[3] I. S. Gaal, *On a generalized notion of compactness I-II*, Proc. Nederl. Akad. Wetensch 60 (1957), pp. 421–435.
[4] — *On the theory of (m, n)-compact spaces*, Pacific J. Math. 8 (1958), pp. 721–734.
[5] R. E. Hodel and J. E. Vaughan, *A note on* $[\mathfrak{a}, \mathfrak{b}]$*-compactness*, General Topology and Appl. 4 (1974), pp. 179–189.
[6] N. Howes, *Well-ordered sequences*, Dissertation, Texas Christian University, 1968.
[7] — *A theorem on partially ordered sets and a new theory of convergence*, Notices Amer. Math. Soc. 15 (1968), p. 346.
[8] J. L. Kelley, *General Topology*, Princeton 1955.
[9] A. Miščenko, *Finally compact spaces*, Soviet Math. 3 (4) (1962), pp. 1199–1202.
[10] S. Mrówka, *Compactness and product spaces*, Colloq. Math. 7 (1959), pp. 19–22.
[11] N. Noble, *Products with closed projections II*, Trans. Amer. Math. Soc. 160 (1971), pp. 169–183.
[12] J. E. Rubin, *Set Theory for the Mathematician*, San Francisco 1967.
[13] Yu. M. Smirnov, *On topological spaces, compact in a given interval of powers*, Akad. Nauk SSSR Izvest. Ser. Mat. 14 (1950), pp. 155–178.
[14] J. E. Vaughan, *Product spaces with compactness-like properties*, Duke Math. J. 39 (1972), pp. 611–617.
[15] — *Some recent results in the theory of* $[\mathfrak{a}, \mathfrak{b}]$*-compactness*, Preceedings of the Second Pittsburgh International Conference on General Topology and its Applications, Lecture Notes in Mathematics 378, Berlin 1974, pp. 534–550.

Accepté par la Rédaction le 24. 10. 1973

On the descriptive set theory of the lexicographic square

by

A. J. Ostaszewski (London)

Abstract. Analytic and descriptive Borel subsets of the lexicographic square S are characterized. A sigma-compact subset is found not to be descriptive Borel. All analytic subsets are seen to be images under a three-valued semi-continuous mapping from the set I of irrationals and some are not two-valued such images. A first-countable separable compact subset is seen to be a two-valued such image of I but not single-valued. Two Borelian hierarchies in S (one derived from compact sets, the other from descriptive Borel sets) are studied. An absolutely closed space which is not sigma-descriptive Borel is constructed.

Introduction and definitions. Let S be the unit square $[0,1]^2$ ordered lexicographically (so that $\langle x_1, x_2\rangle < \langle x_1', x_2'\rangle$ if and only if either $x_1 < x_1'$ or both $x_1 = x_1'$ and $x_2 < x_2'$) and endowed with the topology generated by this ordering. S is compact and first-countable (compare [4, pp. 52–53]). Our investigations below of the analytic and descriptive Borel subsets of S (shortly to be defined) uncover an interesting (perhaps "exemplary") divergence of descriptive set theory in S from the classical situation in Polish spaces. For example, the compact subset $[0,1]\times\{0,1\}$, which is first-countable and separable (it contains $Q\times\{0,1\}$ as a dense subset, where Q denotes the rationals of $[0,1]$), is the image of the set I of irrationals under a two-valued semi-continuous mapping, as indeed is any compact, separable, ordered space, however it is not the image of I under a single-valued, semi-continuous mapping. The compact set $[0,1]\times\{0,\frac{1}{2},1\}$ is the image of I under a three-valued, semi-continuous mapping but not under a two-valued such mapping. The set $S\setminus[0,1]\times\{\frac{1}{2}\}$ is a "naturally occurring" example of a sigma-compact subset which is not descriptive Borel (compare the example given by Z. Frolík in [2, p. 166]).

Let $\mathcal{H}$ be a family of sets in a space X. We denote by Borelian-$\mathcal{H}$ the smallest family of sets of X to include $\mathcal{H}$ and closed under countable unions and countable intersections. We characterize two hierarchies of Borelian-$\mathcal{H}$ sets (see § 3 for definitions), one for $\mathcal{H}$ consisting of the compact sets $\mathcal{K}$ of S, the other for $\mathcal{H}$ consisting of the descriptive Borel sets, finding them cofinal in one another with respect to inclusion. These considerations enable us to make two contributions to the "absolutenees problem". We find a $\mathcal{K}_{\sigma\delta}$ subspace T of S and a topology S^* on the square, larger than S but agreeing on T, which is not Borelian — descriptive Borel in S^*. Thus the property of being Borelian — descriptive Borel is not preserved under topological re-embedding. We discover that S^* is absolutely closed i.e. it is a closed subspace of any Hausdorff space containing it. S^* however is not a countable union of its descriptive Borel subspaces.

For convenience we introduce the following definitions. As usual we identify $\boldsymbol{I}$ with the set of infinite sequence $\boldsymbol{i} = (i_1, i_2, \ldots, i_n, \ldots)$ of positive integers and $\boldsymbol{i}|n$ denotes the initial segment $(i_1, \ldots, i_n)$. $\boldsymbol{I}(\boldsymbol{i}|n)$ is the set of sequences $\boldsymbol{j}$ in $\boldsymbol{I}$ with $\boldsymbol{j}|n = \boldsymbol{i}|n$. We shall say that a multivalued mapping K from $\boldsymbol{I}$ to a Hausdorff space X is *analytic* if, for each $\boldsymbol{i}$ in $\boldsymbol{I}$, $K(\boldsymbol{i})$ is compact in X and the mapping is *semi-continuous*, i.e. if G is open in X and $K(\boldsymbol{i}) \subseteq G$ for some $\boldsymbol{i}$ in $\boldsymbol{I}$, then there is an integer n so that $K(\boldsymbol{j}) \subseteq G$ for all $\boldsymbol{j}$ in $\boldsymbol{I}$ with $\boldsymbol{j}|n = \boldsymbol{i}|n$. If $\boldsymbol{J} \subseteq \boldsymbol{I}$, then $K[\boldsymbol{J}]$ denotes the set $\bigcup_{\epsilon \boldsymbol{J}} K(\boldsymbol{j})$. The analytic mapping will be termed *descriptive*, if $K(\boldsymbol{i}) \cap K(\boldsymbol{j}) = \emptyset$ whenever $\boldsymbol{i}, \boldsymbol{j}$ are distinct elements of $\boldsymbol{I}$. Thus a set in X is *analytic* (*descriptive Borel*) if for some analytic (descriptive) mapping K the set may be represented as $K[\boldsymbol{I}]$. (Compare C. A. Rogers [7]). K will be called *single-valued* if, for each $\boldsymbol{i}$ in $\boldsymbol{I}$, $K(\boldsymbol{i})$ consists of at most one point.

A subset T of S will be called *vertical*, if for some x in $[0, 1]$, $T \subseteq \{x\} \times [0, 1]$. For any subset A of S and for x in $[0, 1]$ we put $A^x = A \cap (\{x\} \times [0, 1])$.

(a, b) ambiguously denotes the open interval in S or $[0, 1]$ (depending on context) with end-points a, b.

1. Fundamental characterization theorems. As a first step towards our characterization we establish:

1.1. PROPOSITION. *If A is analytic in S, then there is an analytic mapping $\tilde{A}$ such that $A = \tilde{A}[\boldsymbol{I}]$ and each set $\tilde{A}(\boldsymbol{i})$ is vertical. Moreover if A is descriptive Borel, $\tilde{A}$ is descriptive.*

Proof. Let $A = K[\boldsymbol{I}]$ with K analytic. Also write $[0, 1] = D[\boldsymbol{I}]$, where D is descriptive and each $D(\boldsymbol{i})$ consists of at most one point. Now define a compact-valued mapping H by

$$H(\boldsymbol{i}) = D(\boldsymbol{i}) \times [0, 1] .$$

We claim that H is descriptive. Clearly $H(\boldsymbol{i}) \cap H(\boldsymbol{j}) = \emptyset$, if $\boldsymbol{i} \neq \boldsymbol{j}$. Now suppose G is open in S and that

$$H(\boldsymbol{i}) \subseteq G .$$

If $H(\boldsymbol{i}) = \emptyset$, then $D(\boldsymbol{i}) \subseteq \emptyset$ and so, for some n, $D[\boldsymbol{I}(\boldsymbol{i}|n)] \subseteq \emptyset$ (since D is semi-continuous). For this n we have of course $H[\boldsymbol{I}(\boldsymbol{i}|n)] \subseteq \emptyset \subseteq G$. So suppose that $D(\boldsymbol{i}) = \{x\}$ for some x in $(0, 1)$. Since $\langle x, 0\rangle \in G$, there is a basic open interval of S say $(\langle a, r\rangle, \langle x, s\rangle)$ about $\langle x, 0\rangle$ contained in G. Analogously there is an interval about $\langle x, 1\rangle$ contained in G. We deduce that there are numbers a, b in $[0, 1]$ such that $a < x < b$ and

$$H(\boldsymbol{i}) \subseteq (a, b) \times [0, 1] \subseteq G .$$

Now since D is semi-continuous, there is n so that $D[\boldsymbol{I}(\boldsymbol{i}|n)] \subseteq (a, b)$. Thus

$$H[\boldsymbol{I}(\boldsymbol{i}|n)] \subseteq (a, b) \times [0, 1] \subseteq G .$$

If $D(\boldsymbol{i}) = \{x\}$ and x is 0 or 1, a slight modification to the above argument establishes also semi-continuity at $\boldsymbol{i}$.

