2^0 In our example both of the spaces X and Y are 1-dimensional, hence they can be embedded into 3-dimensional Euclidean space. We do not know if it is possible to construct an example of this kind taking a subspace of the plane as Y. We do not know also whether Y would be the Knaster–Kuratowski Broom.

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Added in proof. Recently the second of the authors showed, modifying the present construction, that Y can be taken as a subspace of the plane. We have also proved that if we replace in the construction of Knaster–Kuratowski Broom the rational and irrational numbers of the x-axis by two disjoint subsets of irrationals of the second category, then we obtain the space Y with a dispersion point which is not an open-perfect image of any hereditarily disconnected space.

References

[1] N. Bourbaki, *Topologie Generale*, Paris 1961.
[2] R. Engelking, *Outline of General Topology*, Amsterdam–Warszawa 1968.
[3] A. P. Kombarov, *О наследственно несвязных пространствах*, Вест. Моск. Унив. 4 (1971), pp. 21–25.
[4] K. Kuratowski, *Topology* I, II, New York–London–Warszawa 1968.

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The non-existence of Σ_2^1 well-orderings of the Cantor set

by

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Abstract. It is shown the existence of a Σ_2^1 well-ordering of the Cantor set implies that all reals are constructible. This is the converse of a theorem of Gödel.

Throughout this paper we assume the existence of a non-constructible real. With that in hand, let us set forth some notation. A finite sequence s is an extension of t if t is an initial subsequence of s. A tree is a set of finite sequences of 0's and 1's containing every initial subsequence and at least one proper extension of each of its members. For a a function with domain the set of non-negative integers, $\bar{a}(n)$ is the sequence $\langle a(0), a(1), \ldots, a(n-1)\rangle$. A path through the tree P is a function a such that $\bar{a}(n)$ is in P for every n. $[P]$ is the set of paths through P. It is easily shown that $[P]$ is a closed subset of 2^N and that every closed subset of 2^N is the set of paths through a unique tree. The tree corresponding to a closed set is its code; a closed set with a constructible code is constructibly coded. A closed set is perfect iff every sequence in its code has at least two imcompatible extensions in the code.

Let B be the Boolean algebra corresponding to forcing with constructibly coded perfect sets ordered by the subset relation. B is a complete Boolean algebra containing the constructible trees as a dense subset. There are several ways to represent B; one is as the regular open sets in the space 2^N-L with the topology generated by the constructibly coded $[P]$'s.

We are going to be using B-valued set theory. In that set theory there is a canonical generic function S in 2^N. (In the system presented in [6], S is $\{\langle\check{n}, P\rangle\colon \forall s \in P\ [\text{length}\ (s) \leqslant n \vee s_n = 1]\}$.) We are also going to be using another Boolean extension of set theory M, in which every constructible tree P has a path generic over V with respect to B. The Truth Lemma [11] states that for a generic and φ a formula in the forcing language, $V(a)$ satisfies φ iff there is a condition P with $a \in [P]$ and $P \Vdash \varphi$. In interpreting the forcing language for $V(a)$, S is a name for a. Thus if $\varphi(x)$ is a Σ_2^1 or Π_2^1 formula with possible unlisted constructible parameters and α is generic, the statement "$\varphi(\alpha)$ iff there is a constructible tree P with $\alpha \in [P]$ and $P \Vdash \varphi(S)$" has value one in the model M.

LEMMA 1. $\|S \notin L \wedge \omega_1^{(L(S))} = \omega_1^L\|_B = 1$.

Proof. See Sacks [8].

LEMMA 2. *If* $\varphi(x)$ *is* Σ_2^1 *and* P *is a condition,* $[P] \subseteq \{\alpha: \varphi(\alpha)\}$ *implies* $P \Vdash \varphi(S)$.

Proof. Let M be a Boolean extension of V in which every constructible tree has a path generic over V with respect to B. Suppose that $[P] \subseteq \{\alpha: \varphi(\alpha)\}$ but P does not force $\varphi(S)$; then since we are using weak forcing there is a condition Q extending P with $Q \Vdash \sim\varphi(S)$. Then $[Q] \cap \cap \{\alpha: \varphi(\alpha)\} = [Q]$ and so $[Q] \cap \{\alpha: \varphi(\alpha)\}$ has a non-constructible element β (via the assumption in the first sentence of this paper). By the absoluteness Lemma $\varphi(\beta)$ is valid in M, and so in M $[Q] \cap \{\alpha: \varphi(\alpha)\}$ has a constructibly coded perfect subset $[R]$. (This is the exact statement of the perfect set Theorem [5].) Since R is an extension of Q, $R \Vdash \sim\varphi(S)$; pick any generic $\alpha \in [R]$ and the contradiction is immediate.

Lemma 2 has a converse of sorts which we shall call Lemma 3 even though it is not used in anything that follows. Lemmas 2 and 3 between them say that for φ a Σ_2^1 formula, $[P] \subseteq \{\alpha: \varphi(\alpha)\}$ and $P \Vdash \varphi(S)$ bear roughly the same relation to each other as strong and weak forcing.

