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On level sets of Darboux functions

by

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Abstract. K. M. Garg [2] has found the necessary conditions that a set of real numbers be the set of all points y such that the level sets $f^{-1}(y)$ are single point sets, dense in the selves sets, closed sets, connected sets and perfect sets, where f is a free choice Darboux function. The aim of this paper is to give a proof that all Garg's conditions are not only necessary but also sufficient.

Let f be a real function of a real variable. The set $\{x: f(x) = y\} = f^{-1}(y)$ will be referred to as the level set of f corresponding to the value y. Generally we are not able to draw conclusions as to the properties of functions from the properties of their level sets. E.g., it is well known [5] that all level sets may be closed sets, even one point sets, whereas the function itself is not measurable in the sense of Lebesgue. However, under certain additional stipulations as to the function there follow from an appropriate regularity of sufficiently many level sets strong conclusions about the function itself. E.g. if the function f possesses the Darboux property, and the set of those values y for which the level sets $f^{-1}(y)$ are closed is dense, then f is continuous [3].

Special families of level sets of continuous functions have been discussed in [4], [1] and [2].

Let us denote by 1 the family of all single-point sets on the real axis. Furthermore, let us denote by d the family of all sets dense in themselves, by k the family of all closed sets, by c the family of all connected sets, by p the family of all perfect sets and finally by ∞ the family of all infinite sets. This rather non-typical notation will be adopted here because of the notation adopted in other papers on level sets. If $*$ is a family of sets and f a fixed function, we shall put $Y_*(f) = \{y: f^{-1}(y) \in *\}$. Let us denote, in the usual way, by G_δ the family of all sets of the form $\bigcap_{n=1}^{\infty} G_n$ where G_n are open sets, and by $F_{\sigma\delta}$ the family of all sets of the form $\bigcap_{k=1}^{\infty}\bigcup_{n=1}^{\infty} F_{k,n}$ where $F_{k,n}$ are closed sets. Let F^- denote the family of all sets of the form $A\setminus B$, where A is a closed set and B is a subset of the set of all end-points of components of the complement of A. Every point of B is hence an end-point of an open interval disjoint with A. Let F^+ denote the family of all sets of the form $A\setminus B$ where A is closed and B is at most denumerable. Evidently $F^-\subset F^+$. Finally, let G_δ^+ denote the family of all sets of the form $A\cup B$ where $A\in G_\delta$ and B is at most denumerable. We can see that $F^+\subset G_\delta$ and $G^+\subset F_{\sigma\delta}$. Let C be the family of all continuous functions, D the family of all Darboux functions, i.e. of all functions mapping connected sets onto connected sets.

It has been proved by Sierpiński in [4] that $\{Y_p(f): f\in C\}\subset F_{\sigma\delta}$ and that $\{Y_1(f): f\in C\}\subset G_\delta$. Garg has obtained an even stronger theorem: $\{Y_p(f): f\in D\}\subset G^+$ and $\{Y_1(f): f\in D\}\subset F^-$ ([2], Th. 10 and 9). These results of Sierpiński and Garg have the form of inclusions of classes of sets. They quote necessary conditions to be satisfied by the set E so as to ensure the existence of a continuous function or a Darboux function f such that $E=Y_*(f)$. A complete characterization of a class $Y_*(f)$ can be found only in Borsuk's paper [1], where he proves that

$$\{Y_\infty(f): f\in C\}=G_\delta^+ .$$

The aim of this paper is to give a complete characterization of the remaining classes of sets $Y_*(f)$ for continuous functions and Darboux functions. We shall prove that in all cases investigated by Garg the symbol of inclusion can be replaced by the symbol of equality.

THEOREM 1. $F^-\subset\{Y_c(f): f\in C\}\cap\{Y_1(f): f\in C\}$.

Proof. Let $E\in F^-$. Then $E=A\setminus B$, where according to the definition of the class F^-, A is a closed set and B consists of the end-points of open intervals disjoint with A. Let $R\setminus A=\bigcup_i(a_i, b_i)$ and $(a_i, b_i)\cap(a_j, b_j)\neq\emptyset$ for $i\neq j$.