Now define $\tilde{A}$ by

$$\tilde{A}(\boldsymbol{i}) = H(i_1, i_3, \ldots, i_{2n-1}, \ldots) \cap K(i_2, i_4, \ldots, i_{2n}, \ldots) .$$

A standard argument will show that $\tilde{A}$ is analytic. Moreover, if A is descriptive Borel, we may assume that $K(\boldsymbol{i}) \cap K(\boldsymbol{j}) = \emptyset$, whenever $\boldsymbol{i} \neq \boldsymbol{j}$. Since H enjoys a similar property, $\tilde{A}$ is descriptive.

Remark. The argument above also shows that if $x \in [0, 1]$, then $S \setminus \{x\} \times [0, 1]$ is descriptive Borel in S.

We wish to reduce the study of analytic subsets of S to those of the real line. This reduction will be achieved partly by the following two propositions.

DEFINITIONS. Let $\tilde{A}\colon \boldsymbol{I} \to S$ be a semi-continuous mapping. By an *exceptional point of the representation* $\tilde{A}$ we mean a real number x to which there corresponds a vector $\boldsymbol{i}$ in $\boldsymbol{I}$ such that $\tilde{A}(\boldsymbol{i})$ meets $\{x\} \times [0, 1]$, and

$$|\tilde{A}(\boldsymbol{i}) \cap \{x\} \times \{0, 1\}| \leqslant 1 .$$

By an *exceptional point* of a subset Z of S we shall mean any real number x such that Z meets $\{x\} \times [0, 1]$, and

$$|Z \cap \{x\} \times \{0, 1\}| \leqslant 1 .$$

1.2. PROPOSITION. *Let $\tilde{A}\colon \boldsymbol{I} \to S$ be a vertical semi-continuous compact-valued mapping. Then there are at most countably many exceptional points of the representation $\tilde{A}$ and so any analytic set in S has at most a countable number of exceptional points.*

Proof. (See Skula [9] for a different proof.) Write $A = \tilde{A}[\boldsymbol{I}]$. Let

$$\boldsymbol{J} = \{\boldsymbol{i} \in \boldsymbol{I}\colon \tilde{A}(\boldsymbol{i}) \subseteq S \setminus [0, 1] \times \{0, 1\}\} ,$$

then $\boldsymbol{J}$ is open in $\boldsymbol{I}$. For if $\boldsymbol{j} \in \boldsymbol{J}$, then since $S \setminus [0, 1] \times \{0, 1\}$ is open and

$$\tilde{A}(\boldsymbol{j}) \subseteq S \setminus [0, 1] \times \{0, 1\} ,$$

there is an integer n so that

$$\tilde{A}[I(\boldsymbol{j}|n)] \subseteq S \setminus [0,1] \times \{0,1\}$$

and so the Baire interval $I(\boldsymbol{j}|n)$ lies wholly in $\boldsymbol{J}$. Thus $\tilde{A}[\boldsymbol{J}]$ is analytic in S, hence by a theorem of M. Sion ([8]) is a Lindelöf set. Now $\tilde{A}[\boldsymbol{J}]$ is covered by the collection of open sets $\{x\} \times (0,1)$ for $0 \leqslant x \leqslant 1$. So it is covered by a countable subcollection; that is, for countably many reals x at most, do we have $\emptyset \neq \tilde{A}(\boldsymbol{i}) \subseteq \{x\} \times (0,1)$ for some $\boldsymbol{i}$ in $\boldsymbol{I}$. But the set $\tilde{A}[\boldsymbol{J}]$ covers all the sets A^x for which $A^x \subseteq \{x\} \times (0,1)$ (for if $y \in A^x$, then, for some $\boldsymbol{i}$ in $\boldsymbol{I}$, $y \in \tilde{A}(\boldsymbol{i}) \subseteq A^x$, since each $\tilde{A}(\boldsymbol{i})$ is vertical). Thus a fortiori there are countably many points x at most for which $A^x \subseteq \{x\} \times (0,1)$.

To complete the proof we consider the set

$$\boldsymbol{J}_0 = \{\boldsymbol{j} \in \boldsymbol{I}\colon \langle x, 0\rangle \in \tilde{A}(\boldsymbol{j}) \text{ and } \langle x, 1\rangle \notin \tilde{A}(\boldsymbol{j}) \text{ for some } x \in [0,1]\}.$$

For $\boldsymbol{j}$ in $\boldsymbol{J}_0$ we define $u(\boldsymbol{j})$ to be the number x in $[0,1]$ such that $\langle x, 0\rangle \in \tilde{A}(\boldsymbol{j})$. We claim that u is continuous on $\boldsymbol{J}_0$ and has a local maximum at each point of $\boldsymbol{J}_0$. For let $\boldsymbol{j}^* \in \boldsymbol{J}_0$ and let $\delta > 0$ be given, then

$$\tilde{A}(\boldsymbol{j}^*) \subseteq (\langle x-\delta, 0\rangle, \langle x, 1\rangle),$$

where we suppose that $0 < x$ and δ is so small that $0 < x - \delta$. Then, for some n,

$$\tilde{A}[I(\boldsymbol{j}^*|n)] \subseteq (\langle x-\delta, 0\rangle, \langle x, 1\rangle).$$

Hence, if $\boldsymbol{j} \in I(\boldsymbol{j}^*|n) \cap \boldsymbol{J}_0$, then

$$x - \delta < u(\boldsymbol{j}) \leqslant x = u(\boldsymbol{j}^*).$$

If $x = 0$, we may deduce instead that for some n

$$0 = u(\boldsymbol{j}) \leqslant u(\boldsymbol{j}^*) = 0,$$

for all $\boldsymbol{j}$ in $I(\boldsymbol{j}^*|n) \cap \boldsymbol{J}_0$.

Now $\boldsymbol{J}_0$ is a separable metric space so by what we have just shown and in view of a lemma we shall shortly prove $u[\boldsymbol{J}_0]$ is at most countable. It follows that the set of points x such that for some $\boldsymbol{i}$ (necessarily in $\boldsymbol{J}_0$) $\langle x, 0\rangle \in \tilde{A}(\boldsymbol{i})$ and $\langle x, 1\rangle \notin \tilde{A}(\boldsymbol{i})$ is countable at most. A fortiori the (smaller) set of points x such that $\langle x, 0\rangle \in A$ and $\langle x, 1\rangle \notin A$ is at most countable. An analogous proof demonstrates that the remaining exceptional points of the representation $\tilde{A}$ and exceptional points of A form at most countable sets. So to establish our proposition we must prove:

1.3. Lemma. *Let E be a separable metric space and u a real-valued function defined on E, having a local maximum at all points of E. Then $u[E]$ is at most countable.*

Proof. I am indebted to Mr H. Kestelman for the following proof. Suppose that $u[E]$ is uncountable. Since E has a countable base for its topology, E has at most a countable number of points which are not points of condensation. We may assume that every point of E is a point of condensation (otherwise replace E throughout by the set of its points of condensation). For each x in E choose a ball $B(x)$ of radius less than 1 such that, for all y in $B(x)$, $u(y) \leqslant u(x)$ and write, for n a positive integer,

$$E_n = \{x \in E\colon \text{radius } B(x) > 1/n\},$$

then

$$E = \bigcup_{n=1}^{\infty} E_n \quad \text{and} \quad u[E] = \bigcup_{n=1}^{\infty} u[E_n].$$

Consequently for some integer N, $u[E_N]$ is uncountable. Now choose a subset F of E_N so that

$$u[F] = u[E_N]$$

and u is one-to-one on F. F is uncountable hence (since E is separable, metric) has a point of condensation, ξ say. So there is a point η in F distinct from ξ at a distance less than $1/2N$. Of course, by definition of F, $u(\xi) \neq u(\eta)$. On the other hand ξ is in E_N, so that the radius of $B(\xi)$ is at least $1/N$ and hence $\eta \in B(\xi)$ with the result that $u(\eta) \leqslant u(\xi)$. But η is also in E_N and we deduce that $\xi \in B(\eta)$ from which it follows that $u(\xi) \leqslant u(\eta)$. We are thus led to the contradictory conclusion that $u(\xi) = u(\eta)$. $u[E]$ is accordingly countable.