Using the Kondo–Addison Uniformization Theorem [7], any Σ_2^1 set A can be written as the domain of a Π_2^1 function f_A. Furthermore in ZF set theory it can be proven that f_A is a function and A is its domain.

DEFINITION. A Π_1^1 set is *large* if A has a perfect subset; a Σ_2^1 set A is *large* if f_A has a large graph.

Note that the statement "A is large" is Σ_2^1. Furthermore if A is large it has a perfect subset [4], but not necessarily vice versa. In the presence of a non-constructible real, the perfect set theorem [5] states that A is large iff it has a non-constructible element.

LEMMA 3. *If* φ *is* Σ_2^1 *and* $P \Vdash \varphi(S)$ *then* $[P] \cap \{\alpha: \varphi(\alpha)\}$ *is large.*

Proof. Again let M be a Boolean extension of V in which every constructible tree has a generic path. In M it is valid that $[P] \cap \{\alpha: \varphi(\alpha)\}$ has a non-constructible element; any generic path through P will do. Thus it is also valid in M that $[P] \cap \{\alpha: \varphi(\alpha)\}$ is large. This being a Σ_2^1 statement, it is true in V, completing the proof of the Lemma.

LEMMA 4. *If* $\varphi(x)$ *is* Π_2^1 *and* $P \Vdash \varphi(S)$, *then every non-constructible path through* P *satisfies* φ.

Proof. Otherwise $[P] \cap \{\alpha: \sim\varphi(\alpha)\}$ would have a non-constructible element, and hence a constructibly coded perfect subset, violating Lemma 2.

The class $L(\alpha)$ of sets hereditarily constructible from α is often defined to be the denotation of certain ranked terms $\tau(\alpha, \sigma_1, \ldots, \sigma_n)$ where the σ_i are ordinals. These terms are such that within any transitive model for Kripke–Platek set theory containing α and each σ_i, $\tau(\alpha, \sigma_1, \ldots, \sigma_n)$ has the same value as it has in the universe. If t is a well-ordering of integers, let $|t|$ be its order type.

LEMMA 5. *If* $t_1, \ldots, t_n$ *are well-orderings of integers, the predicate* $\beta = \tau(\alpha, |t_1|, \ldots, |t_n|)$ *is* Δ_2^1 *in the parameters* $\alpha, \beta, t_1, \ldots, t_n$.

Proof. $\beta = \tau(\alpha, |t_1|, \ldots, |t_n|)$ iff it is true in any or all countable transitive models for Kripke–Platek set theory containing the parameters. This is in turn equivalent to its being true in any or all well-founded models for Kripke–Platek set theory containing surrogates for the parameters. This latter condition is easily seen to be Δ_2^1.

In order to illustrate how these lemmas can be used to elucidate perfect set forcing, let us give a new proof of an old theorem from [8].

THEOREM 1. *The statement "S has minimal degree of constructibility" has value one.*

Proof. Suppose otherwise. Then there is a term τ and ordinals $\sigma_1, \ldots, \sigma_n$ and a condition P such that

$$P \Vdash S \notin L(\tau(S, \sigma_1, \ldots, \sigma_n)) \wedge \tau(S, \sigma_1, \ldots, \sigma_n) \notin L .$$

Since $\omega_1^L = \omega_1^{L(S)}$ (Lemma 1), we may assume that $\sigma_1, \ldots, \sigma_n$ are all constructibly countable. Therefore by Lemma 5 and the well-known theorem that "$\alpha \in L(\beta)$" is Σ_2^1 [10], the predicate $R(\alpha)$ defined by $\alpha \notin L(\tau, \sigma_1, \ldots, \sigma_n) \wedge \wedge \tau(\alpha, \sigma_1, \ldots, \sigma_n) \notin L$ is Π_2^1 in constructible parameters. From Lemma 4 every non-constructible member of $[P]$ satisfies R. Let α_0 be a non-constructible element of $[P]$ and let β_0 be $\tau(\alpha_0, \sigma_1, \ldots, \sigma_n)$. The set $\{\alpha \in [P]: \tau(\alpha, \sigma_1, \ldots, \sigma_n) = \beta_0\}$ is Σ_2^1 in β_0 and constructible parameters, non-empty, and has no element in $L(\beta_0)$, contradicting the Absoluteness Lemma. Thus our original assumption is false and the theorem is proven.

THEOREM 2. *If there is a non-constructible real, there is no* Σ_2^1 *well-ordering of* 2^N.