If $E=\emptyset$, we take $f(x)=x\sin x$. In this case evidently $Y_c(f)=Y_1(f)=\emptyset=E$.

If $E\neq\emptyset$, we take $f(x)=x$ for $x\in A$ and then we define f on each component of $R\setminus A$.

1. If $a_i\notin E$, $b_i\notin E$, $b_i\neq+\infty$ and $a_i\neq-\infty$, then f is on $\langle a_i, b_i\rangle$ continuous and piecewise linear and the points $A(a_i, a_i)$, $B(\frac{2}{3}a_i+\frac{1}{3}b_i, b_i)$, $C(\frac{1}{3}a_i+\frac{2}{3}b_i, a_i)$ and $D(b_i, b_i)$ are the vertices of its graph.

2. If $a_i\in E$ and $b_i\in E$, then we define f on (a_i, b_i) in such a way that its graph is a broken line with infinitely many sides. The vertices of this broken line are situated on the sides of the rhombus $AB'C'D$, where A and D have the same coordinates as in the previous case,

$$B'(\tfrac{2}{3}a_i+\tfrac{1}{3}b_i,\ \tfrac{1}{3}a_i+\tfrac{2}{3}b_i)\quad\text{and}\quad C'(\tfrac{1}{3}a_i+\tfrac{2}{3}b_i,\ \tfrac{2}{3}a_i+\tfrac{1}{3}b_i)\,.$$

The sides of the broken line have angular coefficients aqual to 10 or -10. B is one of the vertices of the broken line.

3. If $a_i\notin E$, $b_i\in E$ and $a_i\neq-\infty$, we define f on (a_i, b_i) as in case 2 replacing the quadrangle $AB'C'D$ by the triangle AB^*D and the given vertex B' by B^*. We take $B^*(\frac{1}{2}(a_i+b_i), a_i)$.

4. If $a_i\in E$, $b_i\notin E$ and $b_i\neq+\infty$, we define f on (a_i, b_i) as in case 2 replacing the quadrangle $AB'C'D$ by the triangle $AB^{**}D$ and the given vertex B' by B^{**}. We take $B^{**}(\frac{1}{2}(a_i+b_i), b_i)$.

5. If $b_i=+\infty$, then for $x>a_i$ we define the function f so that its graph is a broken line with infinitely many sides. The angular coefficients of the sides are equal to 10 or -10 and the vertices of the broken line are situated on the lines

$$y=\tfrac{1}{5}(x-a_i)\quad\text{and}\quad y=a_i\quad\text{if}\quad a_i\notin E$$

or on the lines

$$y=\tfrac{1}{10}(x-a_i)\quad\text{and}\quad y=\tfrac{1}{5}(x-a_i)\quad\text{if}\quad a_i\in E\,.$$

6. If $a_i=-\infty$, then for $x<b_i$ we define the function f as in case 5, replacing the straight lines occurring there by others:

$$y=\tfrac{6}{5}(x-b_i)\quad\text{and}\quad y=b_i\quad\text{if}\quad b_i\notin E\,,$$

and

$$y=\tfrac{6}{5}(x-b_i)\quad\text{and}\quad y=\tfrac{11}{10}(x-b_i)\quad\text{if}\quad b_i\in E\,.$$

It is easy to verify that the function f defined in such a way is continuous, and that the sets $f^{-1}(y)$ are single point sets if and only if $y\in E$. Similarly these sets are connected if and only if $y\in E$. Therefore we have $E=Y_c(f)=Y_1(f)$ and the proof is completed.

Garg showed in [2] (Th. 8 and Lemma 7) that $\{Y_c(f): f\in D\}\subset F^-$. Since $\{Y_c(f): f\in C\}\subset\{Y_c(f): f\in D\}$, we imply from Garg's theorem and from the above Theorem 1.

COROLLARY 1. $\{Y_c(f): f\in C\}=\{Y_c(f): f\in D\}=F^-$.

THEOREM 2. $F^-\subset\{Y_k(f): f\in D\}$.