1.4. Proposition. *Let A be analytic (descriptive Borel) in S.*

(i) *If $A_0 \times \{0\} = ([0,1] \times \{0\}) \cap A$, then A_0 is analytic (Borel) in the usual topology of $[0,1]$.*

(ii) *If $\{x\} \times A_x = A^x$, then A_x is analytic (Borel) in $[0,1]$.*

Proof. Write $A = \tilde{A}[\boldsymbol{I}]$, where $\tilde{A}$ is vertical, semi-continuous, compact-valued (descriptive, if A is assumed descriptive Borel). Let $\{x_n\}$ enumerate the exceptional points of the representation $\tilde{A}$. For each integer n, let

$$\boldsymbol{H}_n = \{\boldsymbol{i} \in \boldsymbol{I}\colon \tilde{A}(\boldsymbol{i}) \cap (\{x_n\} \times [0,1]) \neq \emptyset\}.$$

The latter set is closed by the semi-continuity of A (since $\boldsymbol{I} \setminus \boldsymbol{H}_n = \{\boldsymbol{i} \in \boldsymbol{I}\colon \tilde{A}(\boldsymbol{i}) \subseteq S \setminus (\{x_n\} \times [0,1])\}$). Write $\boldsymbol{J} = \boldsymbol{I} \setminus \bigcup_{n=1}^{\infty} \boldsymbol{H}_n$, which is $\mathcal{G}_\delta$ in $\boldsymbol{I}$. Let U be open in $[0,1]$ and let $\boldsymbol{j} \in \boldsymbol{J}$. Suppose that

$$\tilde{A}(\boldsymbol{j}) \cap ([0,1] \times \{0\}) \subseteq U \times \{0\}.$$

Two cases arise. If $\tilde{A}(\boldsymbol{j}) = \emptyset$, then, for some integer n, $\tilde{A}[I(\boldsymbol{j}|n)] = \emptyset$ (by the semi-continuity of $\tilde{A}$). If, however, $\tilde{A}(\boldsymbol{j}) \neq \emptyset$, we can choose x so that $\tilde{A}(j) \subseteq \{x\} \times [0,1]$. Then, x is not an exceptional point for the representation $\tilde{A}$ (since $j \in J$). Hence $\{x\} \times \{0,1\} \subseteq \tilde{A}(j)$ and $x \in U$. We deduce that $\tilde{A}(j) \subseteq U \times [0,1]$ and since $\tilde{A}$ is semi-continuous, there is an integer n so that $\tilde{A}[I(j|n)] \subseteq U \times [0,1]$. Thus the mapping

$$j \to \tilde{A}(j) \cap ([0,1] \times \{0\})$$

for $j \in J$ is semi-continuous from J to the set $[0,1] \times \{0\}$ endowed with the usual (order) topology. The mapping is disjoint-valued if $\tilde{A}$ was. Since J is Borel in I, the set $\tilde{A}[J] \cap ([0,1] \times \{0\})$ is analytic (Borel) in the usual topology of $[0,1] \times \{0\}$ and differs from $A_0 \times \{0\}$ by a set which is at most countable and consists of exceptional points of the representation $\tilde{A}$.

The assertion (ii) is clear because the subspace topology of $\{x\} \times [0,1]$ in S is homeomorphic to the usual topology of $[0,1]$.

We are now in a position to characterize the analytic subsets of S.

THEOREM 1. *A necessary and sufficient condition that a subset A of S be analytic in S is that it may be expressed in the form*

$$\bigcup_{x \in E} \{x\} \times A_x \cup \bigcup_{x \in P} \{x\} \times A_x,$$

where (i) *E is at most countable and all the sets A_x for x in E are analytic;*

(ii) *P is analytic in* $[0,1]$, *disjoint from E and for each x in P the set A_x is analytic in* $[0,1]$ *and contains both* 0 *and* 1.

Proof. We begin by establishing the necessity of such conditions. Suppose then that A is analytic. By Proposition 1.2 the set E of exceptional points of A is at most countable. We have already remarked at the end of Proposition 1.1 that the sets $S \setminus (\{x\} \times [0,1])$ are analytic. Hence the set

$$A \cap \bigcap_{x \in E} S \setminus (\{x\} \times [0,1]),$$

which we denote by A^1, is analytic (E being at most countable). By Proposition 1.4, if P satisfies $P \times \{0\} = A^1 \cap ([0,1] \times \{0\})$, then P is analytic in $[0,1]$ and disjoint from E. Now if $\langle x, y \rangle \in A^1$, then, since x is not an exceptional point of A, both points $\langle x, 0 \rangle$ and $\langle x, 1 \rangle$ belong to A and also to A^1. Further by Proposition 1.4 for each x the set A_x is analytic. If $x \in P$, then both $0, 1$ are in A_x.

Now we establish the sufficiency of the conditions. Since P is analytic in $[0,1]$, we may write $P = K[I]$, where K is a single-valued semi-continuous mapping (each $K(i)$ consisting of at most one point). Now, for given i in I, $(K(i) \times [0,1]) \cap A$ is congruent to an analytic set in $[0,1]$ which contains 0 and 1 (by (ii)). We may express this set in $[0,1]$ as $\bigcup_{j \in I} K'(i,j)$,

where for each i the map $j \to K'(i,j)$ is semi-continuous and compact-valued. Put

$$K(i,j) = \{0,1\} \cup K'(i,j).$$

Then $j \to K(i,j)$ is also a semi-continuous compact-valued mapping. For given i, j write $A^*(i,j) = K(i) \times K(i,j)$. We claim that the mapping A^* is semi-continuous from $I \times I$ to S (and is of course compact-valued). Suppose then that G is open in S and for some i, j in I

$$A^*(i,j) \subseteq G.$$

Two cases arise. If $K(i) = \emptyset$, then, for some n, $K[I(i|n)] \subseteq \emptyset$, whence

$$A^*[I(i|n) \times I(i|n)] \subseteq \emptyset \subseteq G.$$

So we may suppose that (since K is single-valued), for some x, $K(i) = \{x\}$. Now $\{0,1\} \subseteq K(i,j)$, consequently $\langle x, 0 \rangle$ and $\langle x, 1 \rangle$ are elements of G. By the argument of Proposition 1.1 there is a set U open in $[0,1]$ containing x such that

$$A^*(i,j) \subseteq (U \setminus \{x\} \times [0,1] \cup (G \cap \{x\} \times [0,1]) \subseteq G.$$

From the relation $K(i) = \{x\} \subseteq U$ follows that, for some integer n $K[I(i|n)] \subseteq U$. Let G_x satisfy $\{x\} \times G_x = G^x$, then G_x is open in $[0,1]$ and $K(i,j) \subseteq G_x$. But then we have, for some integer m, $\bigcup_{k \in I(j|m)} K(i,k) \subseteq G_x$. We conclude that

$$A^*[I(i|n) \times I(j|m)] \subseteq (K[I(i|n)] \setminus \{x\}) \times [0,1] \cup \{x\} \times \bigcup_{k \in I(j|m)} K(i,k)$$
$$\subseteq (U \setminus \{x\}) \times [0,1] \cup G^x \subseteq G.$$

Finally, if we put $A^1(i_1, i_2, \ldots) = A^*(i_1, i_3, \ldots, i_{2n-1}, \ldots; i_2, i_4, \ldots, i_{2n} \ldots)$, then A^1 is an analytic mapping and $A^1[I]$ is an analytic subset of S. To obtain A we must add to this set a countable number of sets $\{x\} \times A_x$, for x in E, which are analytic in $\{x\} \times [0,1]$ both in the usual sense and in the sense of S. We discover thus that A is an analytic subset of S as required.

The descriptive Borel subsets of S have a finer characterization.

THEOREM 2. *A necessary and sufficient condition for a subset A of S to be descriptive Borel in S is that it may be expressed in the form*

$$\bigcup_{x \in E} \{x\} \times A_x \cup \bigcup_{x \in B} \{x\} \times A_x, \tag{1}$$

where (i) *E is at most countable and for x in E, A_x is Borel in* $[0,1]$;

(ii) *B is Borel in* $[0,1]$, *disjoint from E and, for each x in B, the set A_x is compact and contains both* 0 *and* 1.

Proof. We first establish the necessity of the conditions. Let A be descriptive Borel in S. By Proposition 1.1 we may write $A = \tilde{A}[I]$, where $\tilde{A}$ is vertical and descriptive. Let E be the set of exceptional points of the representation $\tilde{A}$. Then E is at most countable. Put

$$A^1 = A \cap \bigcap_{x \in E} S \setminus (\{x\} \times [0,1]),$$

then A^1 is descriptive Borel (see remark after Proposition 1.1). For each x in E we put

$$J(x) = \{i \in I\colon \tilde{A}(i) \subseteq S \setminus (\{x\} \times [0,1])\},$$

then $J(x)$ is open (by the semi-continuity of $\tilde{A}$). Let

$$J = \bigcup_{x \in E} J(x).$$

Then J is $\mathcal{G}_\delta$ in I and $A^1 = \tilde{A}[J]$. Let $\langle x, y\rangle \in A^1$. We shall show that there is a unique vector i (necessarily in J) such that $\{x\} \times \{0, y, 1\} \subseteq \tilde{A}(i)$. Since $\tilde{A}$ is disjoint-valued, there is a unique vector i such that $\langle x, y\rangle \in \tilde{A}(i)$. By assumption x is not an exceptional point of $\tilde{A}$, so $\{x \times \{0,1\} \subseteq \tilde{A}(i)$.

Now suppose $j \in I$ and $\tilde{A}(j) \cap \{x\} \times [0,1] \neq \emptyset$, then, since x is not an exceptional point for the representation $\tilde{A}$ we have $\{x\} \times \{0,1\} \subseteq \tilde{A}(j)$, so by the disjointness of $\tilde{A}$ we have $j = i$. Hence $A^1 \cap \{x\} \times [0,1] = \tilde{A}(i)$. A_x is accordingly compact and contains both 0 and 1. By Proposition 1.4 if $B \times \{0\} = A^1 \cap ([0,1] \times \{0\})$, then B is Borel in $[0,1]$ and is disjoint from E. The condition is thus shown to be necessary.