Proof. Suppose otherwise that $<$ is a Σ_2^1 formula which well-orders 2^N. We claim that the Boolean value of "In $L(S)$ $<$ well-orders 2^N." is one. First note that by writing down completely "$<$ well-orders 2^N.", we see that it is of the form $\varphi \wedge \forall \alpha, \beta[\alpha = \beta \vee \alpha < \beta \vee \beta < \alpha]$ where φ is Π_2^1. Two applications of the Absoluteness Lemma reveal that since φ is true in V, it is valid in $V(B)$ and hence valid in $L(S)$. So the only way our claim can be false that for terms τ_1, τ_2 and constructibly countable ordinals $\sigma_1, \ldots, \sigma_n$ and a condition P the following is satisfied:

$$P \Vdash \tau_1(S, \sigma_1, \ldots, \sigma_n) \nless \tau_2(S, \sigma_1, \ldots, \sigma_n) \wedge \tau_2(S, \sigma_1, \ldots, \sigma_n) \nless \tau_1(S, \sigma_1, \ldots, \sigma_n) .$$

By Lemma 5 the statement forced is Π_2^1 in constructible parameters and so must be satisfied by every non-constructible path through P. Since there is such a path, this contradicts our original assumption that $<$ is a linear ordering, and establishes the claim.

It is easy to see that the only elements of B invariant under all automorphisms of B are 0 and 1. From this it follows that in $L(S)$ all definable sets are constructible [11], [6, Theorem 6.8]. However it must also be valid that the first non-constructible element in the ordering $<$ is definable and non-constructible.

References

[1] H. Friedman, *Minimality in the Δ_2^1 degrees*, Fund. Math. 81 (1974), pp. 183–197.
[2] — PCA *well-orderings of the line*, J. S. L. 39 (1974), pp. 79–80.
[3] K. Gödel, *The Consistency of the Continuum Hypothesis*, Princeton, N. J. 1940.
[4] R. Mansfield, *On the possibility of a Σ_2^1 well-ordering of the Baire Space*, J.S.L. 38 (1973), pp. 396–398.
[5] — *Perfect subsets of definable sets*, Pac. Jour. Math., 35 (1970), pp. 451–457.
[6] — and J. Dawson, *Boolean valued set theory and forcing* (to appear).
[7] D. A. Martin, *Projective sets and cardinal numbers* (to appear), J.S.L.
[8] G. Sacks, *Forcing with perfect closed sets*, Proc. Sym. Pure Math. 13 A.M.S., Providence, R. I. (1971), pp. 331–335.
[9] J. Shoenfield, *Mathematical Logic*, Menlo Park, Calif. 1967.
[10] — *The problem of predicativity*, Essays on the Foundations of Mathematics, Jerusalem, pp. 132–139.
[11] — *Unramified forcing*, Proc. Sym. Pure Math. 13, A.M.S., Providence, R. I. (1971), pp. 357–381.
[12] R. M. Solovay, *A model of set theory in which every set of reals is Lebesgue measurable*, Ann. of Math. 92 (1970), pp. 1–56.

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On longest paths in connected graphs *

by

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Abstract. It is shown that a connected graph of order $p \geqslant 4$ contains a path of length k, where $1 \leqslant k \leqslant p-1$, if for every integer j with $1 \leqslant j < k/2$, the number of vertices of degree not exceeding j is less than j. Furthermore, a tree of order $p \geqslant 4$ has diameter at least k, where $3 \leqslant k \leqslant p-1$, if the number of vertices of degree one is less than $\{2(p-1)/(k-1)\}$.

A *hamiltonian cycle* (*path*) in a graph G is a cycle (path) containing every vertex of G. Pósa [1] proved that if G is a graph of order $p \geqslant 3$ such that for every integer j with $1 \leqslant j < p/2$, the number of vertices of degree not exceeding j is less than j, then G contains a hamiltonian cycle. In this article, we establish an analogous result for graphs with hamiltonian paths and in fact for graphs containing paths of any specified length.

THEOREM 1. *Let G be a connected graph of order $p \geqslant 4$. Then G contains a path of length k $(1 \leqslant k \leqslant p-1)$ if for every integer j with $1 \leqslant j < k/2$ the number of vertices of degree not exceeding j is less than j.*

Proof. Since G is connected and $p \geqslant 4$, the theorem is true for $k = 1$ and $k = 2$. Henceforth we assume $k \geqslant 3$. Suppose the length of a longest path in G is n where $2 \leqslant n < k$. If P is a longest path in G, let S_P denote $\deg u + \deg v$, where u and v are the endvertices of P. Among all longest paths in G, choose P: $u_0, u_1, \ldots, u_n$ so that S_P is maximum. Suppose $\deg u_0 \leqslant \deg u_n$.

Since P is a longest path, each of u_0 and u_n is adjacent only to vertices of P. If $u_i u_n \in E(G)$, $0 \leqslant i \leqslant n-1$, then $u_0 u_{i+1} \notin E(G)$; for otherwise the cycle

$$C:\ u_0, u_1, \ldots, u_i, u_n, u_{n-1}, \ldots, u_{i+1}, u_0$$

of length $n+1$ is present in G. The cycle C cannot contain all vertices of G since $n+1 < p$. Since G is connected, there exists a vertex w not on C adjacent to a vertex of C; however this implies G contains a path of length $n+1$ which is impossible. Hence for each vertex of $\{u_0, u_1, \ldots, u_{n-1}\}$

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