Proof. Assume that $E\in F^-$. If $E=\emptyset$, we take as f any arbitrary function mapping every interval onto the whole straight line R. This function evidently has the Darboux property. All the sets $f^{-1}(y)$ are dense and boundary sets and therefore they are not closed. Hence $Y_k(f)=\emptyset=E$.

Let us further assume that $E\neq\emptyset$. According to the definition of the class F^- we have $E=A\setminus B$, where A is closed and B consists of the end-points of open intervals disjoint with A. Let $R\setminus A=\bigcup_i(a_i, b_i)$ where $(a_i, b_i)\cap(a_j, b_j)=\emptyset$ for $i\neq j$.

Take $f(x)=x$ for $x\in A$.

In each interval (a_i, b_i) we choose two sequences: a decreasing sequences x'_n and an increasing sequence x''_n such that $x'_1 < x''_1$, $\lim_{n\to\infty} x'_n = a_i$ and $\lim_{n\to\infty} x''_n = b_i$. The function f will be defined at the points x'_n and x''_n in the following way:

1. If $a_i = -\infty$ and $b_i \in E$, then $f(x'_{2n}) = f(x''_{2n}) = b_i - n$ and $f(x'_{2n-1}) = f(x''_{2n-1}) = b_i$.

2. If $a_i \in E$ and $b_i = +\infty$, then $f(x'_{2n}) = f(x''_{2n}) = a_i$ and $f(x'_{2n-1}) = f(x''_{2n-1}) = a_i + n$.

3. If $a_i \neq -\infty$, $b_i \neq +\infty$, $a_i \notin E$ and $b_i \notin E$, then $f(x'_{2n}) = f(x''_{2n}) = a_i$ and $f(x'_{2n-1}) = f(x''_{2n-1}) = b_i$.

4. If $a_i \in E$ and $b_i \in E$, then $f(x'_{2n-1}) = f(x''_{2n-1}) = \frac{1}{2}(a_i + b_i)$, $f(x'_{2n}) = a_i + \dfrac{1}{2n}(b_i - a_i)$ and $f(x''_{2n}) = b_i - \dfrac{1}{2n}(b_i - a_i)$.

5. If $a_i \in E$, $b_i \notin E$ and $b_i < +\infty$, then $f(x'_{2n}) = f(x''_{2n}) = b_i$, $f(x''_{2n-1}) = \frac{1}{2}(a_i + b_i)$ and $f(x'_{2n-1}) = a_i + \dfrac{1}{2n}(b_i - a_i)$.

6. If $b_i \in E$, $a_i \notin E$ and $a_i > -\infty$, then $f(x'_{2n}) = f(x''_{2n}) = a_i$, $f(x''_{2n-1}) = \frac{1}{2}(a_i + b_i)$ and $f(x''_{2n-1}) = b_i - \dfrac{1}{2n}(b_i - a_i)$.

In each of the intervals $\langle x'_{n+1}, x'_n \rangle$, $\langle x'_1, x''_1 \rangle$, $\langle x''_n x''_{n+1} \rangle$ at the end-points of which the function f has already been defined we define f as a linear function.

7. If $a_i = -\infty$ and $b_i \notin E$, we define f in the interval (a_i, b_i) so that it will map each interval $(\alpha, \beta) \subset (a_i, b_i)$ onto a half-line $(-\infty, b_i\rangle$.

8. If $b_i = +\infty$ and $a_i \notin E$, we define f in the interval (a_i, b_i) so that it will map each interval $(\alpha, \beta) \subset (a_i, b_i)$ onto a half-line $\langle a_i, +\infty)$.

It is easy to verify that the function f defined in such a way has the Darboux property and that its level sets $f^{-1}(y)$ are closed if and only if $y \in E$. We have therefore $E = Y_k(f)$. Our theorem is proved.

Garg has proved ([2], Th. 8 and Lemma 7) that $\{Y_k(f)\colon f \in D\} \subset F^-$. There from and from Theorem 2 we obtain

COROLLARY 2. $\{Y_k(f)\colon f \in D\} = F^-$.

THEOREM 3. $F^+ \subset \{Y_1(f)\colon f \in C\}$.