We turn to the sufficiency of the conditions. Let A be a subset of S satisfying the stated conditions. Since B is Borel in $[0,1]$ we may write $B = K[I]$, where K is descriptive and each set $K(i)$ consists at most of one point. Let $H(i)$ denote the compact set in $[0,1]$ which is congruent to $(K(i) \times [0,1]) \cap \bigcup_{x \in B} \{x\} \times A_x$. $H(i)$ is either empty or contains inter alia both 0 and 1. Write

$$\hat{A}(i) = K(i) \times H(i),$$

then we may argue much as with A^* in Theorem 1 to show that $\hat{A}$ is semi-continuous and compact-valued. Moreover, since K is descriptive, $\hat{A}$ also is. Thus $\hat{A}[I]$ is a descriptive Borel subset of S. For each x in E, $\{x\} \times A_x$ is descriptive Borel in $\{x\} \times [0,1]$ and is disjoint from $\hat{A}[I]$. So A is a disjoint countable union of descriptive Borel sets of S and hence is descriptive Borel as required.

1.5. Corollary. *The set $S \setminus ([0,1] \times \{\frac{1}{2}\})$ is sigma-compact but not descriptive Borel.*

Proof. Put $K_n = S \setminus [0,1] \times \left(-\frac{1}{n+3} + \frac{1}{2}, \frac{1}{2} + \frac{1}{n+3}\right)$. Then K_n is compact and

$$S \setminus ([0,1] \times \{\tfrac{1}{2}\}) = \bigcup_{n=1}^{\infty} K_n.$$

If $S \setminus ([0,1] \times \{\frac{1}{2}\})$ were descriptive Borel, let (1) be a representation subject to the conditions of Theorem 2. Let $x \in [0,1] \setminus E$. Then by Theorem 2 the set

$$\{x\} \times [0,1] \cap S \setminus ([0,1] \times \{\tfrac{1}{2}\})$$

viz.

$$\{x\} \times ([0,1] \setminus \{\tfrac{1}{2}\})$$

should be compact, but this is a contradiction. The claim of the corollary holds good.

2. Small-analytic sets. By a *small-analytic* subset of a space X we mean a set A which may be represented in the form $K[I]$ with K analytic and $K(i)$ finite for each i in I (this widens Definition 4.12 in Z. Frolík [3]). We shall be interested in the cases where K is *single-valued, two-valued* or *three-valued*, that is, when each set $K(i)$ consists at most of one, two or three points respectively.

2.1. Proposition. *Every analytic subset of S is the image of I by a three-valued semi-continuous mapping.*

Proof. The argument resembles the one of Proposition 1.1. We write $(0,1) = E[I]$, where E is a single-valued descriptive mapping. We put for each i in I

$$F(i) = [0,1] \times (\{0,1\} \cup E(i)).$$

Then $F(i)$ is compact. We claim that F is a semi-continuous mapping. So suppose that G is open in S and, for some i in I, $F(i) \subseteq G$. If $E(i) = \emptyset$, then, for some n, $E[I(i|n)] \subseteq \emptyset$ and so $F[I(i|n)] = F(i) \subseteq G$. So we may suppose that $E(i) = \{e\}$, with $0 < e < 1$. We are going to show that there are numbers u, v with

$$[0,1] \times \{e\} \subseteq [0,1] \times (u, v) \subseteq G.$$

We rely on the compactness of $F(i)$. For each number x in $[0,1]$ we choose intervals $I_0(x)$ and $I_1(x)$ open in S and numbers $u(x)$, $v(x)$ so that

(1) $\langle x, 0\rangle \in I_0(x) \subseteq G$ and $\langle x, 1\rangle \notin I_0(x)$,

(2) $\langle x, 1\rangle \in I_1(x) \subseteq G$ and $\langle x, 0\rangle \notin I_1(x)$,

(3) $\langle x, e\rangle \in \{x\} \times (u(x), v(x)) \subseteq G$.

We require moreover $I_0(x)$ to have its left-hand end-point on $[0,1]\times\{0\}$ and $I_1(x)$ to have its right-hand end-point on $[0,1]\times\{1\}$. Thus if $y\neq x$ and $\langle y, e\rangle$ is in $I_0(x)$ (or in $I_1(x)$), then $\{y\}\times[0,1]\subseteq I_0(x)$ (or $\{y\}\times[0,1]\subseteq I_1(x)$). This is possible, since $F(\boldsymbol{i})\subseteq G$. Write

$$I(x) = I_0(x)\cup\{x\}\times\big(u(x), v(x)\big)\cup I_1(x).$$

Then

$$\{x\}\times\{0, e, 1\}\subseteq I(x)\subseteq G.$$

Now the sets $I(x)$ for $0\leqslant x\leqslant 1$ form an open cover of the compact set $F(\boldsymbol{i})$, so for some points $x_1, \ldots, x_n$ in $[0,1]$, $F(\boldsymbol{i})$ is covered by $I(x_1)\cup\ldots\cup I(x_n)$. Put

$$u = \max\{u(x_1), \ldots, u(x_n)\},$$

$$v = \min\{v(x_1), \ldots, v(x_n)\},$$

then

$$\langle x_i, e\rangle\in\{x_i\}\times(u, v)\subseteq G\qquad (i = 1, 2, \ldots, n).$$

Now if x is a number different from all the number $x_1, \ldots, x_n$ we shall show that

$$\langle x, e\rangle\in\{x\}\times(u, v)\subseteq G.$$

Certainly for some x_i we have $\langle x, e\rangle\in I(x_i)$. If $x > x_i$, then it must be that $\langle x, e\rangle\in I_1(x_i)$. By the requirements on the end-points we have immediately that $\{x\}\times(u, v)\subseteq I_1(x_i)\subseteq G$. If $x < x_i$, then $\langle x, e\rangle\in I_0(x)$ and again by the requirement on end-points $\{x\}\times(u, v)\subseteq I(x)\subseteq G$. Finally as $E(\boldsymbol{i})\subseteq(u, v)$, there is an integer n such that $E(\boldsymbol{I}(\boldsymbol{i}|n))\subseteq(u, v)$; from this we deduce that

$$F[\boldsymbol{I}(\boldsymbol{i}|n)]\subseteq[0,1]\times E[\boldsymbol{I}(\boldsymbol{i}|n)]\cup F(\boldsymbol{i})$$
$$\subseteq[0,1]\times(u, v)\cup F(\boldsymbol{i})\subseteq G.$$

Now let A be any analytic subset of S. We may express A as $\tilde{A}[\boldsymbol{I}]$, where $\tilde{A}$ is analytic and each set $\tilde{A}(\boldsymbol{i})$ is vertical. Now write

$$A^+(i_1, i_2, \ldots) = \tilde{A}(i_1, i_3, \ldots, i_{2n-1}, \ldots)\cap F(i_2, i_4, \ldots, i_{2n}, \ldots),$$

then A^+ is analytic and each set $A^+(\boldsymbol{i})$ consists of at most three points.

2.2. Proposition. *The set $[0,1]\times\{0,1\}$ is the image of $\boldsymbol{I}$ under a two-valued but not under a single-valued semi-continuous mapping.*

Proof. If we use the representation from Proposition 1.1 we obtain a two-valued mapping since each vertical subset of $[0,1]\times\{0,1\}$ consists of two points at most. Suppose that $[0,1]\times\{0,1\} = K[\boldsymbol{I}]$ with K single-valued. Put

$$J_0 = \{\boldsymbol{j}\in\boldsymbol{I}:\ (\exists x)\langle x, 0\rangle\in K(\boldsymbol{j})\}.$$

Notice that if $\boldsymbol{j}\in J_0$, then $\langle x, 1\rangle\notin K(\boldsymbol{j})$ whenever $\langle x, 0\rangle\in K(\boldsymbol{j})$. Writing $\{u(\boldsymbol{j})\} = K(\boldsymbol{j})$, for $\boldsymbol{j}\in J_0$, we see that u has a local maximum at each point of J_0 (as in Proposition 1.2). Hence by Lemma 1.3 $K[J_0]$ $(= u[J_0])$ is countable. Analogously $K[\boldsymbol{I}\setminus J_0]$ is countable, so $K[\boldsymbol{I}]$ is countable and this is a contradiction.

As a matter of fact this last proposition has a generalization to compact, separable, order topologies.

2.3. Proposition. *Let $\langle X, <\rangle$ be an ordered set whose order topology is compact and separable. Then X is the image of $\boldsymbol{I}$ under a two-valued semi-continuous mapping but is not a single-valued such image when and only when X has uncountably many pairs of consecutive points.*

Proof. In [5] it is shown that $\langle X, <\rangle$ is order-isomorphic to a set $Y\subseteq[0,1]\times\{0,1\}$, Y ordered lexicographically, such that $Y\cap[0,1]\times\{0\}$ is congruent to a closed set in $[0,1]$ and $\langle x, 0\rangle\in Y$ whenever $\langle x, 1\rangle\in Y$. We write

$$Y\cap([0,1]\times\{0\}) = K[\boldsymbol{I}]\times\{0\},$$

with K a single-valued, semi-continuous mapping into $[0,1]$. Endow Y with the topology $\mathcal{C}$ determined by the (restricted) ordering of Y (not to be confused with the subspace topology of the lexicographic-order topology of $[0,1]\times\{0,1\}$). Put

$$H(\boldsymbol{i}) = Y\cap\big(K(\boldsymbol{i})\times\{0,1\}\big),$$

then H is seen to be semi-continuous and two-valued. $\mathcal{C}$ is homeomorphic to the topology of X and we have our result.

