \cap A \subset Y \setminus \bigcup K_{ji}$ and $Y \cap A$ separates $(x, 0)$ and $(y, 0)$ in Y. By a result in [3], p. 62 $Y \cap A$ contains a perfect set since $Y \cap A$ does not contain a local cutpoint of Y. This contradicts the assumption that $A^{(n)} = \emptyset$. Thus, if A is any set in X_α that separates $(x, 0)$ and $(y, 0)$, then $A^{(n)} \neq \emptyset$. We have proved that the rim-type of X_α is α.

Clearly, $X_\alpha \cap (\{i\} \times [-1, 1])$ is countable for $i = 0, 1$.

Finally, let A be a set in X_α such that $A^{(n)} = \emptyset$. We must prove that there is an arc in $X_\alpha \setminus A$ stretching from $\{0\} \times [-1, 1]$ to $\{1\} \times [-1, 1]$. By the above $Y_0 \setminus A$ is contained in one arc component of $X_\alpha \setminus A$. Since the sequence of continua $f_{j1}(X_n)$, $j = 1, 2, \ldots$ converges to a point and $A^{(n)} = \emptyset$ it follows that there is a natural number j and an ordinal $m < n$ such that $\big(A \cap f_{j1}(X_n)\big)^{(m)} = \emptyset$. By (iii) there exists an arc from $f_{j1}(A_n)$ to $f_{j1}(B_n)$ in $f_{j1}(X_n) \setminus A$. Similarly there is a natural number k and an arc from $f_{km_k}(A_n)$ to $f_{km_k}(B_n)$ in $f_{km_k}(X_n) \setminus A$. Thus, there is an arc in $X \setminus A$ stretching from $\{0\} \times [-1, 1]$ to $\{1\} \times [-1, 1]$.

Case 3. α is the successor of the limit ordinal n. Let $\alpha_1, \alpha_2, \ldots$ be a strictly increasing sequence of ordinal numbers which converges to n such that each α_i is not a limit ordinal. Take everything to be as in Case 2 except that for $j = 1, 2, \ldots$ Y_j is the union of a finite number of disjoint copies of X_{α_j} instead of X_n. The argument of Case 2 can now be used to show that X_α is a planar hereditarily locally connected continuum that satisfies (i)-(iii) and the rim-type of X_α is α.

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Accepté par la Rédaction le 3. 8. 1973