Proof. Let $E \in F^+$. Then $E = A \setminus B$, where A is closed and B is at most denumerable. As the set A is closed, we have $A \in F^-$. According to Theorem 1 there exists a continuous function g such that $A = Y_1(g)$. The set $A \cap B$ is at most denumerable. For $y \in A$ the level sets $g^{-1}(y)$ are single-point sets. The set $g^{-1}(A \cap B)$ is at most denumerable. Let h be a continuous non-decreasing function mapping R onto R such that $h^{-1}(g^{-1}(A \cap B))$ is the union of constancy intervals of h. It is easy to see that the function $f = g[h]$ is the required continuous function for which $E = Y_1(f)$.

It follows from Garg's theorem which has been quoted at the beginning of this paper and from Theorem 3 that

COROLLARY 3. $\{Y_1(f)\colon f \in C\} = \{Y_1(f)\colon f \in D\} = F^+$.

THEOREM 4. $G_\delta^+ \subset \{Y_p(f)\colon f \in C\}$.

Proof. Let $E \in G_\delta^+$. If $E = \emptyset$ it is sufficient to take $f(x) = x$ to obtain $E = Y_p(f)$. Assume now that $E \neq \emptyset$. We shall make an additional assumption: $E \subset (0, 1)$. According to the definition, $E = A \cup B$, where $A \in G_\delta$ and B is at most denumerable. We have $(0, 1) \setminus A \in F_\sigma$, i.e., $(0, 1) \setminus A = \bigcup_{n=1}^{\infty} F_n$ where F_n are closed sets. There may be void sets among the sets F_n. Let P be the set of all even integers, and N the set of all odd integers. Let $g(x) = \operatorname{dist}(x, P)$. Let us form all intervals having two successive integers as their end-points into a sequence $\{I_n\}$. Put $F_n^{-1} = I_n \cap g^{-1}(F_n)$ and let $H = P \cup N \cup \bigcup_{n=1}^{\infty} F_n^{-1}$. It is evident that the sets F_n^{-1} are closed, whereas $R \setminus H$ is an open set. Let (a_i, b_i) $(i = 1, 2, \ldots)$ be the sequence of the components of $R \setminus H$. None of these components involves an integer and therefore the function g is linear on any of these components and the angular coefficient is equal to 1 or -1.

We shall partition each interval (a_i, b_i) by the points $a_i + \dfrac{b_i - a_i}{2^n}$, $b_i - \dfrac{b_i - a_i}{2^n}$ $(n = 1, 2, \ldots)$ into a sequence of intervals $I_{i,r}$ $(r = 1, 2, \ldots)$ with disjoint interiors. Let $I_{i,r} = \langle t_{i,r}, u_{i,r} \rangle$. Let $\varphi_{i,r}$ be a continuous function such that $\varphi_{i,r}(I_{i,r}) = g(I_{i,r})$, $\varphi_{i,r}(t_{i,r}) = g(t_{i,r})$, $\varphi_{i,r}(u_{i,r}) = g(u_{i,r})$ and that for each $y \in \varphi_{i,r}(I_{i,r})$ the set $\varphi_{i,r}^{-1}(y)$ is perfect. The construction of such a function has been described in [6].

We shall define the function f^* as follows: $f^*(x) = g(x)$ for $x \in P \cup N \cup \bigcup_{n=1}^{\infty} (F_n^{-1} \cup \{a_n\} \cup \{b_n\})$ and $f^*(x) = \varphi_{i,r}(x)$ for $x \in I_{i,r}$. The function f^* is evidently continuous. If $y_0 \notin A$, then there exists a number n_0 such that $y_0 \in F_{n_0}$. Then there exists in I_{n_0} exactly one point x_0 such that $g(x_0) = y_0$. It is evident that $x_0 \in F_{n_0}^{-1}$ and x_0 is an isolated point of $f^{*-1}(y_0)$. Therefore this set cannot be dense in itself. If $y_0 \in A$, then there exists for each n exactly one interval I_{n,r_n} such that $g^{-1}(y_0) \cap I_{n,r_n} \neq \emptyset$. We have

$$f^{*-1}(y_0) = \bigcup_{n=1}^{\infty} \varphi_{n,r_n}^{-1}(y_0). \tag{1}$$

The sets $\varphi_{n,r_n}^{-1}(y_0)$ are perfect, included in I_n, and there is no accumulation point of the union (1) which does not belongs to one of the summands of this union. Therefore the set $f^{*-1}(y_0)$ is for $y_0 \in A$ a perfect set. Thus we have proved that $A = Y_d(f^*) = Y_p(f^*)$.