If X has uncountably many pairs of points x_1, x_2 such that there are no points x in X strictly between x_1 and x_2, then Y^1 is uncountable, where

$$Y^1 = \{\langle t, 0\rangle\in Y:\ \langle t, 1\rangle\in Y\}.$$

Now argue as in the last proposition, this time taking

$$J_0 = \{\boldsymbol{j}\in\boldsymbol{I}:\ (\exists t)\langle t, 0\rangle\in K(\boldsymbol{j})\cap Y^1\}.$$

If X has countably many pairs of such "consecutive" points x_1, x_2, put

$$Y_0 = \{\langle t, 0\rangle\in Y:\ \langle t, 1\rangle\notin Y\},$$

then Y_0 is congruent to a Borel subset B of $[0,1]$ and since the $\mathcal{C}$-subspace topology on Y_0 is homeomorphic to that of the set B, Y_0 is seen to be the image of $\boldsymbol{I}$ under a single-valued mapping. It is routine to extend this conclusion to Y, since the missing points are countable in number.

We close this discussion of S with the following observation.

2.4. Proposition. *The set $[0,1]\times\{0,\frac{1}{2},1\}$ is not the image of $\boldsymbol{I}$ under a two-valued semi-continuous mapping.*

Proof. Suppose otherwise and write

$$[0,1]\times\{0,\tfrac{1}{2},1\} = K[I],$$

where K is two-valued semi-continuous. By Proposition 1.1 (or rather its proof) we see that no loss of generality is incurred if we assume that each set $K(\boldsymbol{i})$ is vertical. Put

$$J_0 = \{j\in I\colon (\exists x)K(j) = \{\langle x, 0\rangle, \langle x, \tfrac{1}{2}\rangle\}\},$$
$$J_1 = \{j\in I\colon (\exists x)K(j) = \{\langle x, \tfrac{1}{2}\rangle, \langle x, 1\rangle\}\}.$$

Define u_0, u_1 on J_0 and J_1 respectively by:

$$K(j) = \{\langle u_0(j), 0\rangle, \langle u_0(j), \tfrac{1}{2}\rangle\}, \quad \text{if} \quad j\in J_0,$$
$$K(j) = \{\langle u_1(j), \tfrac{1}{2}\rangle, \langle u_1(j), 1\rangle\}, \quad \text{if} \quad j\in J_1.$$

Then, as in Proposition 1.2, u_0 has a local maximum at each point of J_0 and u_1 has a local minimum at each point of J_1. Hence $u_0[J_0]\cup u_1[J_1]$ is at most countable.

Now by Proposition 1.2 there are countably many exceptional points of the representation K and so countably many points x at most for which $K(\boldsymbol{i}) = \{\langle x, \frac{1}{2}\rangle\}$ for some $\boldsymbol{i}$. Hence for uncountably many x there are vectors $\boldsymbol{i}(x)$ in I such that $K(\boldsymbol{i}(x))$ contains $\langle x, \frac{1}{2}\rangle$ together with another point $\langle x, k(x)\rangle$, where $k(x)\in\{0,1\}$. It follows that $u_0[J_0]\cup u_1[J_1]$ is uncountable. This contradiction shows that no such K exists and our claim is justified.

3. The $\mathfrak{D}^{(\alpha)}$ and the $\mathfrak{K}^{(\alpha)}$ hierarchies. We recall that for any subset A of the unit square and for x in $[0,1]$, A_x denotes the (unique) set such that $\{x\}\times A_x = A\cap(\{x\}\times[0,1])$. We recall also that an ordinal number is said to be *odd* if it may be written as $\lambda+(2n-1)$ with n a positive integer and λ zero or a limit ordinal, otherwise it is said to be *even*.

DEFINITIONS. Let $\mathcal{H}$ be a family of sets in a space X. We define for $\alpha<\omega_1$ sets $\mathcal{H}^{(\alpha)}$ by transfinite induction by the scheme:

$$\mathcal{H}^{(0)} = \mathcal{H},$$

$$\mathcal{H}^{(\alpha)} = \begin{cases} \big(\bigcup_{\beta<\alpha}\mathcal{H}^{(\beta)}\big)_\sigma, & \text{if } \alpha \text{ is odd},\\ \big(\bigcup_{\beta<\alpha}\mathcal{H}^{(\beta)}\big)_\delta, & \text{if } \alpha \text{ is even and } 0<\alpha.\end{cases}$$

We define $\mathfrak{K}(X)$ to be the family of compact subsets of X and $\mathfrak{D}(X)$ to be the family of descriptive Borel subsets of X. $\mathcal{F}(X)$ denotes the closed sets.

In this section we shall study the hierarchy $\langle\mathfrak{D}^{(\alpha)}\colon \alpha<\omega_1\rangle$ where $\mathfrak{D}^{(\alpha)} = \mathfrak{D}(S)^{(\alpha)}$. We give a characterization of these sets in the manner of § 1 and deduce a hierarchy theorem (which generalizes the result of Corollary 1.5). We remark that, of course, if X is any space, then $\mathfrak{K}(X)^{(\alpha)}\subseteq\mathfrak{D}(X)^{(\alpha)}$ for all α and that if X is descriptive Borel, then $\mathcal{F}(X)^{(\alpha)}\subseteq\mathfrak{D}(X)^{(\alpha)}$. Moreover the $\mathfrak{D}(X)^{(\alpha)}$ hierarchy is absolute in the sense that, if X be embedded topologically in a space Y, then $\mathfrak{D}(X)^{(\alpha)}\subseteq\mathfrak{D}(Y)^{(\alpha)}$. It is unfortunate that the $\mathcal{F}^{(\alpha)}$ hierarchy has no absoluteness property (X need not be Borelian-$\mathcal{F}(Y)$).

In the following lemma we collect together some results which we shall need in this section. The inductive arguments which establish them are routine and accordingly are omitted.

3.1. LEMMA.

(1) *Let $\mathcal{H}$ be a family of subsets of S. If A belongs to $\mathcal{H}^{(\alpha)}$, then A_x is a member of $\{H_x\colon H\in\mathcal{H}\}^{(\alpha)}$.*

(2) *If $H\in\mathfrak{D}$ and $H'\in\mathfrak{D}^{(\alpha)}$ are disjoint, then $H\cup H'$ belongs to $\mathfrak{D}^{(\alpha)}$.*

(3) *If A is in $\mathfrak{D}^{(\alpha)}$, then there is a countable subfamily $\mathcal{H}$ of $\mathfrak{D}$ such that A is in $\mathcal{H}^{(\alpha)}$.*

(4) *If K belongs to $\mathfrak{K}(X)$ and K' belongs to $\mathfrak{K}(X)^{(\alpha)}$, then both $K\cap K'$ and $K\cup K'$ belong to $\mathfrak{K}(X)^{(\alpha)}$.*

(5) *If H belongs to $\mathfrak{K}(X)^{(1)}$ (i.e. is sigma-compact) and K' belongs to $\mathfrak{K}(X)^{(\alpha)}$ with $\alpha\geqslant 1$, then $H\cap K'$ belongs to $\mathfrak{K}(X)^{(\alpha)}$.*

(6) *If K is compact in X and $H\subseteq K$, then H belongs to $\mathfrak{K}(X)^{(\alpha)}$, when and only when H belongs to $\mathfrak{K}(K)^{(\alpha)}$.*

3.2. PROPOSITION. *Let $A\in\mathfrak{D}^{(\alpha)}$ with $\alpha<\omega_1$, then apart from at most a countable set of numbers x in $[0,1]$ we have that*

(i) *if $\langle x, y\rangle\in A$ for some y then both $\langle x, 0\rangle$ and $\langle x, 1\rangle$ are points of A;*

(ii) *$A_x\in\mathfrak{K}([0,1])^{(\alpha)}$.*

Proof. Choose a countable family $\mathcal{H}\subseteq\mathfrak{D}$ so that $A\in\mathcal{H}^{(\alpha)}$. By Proposition 1.2, if $H\in\mathcal{H}$, then there is an at most countable set $E(H)$ in $[0,1]$ consisting of exceptional points of H. Thus if $x\notin E(H)$ and for some y in $[0,1]$ the point $\langle x, y\rangle$ lies in H, then both points $\langle x, 0\rangle$ and $\langle x, 1\rangle$ lie in H and H_x is compact. Write E for the union of the sets $E(H)$ for H in $\mathcal{H}$. We shall show that if $x\notin E$ and $\langle x, y\rangle\in A$ for some y, then $\{\langle x, 0\rangle, \langle x, 1\rangle\}\subseteq A$. We prove the following by induction on γ $(<\omega_1)$:

($*$) If $\mathcal{H}$ is a family of sets in S, $M\in\mathcal{H}^{(\gamma)}$, $\langle x, y\rangle\in M$ and $\{\langle x, 0\rangle, \langle x, 1\rangle\}\nsubseteq M$, then for some set H in $\mathcal{H}$ $\langle x, y\rangle\in H$ and $\{\langle x, 0\rangle, \langle x, 1\rangle\}\nsubseteq H$.

If $\gamma = 0$, the assertion ($*$) is trivial. Suppose that ($*$) is true for all sets M and all ordinals γ less than β. We prove ($*$) for $\gamma=\beta$ and given M. If β is odd, then there are sets $H_1, H_2, \ldots, H_n, \ldots$ belonging to $\bigcup_{\gamma<\beta}\mathcal{H}^{(\gamma)}$ so that

$$M = \bigcup_{n=1}^{\infty} H_n.$$

Then for some integer n^* we have $\langle x, y\rangle \in H_{n^*}$ and moreover we cannot have $\{\langle x, 0\rangle, \langle x, 1\rangle\} \subseteq H_{n^*}$. Now for some γ^* less than β, $H_{n^*} \in \mathcal{H}^{(\gamma^*)}$, so applying $(*)$ with H_{n^*} for M and γ^* for γ we obtain the required conclusion by virtue of the inductive hypothesis. If β is even, then there are sets $H_1, H_2, \ldots, H_n, \ldots$ belonging to $\bigcup_{\gamma<\beta} \mathcal{H}^{(\gamma)}$ such that

$$M = \bigcap_{n=1}^{\infty} H_n .$$

If it were the case that $\{\langle x, 0\rangle, \langle x, 1\rangle\} \subseteq H_n$ for each n, we would deduce that $\{\langle x, 0\rangle, \langle x, 1\rangle\} \subseteq M$. Consequently there is an integer n^* such that $\{\langle x, 0\rangle, \langle x, 1\rangle\} \nsubseteq H_{n^*}$. But $\langle x, y\rangle \in H_{n^*}$ and if $H_{n^*} \in \mathcal{H}^{(\gamma^*)}$ (where $\gamma^* < \beta$) we may apply $(*)$ with H_{n^*} for M and γ^* for γ to obtain the desired conclusion (again by virtue of the inductive hypothesis).

We have thus established $(*)$. From it we deduce immediately that if $\langle x, y\rangle \in A$ and $\{\langle x, 0\rangle, \langle x, 1\rangle\} \nsubseteq A$, then, for some H in $\mathcal{H}$, $\langle x, y\rangle \in H$ and $\{\langle x, 0\rangle, \langle x, 1\rangle\} \nsubseteq H$. It follows that x belongs to $E(H)$ and so to E. This proves part (i) of the proposition.

By Lemma 2.1 we have that $A_x \in \{H_x\colon H \in \mathcal{H}\}^{(\alpha)}$. However if $x \notin E$ and $H \in \mathcal{H}$, then $x \notin E(H)$ so that H_x is compact in $[0, 1]$. Thus part (ii) of the proposition is also proved.

THEOREM 3. *A necessary and sufficient condition for a subset B of S to belong to $\mathfrak{D}^{(\alpha)}$ (for $\alpha < \omega_1$) is that*

$$B = \bigcup_{x\in E} \{x\} \times B_x \cup \bigcup_{x\in D} \{x\} \times B_x , \tag{1}$$

where (i) *E is at most countable in $[0, 1]$ and each set B_x for x in E is Borel in $[0, 1]$;*

(ii) *D is Borel in $[0, 1]$ and is disjoint from E;*

(iii) *B_x is a member of $\mathcal{K}([0, 1])^{(\alpha)}$ and $\{0, 1\} \subseteq B_x$ for each x in D.*

Proof. We show first the necessity of this condition. Let $B \in \mathfrak{D}^{(\alpha)}$. Choose E as in the proof of the last proposition and put $B^1 = \{x\colon \langle x, 0\rangle \in B\}$. By a routine induction on α we may show that B^1 is Borel in $[0, 1]$ (the non-trivial case $\alpha = 0$ is given by Proposition 1.4). Let $D = B^1 \setminus E$, then D is Borel in $[0, 1]$ and is disjoint from E. By choice of E all the sets B_x for x in D are members of $\mathcal{K}([0, 1])^{(\alpha)}$ and satisfy $\{0, 1\} \subseteq B_x$. If on the other hand x is in E then $B_x \in \{D_x\colon D \in \mathfrak{D}\}^{(\alpha)}$ is readily shown to be Borel in $[0, 1]$.

We now prove the sufficiency of our condition. Let B be a subset of S represented in (1) subject to the conditions (i), (ii), (iii) on E and D. Since E is at most countable and each set $\{x\} \times B_x$ is in $\mathfrak{D}^{(0)}$ so is their union. The two summands in (1) are disjoint (by (ii)) hence by Lemma 3.1 (2) it will suffice to show that

$$\bigcup_{x\in D} \{x\} \times B_x$$

is a member of $\mathfrak{D}^{(\alpha)}$. This we do by induction on α. The case $\alpha = 0$ is given to us by Theorem 2 since D is Borel and $\{0, 1\} \subseteq B_x$ for each x in D. Suppose then, that our assertion is true for all ordinals less than β; we prove it also for β. For each x in D there are sets B_x^n in $\bigcup_{\alpha<\beta} \mathcal{K}([0, 1])^{(\alpha)}$ such that

$$B_x = \begin{cases} \bigcup_{n=1}^{\infty} B_x^n , & \text{if } \beta \text{ is odd} , \\ \bigcap_{n=1}^{\infty} B_x^n , & \text{if } \beta \text{ is even} . \end{cases}$$

Since $\{0, 1\} \subseteq B_x$ and $\{0, 1\}$ is compact, we may by Lemma 3.1 (4) assume that $\{0, 1\} \subseteq B_x^n$ for each n. Now we make use of the identities

$$\bigcup_{x\in D} \{x\} \times \Big(\bigcup_{n=1}^{\infty} B_x^n\Big) = \bigcup_{n=1}^{\infty} \bigcup_{x\in D} \{x\} \times B_x^n ,$$

$$\bigcup_{x\in D} \{x\} \times \Big(\bigcap_{n=1}^{\infty} B_x^n\Big) = \bigcap_{n=1}^{\infty} \bigcup_{x\in D} \{x\} \times B_x^n ,$$

and of the inductive hypothesis applied to the various sets $\bigcup_{x\in D} \{x\} \times B_x^n$ to deduce the required result.

A close look at the last part of the above proof shows that the following is true:

3.3. PROPOSITION. *If K is a compact subset of $[0, 1]$ and for each x in K the set $B(x)$ is a member of $\mathcal{K}([0, 1])^{(\alpha)}$ and $\{0, 1\} \subseteq B(x)$, then*

$$\bigcup_{x\in K} \{x\} \times B(x)$$

is a member of $\mathcal{K}(S)^{(\alpha)}$.

Proof. The case $\alpha = 0$ is easy. Argue thereafter as above.

We move on to the promised hierarchy theorem.

THEOREM 4. *For each $\alpha < \omega_1$ there is a member of $\mathcal{K}(S)^{(\alpha)}$ and so of $\mathfrak{D}^{(\alpha)}$ which is not in $\mathfrak{D}^{(\beta)}$ (and a fortiori $\mathcal{K}(S)^{(\beta)}$) for $\beta < \alpha$.*

Proof. Fix α. Let T^1 be a subset of $[\frac{1}{3}, \frac{2}{3}]$ which is in $\mathcal{K}([\frac{1}{3}, \frac{2}{3}])^{(\alpha)}$ but not in $\mathcal{K}([\frac{1}{3}, \frac{2}{3}])^{(\beta)}$ for any β less than α. By applying Lemma 3.1 (5) and (6) we see at once that the set $T = T^1 \cup \{0, 1\}$ is in $\mathcal{K}([0, 1])^{(\alpha)}$ but not in $\mathcal{K}([0, 1])^{(\beta)}$ for any $\beta < \alpha$. By Proposition 3.3, $[0, 1] \times T$ is a member $\mathcal{K}(S)^{(\alpha)}$. Call this set B. We claim B is not a member of $\mathfrak{D}^{(\beta)}$ for any $\beta < \alpha$. Suppose, if possible, that our set B belongs to $\mathfrak{D}^{(\beta)}$ with $\beta < \alpha$. Let (1) be a representation of B subject to (i), (ii), (iii) of Theorem 3. Let $x \in [0, 1] \setminus E$. Then the set B_x, that is T, is a member of $\mathcal{K}([0, 1])^{(\beta)}$ and this is a contradiction.

It is natural to seek a characterization of the $\mathcal{K}(S)^{(\alpha)}$ sets. However we face an obstacle in that the decomposition used so far "raises the index" at the first two levels. We offer a characterization of the $\mathcal{K}(S)^{(\alpha)}$ sets for $\alpha \geqslant 2$.

3.4. PROPOSITION. *Suppose that K is a compact subset of S and that $C\times\{0\} = K \cap ([0,1]\times\{0\})$. Then C is $\mathcal{K}_{\sigma\delta}$ in $[0,1]$.*

Proof. Let K_i be defined by the relation

$$K_i\times\{i\} = K\cap([0,1]\times\{i\}) \qquad (i=0,1).$$

We claim that $K_0\cup K_1$ is compact in $[0,1]$. For if $\{x_n\}$ is a strictly monotone sequence of points of $K_0\cup K_1$ we show that the sequence has a limit in $K_0\cup K_1$. We may assume without loss of generality that the sequence is strictly increasing. For each n there is thus a point $y_n\in\{0,1\}$ such that $\langle x_n, y_n\rangle\in K$. Moreover the sequence is increasing in S, hence for some number x in $[0,1]$ we have $\langle x, 0\rangle = \sup\langle x_n, y_n\rangle$ and so $\sup x_n = x\in K_0$. Thus $K_0\cup K_1$ is compact. But by Proposition 1.2 the set $K_1\setminus K_0$ or

$$\{x\colon\ \langle x,0\rangle\notin K \text{ and } \langle x,1\rangle\in K\}$$

is at most countable. Hence K_0 differs from a compact set by at most a countable set, hence is itself $\mathcal{K}_{\sigma\delta}$ in $[0,1]$.