The function f^* maps the line R onto the closed interval $\langle 0, 1\rangle$. We shall modify it so that it will map R onto the open interval $(0,1)$, remaining continuous all the time.

Let $\{p_n\}$ be the sequence of all integers. Let (a_{r_n}, b_{r_n}) and (a_{s_n}, b_{s_n}) be the components of the set $R \setminus H$ such that $b_{r_n} = p_n = s_{r_n}$. Such components exist because integers are isolated points of the set H. For $p_n \in N$ we put $\delta_n = \frac{1}{2}\min\left(\frac{1}{n}, \operatorname{dist}(1, F_{r_n} \cup F_{s_n})\right)$. Let us take $f^{**}(x) = f^*(x)$ for $x \in P$ and $f^{**}(x) = \min(1-\delta_n, f^*(x))$ for $x \in (p_n-1, p_n+1)$. Let $p_n \in P$, let $\varepsilon_n = \frac{1}{2}\min\left(\frac{1}{n}, \operatorname{dist}(0, F_{r_n} \cup F_{s_n})\right)$. Let $g(x) = f^{**}(x)$ for $x \in N$ and $g(x) = \max(\varepsilon_n, f^{**}(x))$ for $x \in (p_n-1, p_n+1)$. The function g defined in such a way is continuous and maps the line R onto the interval $(0,1)$. Just as in the case of the function f^* there exists for $y_0 \notin A$ at last one isolated point of the level set $g^{-1}(y_0)$. For $y_0 \in A$ we get the level set $g^{-1}(y_0)$ from the level set $f^{*-1}(y_0)$, replacing in (1) a finite number of the summands of the union by void sets or closed intervals. So these level sets are perfect and we have also for this function $A = Y_d(g) = Y_p(g)$.

Because of the continuity of g its level sets are closed sets. Each of them contains an at most denumerable set of isolated points. Let $Z(y)$ be the set of isolated points of $g^{-1}(y)$. Let $W = \bigcup_{y \in B} Z(y)$. The set W is at most denumerable. Let h be a continuous non-decreasing function mapping R onto R such that $h^{-1}(W)$ is the union of all constancy intervals of h. Take $f = g[h]$. As can be seen, for $y \in B$ the level sets $f^{-1}(y)$ are perfect sets. The same holds for $y \in A$. The function does not possess any other level sets dense in themselves. Therefore $E = Y_d(f) = Y_p(f)$.

Let us now drop the assumption $E \subset (0,1)$. If this assumption is not true, then let $\varphi(x) = \frac{1}{\pi}\operatorname{arctg} x + \frac{1}{2}$. Let $E^* = \varphi^{-1}(E)$. Evidently $E^* \subset (0,1)$ and $E^* \in G_\delta$. In accordance with what has already been demonstrated there exists a continuous function ψ mapping R onto $(0,1)$ and such that $E^* = Y_d(\psi) = Y_p(\psi)$. Take $f = \varphi^{-1}[\psi]$. Then of course $E = Y_d(f) = Y_p(f)$ and the proof is complete.

K. M. Garg demonstrated in [3], Th. 10 that

$$\{Y_p(f)\colon f \in D\} \cup \{Y_d(f)\colon f \in D\} \subset G_\delta^+.$$

As we have $\{Y_p(f)\colon f \in C\} \subset \{Y_d'(f)\colon f \in D\}$, therefore from Garg's theorem and from Theorem 4 we get

COROLLARY 4. $\{Y_p(f)\colon f \in C\} = \{Y_p(f)\colon f \in D\} = \{Y_d(f)\colon f \in D\} = G^+$.

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