If in the above proposition K is replaced by a sigma-compact set, then the corresponding set C will differ from a sigma-compact set by a countable set of points, i.e. will also be $\mathcal{K}_{\sigma\delta}$ in $[0,1]$.

A routine induction will now show the following:

3.5. PROPOSITION. *If B is in $\mathcal{K}(S)^{(\alpha)}$ then the set B' such that $B'\times\{0\} = B\cap([0,1]\times\{0\})$ is in $\mathcal{K}([0,1])^{(\alpha)}$ provided $\alpha\geqslant 2$.*

3.6. PROPOSITION. *If B is in $\mathcal{K}([0,1])^{(\alpha)}$ and, for each $x\in K$, $B(x)$ is in $\mathcal{K}([0,1])^{(\alpha)}$, and contains both 0 and 1, then*

$$\bigcup_{x\in B}\{x\}\times B(x)$$

is in $\mathcal{K}(S)^{(\alpha)}$.

Proof. The case $\alpha=0$ is trivial. The inductive argument at odd ordinals α is modeled after the method of Proposition 2.3 but we must also decompose B into a countable union. For even ordinals α we argue thus. Say $B=\bigcap_{m=1}^{\infty}B_m$ and the sets $B_1,\ldots,B_m,\ldots$ are in $\bigcup_{\beta<\alpha}\mathcal{K}([0,1])^{(\beta)}$, while $B(x)=\bigcap_{n=1}^{\infty}B^n(x)$ for $x\in B$ with $B^1(x),\ldots,B^n(x),\ldots$ also in $\bigcup_{\beta<\alpha}\mathcal{K}([0,1])^{(\beta)}$. Define sets $B_m^n(x)$ as follows

$$B_m^n(x)=\begin{cases} B^n(x), & \text{if } x\in B,\\ \{0,1\}, & \text{if } x\in B_m\setminus B.\end{cases}$$

Now

$$\bigcup_{x\in B}\{x\}\times B(x)=\bigcap_{n=1}^{\infty}\bigcup_{x\in B}\{x\}\times B^n(x)=\bigcap_{n=1}^{\infty}\bigcap_{m=1}^{\infty}\bigcup_{x\in B_m}\{x\}\times B_m^n(x).$$

Applying Lemma 3.1 (4) and the inductive hypothesis we settle the even-ordinal case.

THEOREM 5. *Let $2\leqslant\alpha<\omega_1$. A necessary and sufficient condition that a set B in S be in $\mathcal{K}(S)^{(\alpha)}$ is that*

$$B=\bigcup_{x\in E}\{x\}\times B_x\cup\bigcup_{x\in D}\{x\}\times B_x, \tag{2}$$

where (i) *E is countable and for $x\in E$, B_x is in $\mathcal{K}([0,1])^{(\alpha)}$;*
(ii) *D is in $\mathcal{K}([0,1])^{(\alpha)}$ and $D\cap E=\emptyset$;*
(iii) *for each $x\in D$ the set B_x is in $\mathcal{K}([0,1])^{(\alpha)}$ and $\{0,1\}\subseteq B_x$.*

Proof. We demonstrate the necessity of this condition. Suppose B is in $\mathcal{K}(S)^{(\alpha)}$. Since B is thus also in $\mathcal{D}^{(\alpha)}$, we discover by Proposition 3.2 that for x outside an at most countable set E, if $B_x\neq\emptyset$, then $\{0,1\}\subseteq B_x$. Hence by Proposition 3.5 the set $D=\{x\colon\ \{0,1\}\subseteq B_x\}\setminus E$ is in $\mathcal{K}([0,1])^{(\alpha)}$ since $[0,1]\setminus E$ is a $\mathcal{G}_\delta$ set in $[0,1]$ and $\alpha\geqslant 2$. By Lemma 3.1 (1) all the sets B_x are in $\mathcal{K}([0,1])^{(\alpha)}$ and the representation is established.

For the converse: let B satisfy (2) subject to (i), (ii) and (iii); routine argument shows that if $\alpha\geqslant 1$ and all the sets B_x are in $\mathcal{K}([0,1])^{(\alpha)}$, then $\bigcup_{x\in E}\{x\}\times B_x$ is in $\mathcal{K}(S)^{(\alpha)}$. By the last proposition the set $\bigcup_{x\in D}\{x\}\times B_x$ is in $\mathcal{K}(S)^{(\alpha)}$ (by virtue of (iii)) and so the union of these two is also in $\mathcal{K}(S)^{(\alpha)}$. This completes the proof.

We return now to the subject of the absoluteness of the $\mathcal{D}(X)^{(\alpha)}$ hierarchy touched on at the beginning of this section.

THEOREM 6. *There is a $\mathcal{K}_{\sigma\delta}$ set T in S and an analytic, Hausdorff space S^* such that T as a subspace of S is also a subspace of S^*, but T does not belong to any of the families $\mathcal{D}(S^*)^{(\alpha)}$ for $\alpha<\omega_1$.*

Proof. Let I denote the set of irrationals in $(0,1)$ and put $T=[0,1]\times(I\cup\{0,1\})$, then, by Theorem 5, T is $\mathcal{K}_{\sigma\delta}$ in S. We now enlarge the topology S to give a space S^* whose subspace $\{x\}\times[\frac13,\frac23]$ is homeomorphic to the space Y constructed in Proposition 3.7 of [6]. By referring to Y we shall be able to show that $\{x\}\times(I\cap[\frac13,\frac23])$ is not Souslin-$\mathcal{K}$ in the subspace $\{x\}\times[\frac13,\frac23]$. We then find that this fact is contradicted, if we assume that T is a member of $\mathcal{D}(S^*)^{(\alpha)}$ for some $\alpha<\omega_1$.

To define S^* we re-topologize $[0,1]^2$ by specifying basic neighbourhoods. If $\langle x,y\rangle\in[0,1]^2\setminus T$, we take a basic neighbourhood of $\langle x,y\rangle$ in the form

$$\{\langle x,y\rangle\}\cup(T\cap U),$$

6*

with U vertical, open in the sense of S and containing $\langle x, y\rangle$. For $\langle x, y\rangle$ in $[0,1]\times I$ we take its neighbourhoods in the form $T\cap U$ with U open in the sense of S and containing $\langle x, y\rangle$. If $\langle x, y\rangle\in[0,1]\times\{0,1\}$, we take basic neighbourhoods of $\langle x, y\rangle$ to be its neighbourhoods in the sense of S. It is readily seen that the criterion for introducing a topology by neighbourhood bases (Engelking [1, p. 34]) is satisfied. The topology of S^* is Hausdorff, since any set open in the sense of S is open in the sense of S^*. The set T is analytic in S and the topologies of S and S^* agree on T, so T is analytic in S^*. The set $[0,1]\setminus I$ is countable and, if y belongs to this set, the topologies of S and S^* agree on the set $[0,1]\times\{0, y, 1\}$, which is thus compact. But from the relation

$$[0,1]^2 = T\cup\bigcup_{y\in[0,1]\setminus I}[0,1]\times\{0,y,1\}$$

we see that S^* is analytic.

Having defined S^*, we show next that much of the descriptive theory of S studied in § 1 may be transferred to S^*. To begin with, notice that, since $[0,1]$ is descriptive Borel in its usual topology, we may write

$$[0,1] = \bigcup_{i\in I}F(i),$$

where $F(i) = \bigcap_{n=1}^{\infty}F(i|n)$ with all the $F(i|n)$ closed in $[0,1]$ and $F(i)\cap F(j) = \emptyset$ whenever $i\neq j$. The set $F(i|n)\times[0,1]$ is closed in S and hence also in S^*. Using the Souslin-$\mathcal{F}(S^*)$ representation

$$[0,1]^2 = \bigcup_{i\in I}\bigcap_{n=1}^{\infty}F(i|n)\times[0,1],$$

we can argue as in Proposition 1.1 to prove that any descriptive Borel set D in S^* may be represented in the form $\bigcup_{i\in I}K(i)$, where each set $K(i)$ is vertical and K is descriptive. Further in S^* the open set $A\times(0,1)$ is certainly not Lindelöf, whenever A is an uncountable subset of $[0,1]$, hence there are at most countably many x in $[0,1]$ for which there is i in I with $K(i)\subseteq\{x\}\times(0,1)$. Since the topologies of S and S^* agree on $[0,1]\times\{0,1\}$ we may deduce (a) that as in Proposition 1.2 there are at most countably many x in $[0,1]$ for which there is i in I such that $K(i)$ contains exactly one of the points $\langle x, 0\rangle$, $\langle x, 1\rangle$; (b) that as in Proposition 1.4, if $D^1\times\{0\} = D\cap([0,1]\times\{0\})$, then D^1 is (descriptive) Borel in $[0,1]$. From this we may deduce as in Theorem 2 that D may be represented in the form:

$$\bigcup_{x\in E}\{x\}\times D_x\cup\bigcup_{x\in B}\{x\}\times D_x, \tag{3}$$

where E is some set which is at most countable, while B is a Borel subset of $[0,1]$ such that, if x is in $[0,1]$, then $\{x\}\times D_x$ is compact in the subspace $\{x\}\times[0,1]$ of S^*. The argument of Proposition 3.2 may now be applied to show that, if D is a set in $\mathcal{D}(S^*)^{(\alpha)}$ for some $\alpha<\omega_1$, then it may be represented in the form (3) subject to E being at most countable and B having the property that for every x in B $\{x\}\times D_x$ is a $\mathcal{K}^{(\alpha)}$-set in the subspace $\{x\}\times[0,1]$ of S^*.

We now notice that, if F is a closed subset of a Hausdorff space X and K is compact in X then $F\cap K$ is compact in the subspace F and continuing this argument inductively, if B is in $\mathcal{K}(X)^{(\alpha)}$, then $F\cap B$ is in $\mathcal{K}(F)^{(\alpha)}$.

Suppose that T is in $\mathcal{D}(S^*)^{(\alpha)}$. Using the representation established above we may choose x so that $\{x\}\times T_x$ is a $\mathcal{K}^{(\alpha)}$ subset of the subspace $\{x\}\times[0,1]$. Now $\{x\}\times[\frac{1}{3},\frac{2}{3}]$ is closed in the subspace $\{x\}\times[0,1]$, so we deduce that $\{x\}\times(T_x\cap[\frac{1}{3},\frac{2}{3}])$ is a $\mathcal{K}^{(\alpha)}$ subset of the subspace $\{x\}\times\times[\frac{1}{3},\frac{2}{3}]$. Consider that the mapping φ: $\langle x, y\rangle\to 3y-1$ between $\{x\}\times\times[\frac{1}{3},\frac{2}{3}]$ and $[0,1]$ is a homeomorphism between $\{x\}\times[\frac{1}{3},\frac{2}{3}]$ as a subspace of S^* and the space Y of [6, Prop. 3.7]. Moreover

$$\varphi\big[\{x\}\times(T_x\cap[\tfrac{1}{3},\tfrac{2}{3}])\big] = \varphi\big[\{x\}\times(I\cap(\tfrac{1}{3},\tfrac{2}{3}))\big] = I.$$

So I is in $\mathcal{K}(Y)^{(\alpha)}$, hence is a Souslin-$\mathcal{K}(Y)$ set. Now the salient feature of the space Y is that I is not a strongly convergent Souslin subset of Y and a fortiori is not a Souslin-$\mathcal{K}(Y)$ set. The reductio ad absurdum establishes our theorem.

If a subspace B of a space X is descriptive Borel (or sigma-descriptive Borel) in X, then B is descriptive Borel (or sigma-descriptive-Borel) in any Hausdorff space Z in which B may be topologically embedded. In particular B is Borel in all such spaces Z. We have just seen that this stronger kind of absoluteness property is not necessarily shared by subspaces which are members of $\mathcal{D}(X)^{(2)}$. One might optimistically hope that, if a space B is Borel in any Hausdorff space of which it is a subspace, then B is sigma-descriptive Borel. This conjecture is false even if B is analytic.

THEOREM 7. *There in an analytic Hausdorff space which is absolutely closed with respect to Hausdorff spaces (i.e. is closed in any Hausdorff space of which it is a subspace), but is not sigma-descriptive Borel.*

Proof. We claim that the space S^* constructed in Theorem 6 has just the required property.

First note that S^* is not sigma-descriptive Borel, otherwise we may argue as in Theorem 6 that for some x in $[0,1]$ the set $\{x\}\times[\frac{1}{3},\frac{2}{3}]$ is a sigma-compact subspace of S^*, but this is not the case (its homeomorph, Y, referred to above certainly is not).

We now consider any topological space Z of which S^* is a subspace and show that $[0,1]^2$ is closed in Z. We shall be making use of the three topologies corresponding to S, S^* and Z. It will be as well to remark that, if η is any point of $[0,1]^2$ and W is an open neighbourhood of η in the sense of S^*, then there is a set G, open in S, such that

$$\eta \in G \quad \text{and} \quad \{\eta\} \cup (G \cap T) \subseteq W.$$

Suppose that $\zeta \in \mathrm{cl}_Z([0,1]^2) \setminus [0,1]^2$. Then there is a directed set $\langle A, \leqslant\rangle$ and a net $\mathfrak{t} = \langle t_a: a \in A\rangle$ of points of $[0,1]^2$ converging to ζ. Now S is a compact space, hence the net $\mathfrak{t}$ contains a subnet converging in the topology of S to a point η in $[0,1]^2$. Since this subnet will converge to ζ in the Hausdorff topology of Z, we may assume for convenience and without loss of generality that this subnet is identical with $\mathfrak{t}$. Now $\zeta \notin [0,1]^2$, so $\zeta \neq \eta$ and we may choose disjoint sets U, V open in Z with $\eta \in U$ and $\zeta \in V$. As the set $U \cap [0,1]^2$ is open in S^*, there is a set G open in S with

$$\eta \in G \quad \text{and} \quad \{\eta\} \cup (G \cap T) \subseteq U.$$

As $\mathfrak{t}$ converges in Z to ζ, there is an element a_1 in A such that for every a, with $a_1 \leqslant a$, $t_a \in V$. On the other hand $\mathfrak{t}$ converges in S to η and $\eta \in G$, so there is an element a_2 in A such that for every a, with $a_2 \leqslant a$, $t_a \in G$. We may choose an element b in A with $a_1 \leqslant b$ and $a_2 \leqslant b$. The point t_b belongs to the set $V \cap [0,1]^2$ which is open in S^*, so there is a set H, open in S, with

$$t_b \in H \quad \text{and} \quad \{t_b\} \cup (H \cap T) \subseteq V \cap [0,1]^2.$$

However $t_b \in G$, so $t_b \in G \cap H$. We deduce that $G \cap H \cap T \neq \emptyset$, but $G \cap H \cap T \subseteq U \cap V$. So U and V are not disjoint and this is a contradiction. Consequently S^* is closed in Z.

Remark 1. An absolutely closed space need not be Lindelöf, so it need not be analytic. For example, let $(X, <)$ be a totally ordered set with dense ordering and compact order topology and suppose that the weight at the point $\xi = \inf X$ is ω_1 in the order topology. Define another topology on X as follows. Let $\langle x_a: a < \omega_1\rangle$ be a monotone decreasing sequence with infimum ξ which is continuous in the sense of the order topology of X. Define basic neighbourhoods of points of X other than ξ to be the same as those of the interval topology of X. Write $T = X \setminus \{x_a: a < \omega_1\}$ and let the basic neighbourhoods of ξ take the form

$$\{x: x < x_a\} \cap T \quad \text{for} \quad a < \omega_1.$$

The space X^* so generated is Hausdorff (compare Example 1 Engelking [1, p. 48]), but is not Lindelöf, since the open cover

$$\{T, \{x: x_a < x\}: a < \omega_1\}$$

does not have a countable subcover. X^* is absolutely closed (argue as in Theorem 7).

Remark 2. It is well known (compare problem D in Engelking [1, p. 161]) that a regular Hausdorff space is absolutely closed with respect to Hausdorff spaces if and only if it is compact.

Acknowledgment. I wish to thank Professor C. A. Rogers, my supervisor, for his encouragement during the preparation of this paper and for his interest in it. I am grateful to the Science Research Council for its financial support.

Added in proof. Professor Roy O. Davies has kindly drawn my attention to Skula's article [9] where, by an argument different to ours, it is shown that $[0,1] \times \{0\}$ is not analytic in $[0,1] \times \{0,1\}$. In a forthcoming paper we shall extend the argument presented here to show that in various senses $[0,1] \times \{0\}$ is not even a projective subset. This answers a question raised by Kurepa and reported in [9].

References

[1] R. Engelking, *Outline of General Topology*, Amsterdam–Warszawa 1968.
[2] Z. Frolík, *A contribution to the descriptive theory of sets and spaces*, Gen. Top. and its Rel. to Mod. Analysis and Algebra I (1961), pp. 157–173.
[3] — *A survey of separable descriptive theory of sets and spaces*, Czech. Math. J. 20 (95) (1970), pp. 406–467.
[4] L. Gillman and M. Jerison, *Rings of Continuous Functions*, 1960.
[5] A. J. Ostaszewski, *A characterization of compact, separable, ordered spaces*, J. London Math. Soc. 7 (1974), pp. 758–760.
[6] — *On Lusin's Separation Principle in Hausdorff spaces*, Proc. London Math. Soc. 27 (1973), pp. 649–666.
[7] C. A. Rogers, *Descriptive Borel sets*, Proc. Royal Soc., A, 286 (1965), pp. 455–478.
[8] M. Sion, *Topological and measure theoretic properties of analytic sets*, Proc. Amer. Math. Soc. 11 (1960), pp. 769–776.
[9] L. Skula *The subsets P^+ and P^- of the split interval*, Publ. de l'Inst. Math. 20 (1966), pp. 121–123.

MATHEMATICS DEPARTMENT
UNIVERSITY COLLEGE, London

Accepté par la Rédaction le 29. 10. 1973